A class of non-expected utility risk measures and implications for asset allocations

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Received 1 May 2000; accepted 11 October 2000

Abstract

This paper discusses a class of risk measures developed from a risk measure recently proposed for insurance pricing. This paper reviews the distortion function approach developed in the actuarial literature for insurance risk. The proportional hazards transform is a particular case. The relationship between this approach to risk and other approaches including the dual theory of choice under risk is discussed. A new class of risk measures with suitable properties for asset allocation based on the distortion function approach to insurance risk is developed. This measure treats upside and downside risk differently. Properties of special cases of the risk measure and links to conventional portfolio selection risk measures are discussed. © 2000 Elsevier Science B.V. All rights reserved.

\textit{JEL classification:} D81; G11

\textit{Keywords:} Risk measure; Non-expected utility; Asset allocation

1. Introduction

The appropriate measure to use to quantify risk for portfolio selection continues to be a subject for debate. Balzer (1994) outlines a range of specific risk measures, discusses the advantages and disadvantages of them and comes to the conclusion that semi-variance is the best measure of investment risk. Markowitz (1952) originally developed a portfolio selection methodology based on mean-variance analysis using the variance of returns as a risk measure although he preferred a semi-variance measure. Markowitz (1959) demonstrated the relationship between a number of different risk measures and utility functions. Lipman (1990) provides a review of expected utility and considers an utility approach to investment strategy incorporating benchmarks.

Expected utility is a standard approach to uncertainty in financial and economic theory. Investor preferences over uncertain investment outcomes are assumed to have an expected utility property. Investors select from alternative investment portfolios by maximizing expected utility. Von Neumann and Morgenstern (1944) develop a set of axioms under which investor preferences given by a utility function will have the expected utility property. Expected utility theory has been criticized because observed risk taking behaviour does not always exhibit the expected utility property (Allais, 1953; Kahneman and Tversky, 1979) and experiments do not conform to the independence axiom.
Wang (1995) has proposed an approach to insurance pricing using the proportional hazards (PH) transform. This approach to insurance risk is related to the dual theory of choice under uncertainty of Yaari (1987). Insurance and investment risks are closely related, so it is of interest to consider this approach to insurance risk in the investment context. These alternative approaches are examples of non-expected utility risk measures. We do not cover all of the alternative non-expected utility risk measures in this paper. Instead we concentrate on the distortion function approach and the dual theory.

This paper begins by outlining the PH transform approach to insurance risk proposed by Wang (1995) and the more general distortion function approach. The dual theory of choice under uncertainty is then outlined and the links between this theory and expected utility theory and the PH transform discussed. Wang and Young (1997) provide a more comprehensive discussion of utility theory and Yaari’s dual theory of risk than is covered here.

The portfolio selection problem is then reviewed for the expected utility case and for the distortion function as a measure of risk. We consider the single-period case. More details on the expected utility approach can be found in books such as Ingersoll (1987) and Pliska (1997).

We then propose a new class of risk measures for portfolio selection based on the dual theory concepts but with desirable properties required for an asset allocation model including risk aversion and diversification. Properties of this risk measure are discussed and relationships between certain members of this class and classical risk measures are covered.

2. Proportional hazards transform

Wang (1995, 1996a) considers a non-negative loss random variable $X \geq 0$ with distribution function $F_X(t)$, decumulative distribution function

$$S_X(t) = \Pr\{X > t\} = 1 - F_X(t) = \bar{F}_X(t),$$

and density function $f_X(t)$.

The PH transform is defined as

$$g(x) = x^r,$$

where $0 < r \leq 1$. The risk-adjusted premium is then

$$H_g(X) = \int_0^\infty g[S_X(t)] \, dt = \int_0^1 S_X^{-1}(q) \, dg(q).$$

For instance if $r = 1$ then $H_g(X) = \int_0^\infty [S_X(t)] \, dt = E[X]$ and we have risk-neutrality. Wang (1996b) also interprets the parameter $r$ as a measure of ambiguity aversion. $H_g(X) - E[X]$ can be interpreted as an insurance risk premium.

For insurance pricing, we require $g$ to be concave and increasing with $g(0) = 0$, $g(1) = 1$ and hence $g(x) \geq x$ and $0 \leq x \leq 1$. The function $g$ is referred to as a distortion function.

The use of the PH transform approach to insurance pricing appears to have been motivated by assuming an analogy between a loss distribution and a survival distribution. The expected loss is treated as equivalent to the expected lifetime.

If $S_T(t)$ is the survival distribution function, $F_T(t)$ the decumulative distribution function, and $f_T(t)$ the density function, then the hazard (or failure) rate function $\lambda(t)$ is defined as

$$\lambda(t) = \frac{f_T(t)}{1 - F_T(t)} = \frac{-(d/dt)S_T(t)}{S_T(t)} = -\frac{d}{dt} \ln [S_T(t)].$$
so that
\[ S_T(t) = \exp \left( - \int_0^t \lambda(s) \, ds \right). \]
The expected lifetime is
\[ E[T] = \int_0^\infty tf_T(t) \, dt = \int_0^\infty [S_T(t)]' \, dt. \]
Now assume a proportional change to the hazard function so that
\[ \lambda^*(t) = r\lambda(t). \]
The survival distribution function under the new hazard rate is given by
\[ -\frac{d}{dt} \ln [S_{T^*}(t)] = r\lambda(t), \]
so that
\[ S_{T^*}(t) = \exp \left( - \int_0^t r\lambda(s) \, ds \right) = [S_T(t)]^r. \]
The expected lifetime is now
\[ E[T^*] = \int_0^\infty [S_T(t)]^r \, dt. \]
We note that the PH transform approach to insurance pricing considers the loss distribution in a similar manner to
the survival distribution. The PH transform distorts the probability of a claim occurring and then uses the distorted
probabilities to calculate the expected value.

For \( X \) and \( Y \) non-negative random variables we have the following properties of the PH transform with \( g(x) = x^r \),
where \( 0 < r \leq 1 \) (Wang, 1996a):
1. \( E[X] \leq H_g[X] \leq \max(X) \);
2. \( H_g(ax + b) = aH_g(X) + b, a \geq 0, b \geq 0 \);
3. \( H_g(X + Y) \leq H_g(X) + H_g(Y) \).
This third property is referred to as sub-additivity. \( H_g(X) \) will be additive in the special case of comonotonic risks.
Risks \( X_1 \) and \( X_2 \) are comonotonic if there exists a risk \( Z \) and non-decreasing real-valued functions \( f \) and \( h \) such
that \( X_1 = f(Z) \) and \( X_2 = h(Z) \) (Wang and Young, 1997). The concept of comonotonic risks is an extension of
perfect correlation.

In general we have the following properties using the distortion function approach:
1. If \( g(p) = p \), for all \( p \in [0, 1] \), then \( H_g[X] = E[X] \).
2. If \( g(p) \geq p \), for all \( p \in [0, 1] \), then \( H_g[X] \geq E[X] \).
3. \( H_g(ax + b) = aH_g(X) + b, a \geq 0, b \geq 0 \).
4. For \( X \) and \( Y \) comonotonic, \( H_g(X + Y) = H_g(X) + H_g(Y) \).
5. For concave \( g \), \( H_g[X] \geq E[X] \), and \( H_g(X + Y) \leq H_g(X) + H_g(Y) \).
6. For convex \( g \), \( H_g[X] \leq E[X] \), and \( H_g(X + Y) \geq H_g(X) + H_g(Y) \).

3. Dual theory of choice

Consider preference over risks. The symbol \( \succ \) will denote preference or riskiness, so that \( X \succ Y \) indicates \( X \) is
preferred to \( Y \). The use of expected utility as a risk measure is derived from five axioms. These are as follows:
1. If risks $X_1$ and $X_2$ have the same cumulative distribution function, then $X_1$ and $X_2$ are equally risky.

2. $\succ$ is reflexive, transitive and connected (weak order).

3. $\succ$ is continuous in the topology of weak convergence.

4. If $S_X \leq S_Y$, then $X \succ Y$.

5. If $X \succ Y$ and $Z$ is any risk then

$$\mathbb{E}[U(X)] = \int_0^\infty S_X(t) \, dU(t) = \int_0^1 U[S_X^{-1}(q)] \, dq.$$ 

Consider two random payments $X$ and $Y$. Under the expected utility property, $X$ is preferred to $Y$ if

$$\mathbb{E}[U(X)] > \mathbb{E}[U(Y)],$$

where $U$ is assumed to be a continuous non-decreasing function. $U$ is a concave function for a risk averse individual and is unique up to a positive affine transformation (Gerber and Pafumi, 1998).

Yaari (1987) develops the following dual theory of choice where this independence axiom is replaced with the dual independence axiom which states that if $X$ is preferred to $Y$, then a lottery that pays $X$ with probability $\alpha$ and $Z$ with probability $(1 - \alpha)$ will be preferred to a lottery that pays $Y$ with probability $\alpha$ and $Z$ with probability $(1 - \alpha)$. Thus, there is independence with respect to probability mixtures of uncertain outcomes.

If investors conform to these axioms then they will prefer strategies that have higher expected utilities. For a non-negative random variable $X$, the expected utility is given by

$$\mathbb{E}[U(X)] = \int_0^\infty S_X(t) \, dU(t) = \int_0^1 U[S_X^{-1}(q)] \, dq.$$ 

An important implication of the independence axiom and expected utility is that the preference function is linear in the probabilities. Experimental evidence has suggested that decision making behaviour does not conform with the independence axiom (Machina, 1982, 1987). The most famous of these violations of the independence axiom is probably the Allais Paradox. This is a particular example of what is referred to as the common consequence effect. An alternative approach is to use preference functions which are not linear in the probabilities such as Yaari’s dual theory. Other non-linear functional forms have also been suggested as referenced in Machina (1987).
4. Investment selection

4.1. Expected utility

The expected utility-based portfolio theory approach to investment selection is to assume that investors maximize expected utility subject to constraints. Panjer et al. (1998, Chapter 8) gives more details. It will be assumed that the investor has a single period investment horizon and that an appropriate utility function exists. Assume that there are \( N + 1 \) assets. The return on the \( i \)th asset, \( i = 0, 1, 2, \ldots, N \), is given by \( R_i \). The investor with initial wealth \( W_0 \) selects a portfolio \( x^T = (x_0, x_1, \ldots, x_N) \) and this selection provides wealth of \( W_1 = W_0(1 + R_x) \) at the end of the period, where \( R_x = \sum_{i=0}^{N} x_i R_i \). The problem is then

\[
\max \quad E[U(W_0[1 + R_x])]
\]

subject to \( \sum_{i=0}^{N} x_i = 1 \), and possibly other constraints such as non-negativity and perhaps a shortfall constraint. Substituting \( x_0 = 1 - \sum_{i=1}^{N} x_i \) into the objective to be maximized, we obtain

\[
\max E \left[ U \left( W_0 \left[ 1 + R_0 + \sum_{i=1}^{N} x_i (R_i - R_0) \right] \right) \right].
\]

Differentiating with respect to each \( x_i \) gives the first-order conditions for an optimum as

\[
E \left[ (R_i - R_0) \frac{\partial}{\partial W_1} U(W_1) \right] = 0 \quad \text{for } i = 1-N.
\]

We assume that \( U \) is strictly increasing (\( (\partial / \partial W_1) U > 0 \)) and risk aversion so that \( U \) is concave (\( (\partial^2 / \partial W_1^2) U < 0 \)). The first-order conditions are necessary and sufficient for a maximum. We also have that

\[
E[U(W_0[1 + R_x])] \leq U(E[W_0[1 + R_x]]),
\]

and therefore

\[
U^{-1} E[U(W_0[1 + R_x])] \leq E[W_0[1 + R_x]],
\]

where \( U^{-1} E[U(W_0[1 + R_x])] \) is the certainty equivalent of the random end of period wealth.

4.1.1. Quadratic utility

A common assumption underlying mean-variance portfolio selection models is that the utility function is quadratic. In this case

\[
U(W) = W - \frac{1}{2b} W^2 \quad \text{with } b > 0, \ W < b,
\]

\[
\frac{\partial}{\partial W} U(W) = 1 - \frac{1}{b} W,
\]

and the first-order conditions are

\[
E \left[ (R_i - R_0) \left( 1 - \frac{1}{b} W_0 \left[ 1 + R_0 + \sum_{j=1}^{N} x_j (R_j - R_0) \right] \right) \right] = 0 \quad \text{for } i = 1-N.
\]
which simplifies to

\[
\left[ 1 - \frac{1}{b} W_0 (1 + R_0) \right] E[R_i - R_0] - \frac{1}{b} W_0 \sum_{j=1}^{N} x_j E[(R_j - R_0) (R_i - R_0)] = 0 \quad \text{for } i = 1-N.
\]

If we let

\[
C = \begin{pmatrix}
E[(R_1 - R_0) (R_1 - R_0)] & \cdots & E[(R_N - R_0) (R_1 - R_0)] \\
E[(R_1 - R_0) (R_2 - R_0)] & \cdots & E[(R_N - R_0) (R_2 - R_0)] \\
\vdots & \ddots & \vdots \\
E[(R_1 - R_0) (R_N - R_0)] & \cdots & E[(R_N - R_0) (R_N - R_0)]
\end{pmatrix},
\]

\[
E = \begin{pmatrix}
E(R_1 - R_0) \\
E(R_2 - R_0) \\
\vdots \\
E(R_1 - R_0)
\end{pmatrix}, \quad x = \begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_N
\end{pmatrix},
\]

then the solution to the first-order conditions can be written as

\[
x = k C^{-1} E,
\]

where

\[
k = \frac{b}{W_0} - (1 + R_0).
\]

Note that the constraint \( \sum_{i=0}^{N} x_i = 1 \) is automatically satisfied.

### 4.1.2. Gaussian returns and exponential utility

In this case we assume that the returns \( R_0, R_1, \ldots, R_N \) have a multivariate normal distribution. Therefore, \( R_x = \sum_{i=0}^{N} x_i R_i \) has a normal distribution as does \( W_1 = W_0 (1 + R_x) \). Assume the vector of expected returns is \( \mu^T = (\mu_0, \mu_1, \mu_2, \ldots, \mu_N) \) with \( E[R_i] = \mu_i \) and the covariance matrix is \( \Sigma = (\sigma_{ij}) \) with \( \sigma_{ij} = \text{Cov}(R_i, R_j) \).

With \( x^T = (x_0, x_1, \ldots, x_N) \), where \( x_i \) is the proportion in asset \( i \) such that \( \sum_{i=1}^{N} x_i = 1 \), we then have

\[
E[W_1] = E \left[ W_0 \left( 1 + \sum_{i=0}^{N} x_i R_i \right) \right] = W_0(1 + \mu^T x),
\]

\[
\text{Var}[W_1] = W_0^2 \text{Var}[R_x] = W_0^2 \sum_{i=1}^{N} \sum_{j=1}^{N} x_i \sigma_{ij} x_j = W_0^2 x^T \Sigma x.
\]

We assume that

\[ U(W) = -\exp(-bW) \quad \text{with } b \geq 0, \ W > 0, \]

so that

\[
\frac{\partial}{\partial W} U(W) = b \exp(-bW) > 0.
\]
The optimization problem is
\[
\max E[-\exp(-bW_0[1 + R_x])],
\]
subject to \( \sum_{i=0}^{N} x_i = 1. \)

Using the moment generating function of the normal distribution we then have the objective
\[
\max[-\exp(-bW_0(1 + \mu^T x) + \frac{1}{2} b^2 W_0^2 x^T \Sigma x)],
\]
which simplifies to
\[
\max \left\{ \frac{2}{bW_0} \mu^T x - x^T \Sigma x \right\},
\]
subject to \( \sum_{i=0}^{N} x_i = 1. \)

Note that for the multivariate normal distribution assumption
\[
Pr(W_0[1 + R_x] \leq 0) > 0,
\]
so that unlimited liability is assumed in this case.

In Panjer et al. (1998) the objective for mean-variance optimization is expressed as
\[
\max f^T x - x^T \Sigma x.
\]
The objective for exponential utility with Gaussian returns is therefore the same as for the standard mean-variance problem with \( r = 1/bW_0 \). Note that as initial wealth \( W_0 \) increases the risk tolerance parameter decreases. This is often considered to be an undesirable feature in determining asset allocations since tolerance to risky investments decreases with increasing initial wealth.

### 4.2. Dual theory and distortion functions

For asset allocation purposes, we would normally use \( H_g(W_0(1 + R_x)) \) as the certainty equivalent of the random end of period wealth.

In order to be consistent with expected utility and risk aversion, we require
\[
H_g(W_0(1 + R_x)) \leq E[W_0[1 + R_x]],
\]
which holds if \( g \) is convex. This contrasts with the insurance case, where the random variable under consideration is a non-negative loss variable, i.e. a negative change in wealth. For the insurance non-negative loss variable the function \( g \) is concave.

Some special cases with convex \( g \) are as follows:

**The PH transform:**
\[
g(x) = x^r, \quad r \geq 1.
\]

**The dual-power transform:**
\[
g(x) = 1 - [1 - x]^{1/r}, \quad r \geq 1.
\]

Under the dual theory the investor is assumed to solve the following problem:
\[
\max H_g(W_0(1 + R_x)),
\]
subject to the same constraints as for the standard portfolio selection problem. Assuming that \( H_g(R_x) \) is defined as in this paper, where we require that \( R_x \geq 0 \), then from the properties of the distortion functions

\[
H_g(W_0(1 + R_x)) = W_0 + W_0H_g(R_x).
\]

To apply this risk measure to asset allocation we will need to define \( H_g(R_x) \) over all possible values of \( R_x \) since asset returns can in general be negative as well as positive. One approach to this problem is to use the extension of \( H_g(X) \) to random variables taking both positive and negative values proposed by Wang et al. (1997) based on the Choquet integral. In this approach, for any random variable \( X \) with decumulative distribution \( S_X(t)(-\infty < t < \infty) \) the Choquet integral is given by

\[
H_g(X) = \int_{-\infty}^{0} [g[S_X(t)] - 1] \, dt + \int_{0}^{\infty} g[S_X(t)] \, dt.
\]

5. A new class of risk measure

5.1. The risk measure

In order to apply a risk measure to the portfolio problem it is necessary to define the measure over positive and negative outcomes. One approach to this problem is given in Wang (1999).

For expected utility, an investor always places some wealth into the risky asset provided its expected return is positive. Thus there is always some diversification with expected utility but this is not necessarily the case with the dual theory where the asset allocation is an “all-or-nothing” strategy.

As an example consider the two asset case with asset 0 a riskless asset and asset 1 a risky asset. The objective is

\[
\max H_g[W_0(1 + x_0 R_0 + (1 - x_0) R_1)] = \max W_0(1 + H_g(R_1)) + x_0 W_0(R_0 - H_g(R_1)).
\]

Now if \( R_0 > H_g(R_1) \) then there is no upper bound to the objective if short selling is allowed in the risky asset. If this short selling is not allowed then the whole of the investor's wealth would be placed in the risk free asset. A similar conclusion holds if \( R_0 < H_g(R_1) \). The riskless asset is now shorted and a long position taken in the risky asset.

\( H_g(R_1) \) can be interpreted as the certainty equivalent of the return. Thus if the risk-adjusted return on the risky asset, \( H_g(R_1) \), exceeds the riskless asset return, then as much as possible would be placed in the risky asset. Yaari (1987) refers to this as “plunging” since the investor will place all their wealth into the risky asset once the risk adjusted return exceeds the risk free rate of return.

The concept of “regret” as a risk measure has also found applications in asset allocation. Dembo and Freeman (1998) describe a risk measure based on the concept of “regret” where the downside of a prospect is treated differently to the upside. Downside outcomes are given a value based on what would be paid to insure against this outcome.

Our approach to adapting the distortion function approach to asset allocation is to consider a point \( \alpha \) in the wealth outcomes where we treat outcomes above and below this point differently. Thus \( \alpha \) could be the return on a benchmark portfolio or the minimum return required to meet a liability on a future date. We take the certainty equivalent of downside outcomes using the distortion function approach treating the downside as a (positive) loss random variable. The certainty equivalent for upside outcomes is also determined using the distortion function approach suitably adapted to investment, rather than loss, random variables.

Recall that a function \( g \) is a distortion function if \( g \) is concave and increasing with \( g(0) = 0 \) and \( g(1) = 1 \). It then follows that \( g(x) \geq x \) for \( 0 \leq x \leq 1 \). In fact for such \( x \), \( g(x) = g(x \cdot 1 + (1 - x) \cdot 0) \geq x \cdot g(1) + (1 - x) \cdot g(0) = x \).

Corresponding to \( g \), we define another distortion function \( h \) which is convex, with \( h(x) \leq x \), increasing and \( h(0) = 0 \) and \( h(1) = 1 \). For instance, \( h(x) = 1 - g(1 - x) \) would be one possible choice.
If \( g \) is a distortion function and \( X \) is a non-negative random variable, then as above, we set

\[
H_g(X) = \int_0^\infty g[S_X(t)] \, dt.
\]

We do not write down conditions under which \( H_g(X) \) is finite for this non-negative random variable. Without some restrictions it is possible that \( H_g(X) = \infty \).

For each choice of \( \alpha \in \mathcal{R} \), and \( g \) and \( h \) distortion functions as defined above, set

\[
H_{a,h,g}(X) = \alpha + H_h((X - \alpha)^+) - H_g((\alpha - X)^+),
\]

where \( a^+ = \max[0, a] \). Note that \( H_{a,h,g}(X) \) is now defined for random variables \( X \) of arbitrary sign. We do not write down conditions under which \( H_{a,h,g}(X) \) is finite for this random variable \( X \). Without some restrictions it is possible that \( H_{a,h,g}(X) \) could be unbounded. From now on we will write simply \( H_{a,h,g}(X) \), but of course it implies that a choice was made for \( \alpha, g, h \). The paper by Wang (1996a) gives an extensive list of possible types of distortion functions that can be employed in the definition of \( H \).

The risk measure proposed contains a wide class of risk measures depending on the choice of \( \alpha, g, h \). For example if we select \( \alpha = 0 \), \( h(x) = 1 - g(1 - x) \), then we obtain the Choquet integral representation given in Wang et al. (1997).

5.2. Properties of the risk measure

Properties that are satisfied by \( H_{a,h,g}(X) \) will follow from properties of \( H_g \) which are listed in various places, for example in Wang et al. (1997), and proved in Denneberg (1994).

**Proposition 1.** For \( X = C \), a constant, we have

\[
H_{a,h,g}(C) = C.
\]

**Proof.**

\[
H_{a,h,g}(C) = \alpha + H_h((C - \alpha)^+) - H_g((\alpha - C)^+) = \alpha + (C - \alpha)^+ - ((\alpha - C)^+ = C
\]

for any real constant \( C \).

**Proposition 2.** For any random variable

\[
H_{a,h,g}(X) \leq E[X].
\]

**Proof.**

\[
H_{a,h,g}(X) = \alpha + H_h((X - \alpha)^+) - H_g((\alpha - X)^+) \leq \alpha + E[(X - \alpha)^+] - E[(\alpha - X)^+]
\]

\[
= E[\alpha + (X - \alpha)^- - (\alpha - X)^+] = E[X].
\]

This argument still holds even if some of the expressions are infinite!

**Proposition 3.** For any random variable \( X \) and \( Y \) with \( X \leq Y \) a.s.

\[
H_{a,h,g}(X) \leq H_{a,h,g}(Y).
\]
Proof. If \( Y \geq X \) then \((X - \alpha)^+ \leq (Y - \alpha)^+ \) and \((\alpha - X)^+ \geq (\alpha - Y)^+ \), so we have \( S_{a(X-\alpha)^+}(t) \geq S_{(Y-\alpha)^+}(t) \) and \( S_{a(Y-\alpha)^+}(t) \geq S_{a(1-\alpha)(Y-\alpha)^+}(t) \). The result follows from the definition of \( H_{a,h,g}(X) \) and the assumptions that \( h \) and \( g \) are non-decreasing functions.

\[ \text{Proposition 4. If} \]
\[ h(x) + g(1-x) = 1 \text{ for } 0 \leq x \leq 1, \]
\[ \text{then} \]
1. \( H_{0,h,g}(X + a) = H_{0,h,g}(X) + a; \)
2. \( H_{a,h,g}(X) \) is concave.

Proof. (1) Direct calculation gives
\[ H_{0,h,g}(X + a) = H_{0,h,g}(X) + \int_0^a [h(S_X(-v)) + g(1 - S_X(-v))] dv = H_{0,h,g}(X) + a. \]
(2) \( H_{a,h,g}(X) \) will be concave if and only if \( H_{0,h,g}(X) \) is concave and this is equivalent to
\[ H_{0,h,g}(X + Y) \geq H_{0,h,g}(X) + H_{0,h,g}(Y), \]
(\#) since \( H_{0,h,g}(\lambda X) = \lambda H_{0,h,g}(X) \) for \( \lambda > 0 \). By (1), (\#) holds when \( X \) and \( Y \) are bounded from below (using the results of Wang (1995)) and in general using approximations like \( X \approx \lim_{n \to \infty} \max[X, -n] \).

\[ \text{Proposition 5. If} \]
\[ \rho(X) \equiv -H_{a,h,g}(X), \]
then, for \( \alpha = 0 \) and \( h(x) + g(1-x) = 1 \), \( \rho \) is a coherent risk measure which satisfies the properties (see Artzner, 1999; Artzner et al., 1999):

(T) Translation invariance, \( \rho(X + \alpha) = \rho(X) - \alpha \).
(S) Subadditivity, \( \rho(X + Y) \leq \rho(X) + \rho(Y) \).
(PH) Positive homogeneity, \( \rho(\lambda X) = \lambda \rho(X) \) for \( \lambda > 0 \).
(M) Monotonicity, \( \rho(X) \geq \rho(Y) \) for \( X \leq Y \).

Proof.

(T) follows from Proposition 4(1).
(S) and (PH) follow from proof of Proposition 4.
(M) follows from Proposition 3.

We can now regard the mapping
\[ X \to H_{a,h,g}(X) \]
as an alternative to an expected utility, and we can define the orderings on random variables by \( X \geq Y \) (\( X \) is weakly-preferred to \( Y \)) if \( H_{a,h,g}(X) \geq H_{a,h,g}(Y) \) and \( X > Y \) (\( X \) is strictly-preferred to \( Y \)) if \( H_{a,h,g}(X) > H_{a,h,g}(Y) \).

We can now use these orderings in portfolio selection. It is a consequence of the concavity of \( H_{a,h,g} \) that such a risk measure will imply diversification for portfolio theory. Diversification is not a property of the distortion function risk measure used in insurance pricing if it is applied directly to asset allocation without modification.
5.3. Non-expected utility property

As mentioned earlier, the risk behaviour of individuals often does not satisfy the axioms of the expected utility theory. In order to address this, a risk measure consistent with actual risk behaviour will need to also satisfy different axioms to those of expected utility. The distortion function approach to pricing satisfies a different set of axioms to those of expected utility. In particular, the independence axiom is not satisfied.

Our risk measure need not satisfy the axioms of expected utility. This example, based on similar examples found in Kahneman and Tversky (1979), demonstrates that it is possible to make a choice of $\alpha, g, h$ so that the corresponding $H_{\alpha, g, h}$ does not obey the substitution axiom or independence axiom of expected utility (see Huang and Litzenberger, 1988).

We choose throughout this example $D_0$ and $g$ a piecewise linear function generated by

$g(0) = 0, \quad g(0.2) = 0.40, \quad g(0.25) = 0.49, \quad g(0.8) = 0.87, \quad g(1.0) = 1.0.$

The slopes for $g$ are as follows:

- on $[0, 0.2]$ it is 2,
- on $[0.2, 0.25]$ it is 1.8,
- on $[0.25, 0.8]$ it is 0.6909, . . . ,
- on $[0.8, 1]$ it is 0.65.

We conclude that $g$ is concave and a legitimate distortion function with arbitrary $h$ and we see that

- $H_{\alpha, g, h}(X_1) = -5220$, if $X_1 = -6000$ with probability 0.8 and $X_1 = 0$ with probability 0.2.
- $H_{\alpha, g, h}(Y_1) = -5000$ if $Y_1 = -5000$ with probability 1.

Thus, $Y_1 \succ X_1$.

Also,

- $H_{\alpha, g, h}(X_2) = -2400$, if $X_2 = -6000$ with probability 0.2 and $X_2 = 0$ with probability 0.8.
- $H_{\alpha, g, h}(Y_2) = -2450$ if $Y_2 = -5000$ with probability 0.25 and $Y_2 = 0$ with probability 0.75.

Thus, $X_2 \succ Y_2$.

But these conclusions cannot follow from an expected utility ordering. For if so, then $E[U(Y_1)] > E[U(X_1)]$ implies $U(-5000) > 0.8U(-6000)$, for some utility function with $U(0) = 0$, while $E[U(X_2)] > E[U(Y_2)]$ implies $0.2U(-6000) > 0.25U(-5000)$ or $U(-5000) < 0.8U(-6000)$, a contradiction.

We can conclude that the new measure $H_{\alpha, g, h}(X)$ does not in general give rise to ordering of risks equivalent to that arising from an expected utility ordering. Thus the risk measure is not expected utility maximizing and can be consistent with violations of the expected utility axioms.

5.4. Examples

We now provide some examples of evaluations of $H_{\alpha, g, h}(X)$ for various distributions for $X$.

5.4.1. Normally distributed wealth

If $X$ is normally distributed $N(\mu, \sigma^2)$, then

$$H_{\alpha, g, h}(X) = \alpha + \sigma \int_{-\infty}^{(\mu-\alpha)/\sigma} h(N(u)) \, du - \sigma \int_{-\infty}^{(\alpha-\mu)/\sigma} g(N(u)) \, du$$

$$= \alpha + \sigma \left[ \int_{-\infty}^{(\mu-\alpha)/\sigma} h(N(u)) \, du - \int_{-\infty}^{(\alpha-\mu)/\sigma} g(N(u)) \, du \right],$$

where

$$N(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} \exp \left( -\frac{z^2}{2} \right) \, dz.$$
Thus

\[ H_{a,h,g}(X) = \alpha + \sigma f \left( \frac{\mu - \alpha}{\sigma} \right), \]

where

\[ f(x) = \int_{-\infty}^{x} h(N(u)) \, du - \int_{-\infty}^{-x} g(N(u)) \, du. \]

5.4.1.1. PH transform and Gaussian returns. The risk measure proposed can use any of the distortion functions used in the insurance literature with Gaussian returns. In the special case with \( h(x) + g(1-x) = 1 \) for \( 0 \leq x \leq 1 \), we obtain

\[ H_{a,h,g}(X) = \alpha + \sigma f(0) \]

with \( f(0) < 0 \). In this case the asset allocation problem is the same as for mean-variance portfolio selection. Further, if \( h(x) \equiv 1 - (1-x)^r \), \( g(x) \equiv x^r \) for \( 0 < r \leq 1 \) then \( f(0) \) is increasing with respect to \( r \). Since,

\[ f(0) = \int_{-\infty}^{0} [1 - (1-N(u))^r + N(u)^r] \, du, \]

larger values of \( r \) are associated with lower risk aversion.

For the asset allocation optimization we define as before

\[ \mu_{W_i} \equiv E[W_0(1 + R_s)] = W_0(1 + \mu^T x), \quad \sigma^2_{W_i} \equiv \text{Var}(W_0(1 + R_s)) = W_0^2 x^T \Sigma x. \]

Then

\[ H_{a,h,g}(W_1) = \alpha + \sigma_{W_i} \left[ f \left( \frac{\mu_{W_i} - \alpha}{\sigma_{W_i}} \right) \right]. \]

where \( W_1 = W_0(1 + R_s) \), \( R_s = \sum_{i=0}^{N} x_i \). The problem is then to maximize \( H_{a,h,g}(W_1) \) subject to \( \sum_{i=0}^{N} x_i = 1 \), and any other constraints on the sign or size of each \( x_i \). The value of \( \alpha \) would be selected to reflect a return where the risk function would be different for returns above and below this point. For instance, this could be taken as the wealth arising from a risk free investment strategy.

This maximization problem for general distortion functions and Gaussian returns can be solved using Lagrange multipliers to show that the optimal asset allocation \( \mathbf{x} \), satisfies

\[ \mathbf{x} = \mathbf{a} - \frac{v f'(z)}{[-f(z) + z f'(z)]} \mathbf{b}, \]

where

\[ f(z) \equiv \int_{-\infty}^{z} h(N(u)) \, du - \int_{-\infty}^{-z} g(N(u)) \, du, \quad \mathbf{a} = \frac{\Sigma^{-1} \mathbf{e}}{\mathbf{e}^T \Sigma^{-1} \mathbf{e}}, \]

\[ \mathbf{b} = \frac{\mathbf{e}^T \Sigma^{-1} \mathbf{\mu}}{\mathbf{e}^T \Sigma^{-1} \mathbf{e}} - \Sigma^{-1} \mathbf{\mu}, \quad v^2 = x^T \Sigma x, \quad z = \frac{\mu_{W_i} - \alpha}{\sigma_{W_i}}, \quad \mathbf{e}^T = (1, 1, \ldots, 1). \]

We note that the allocations \( \mathbf{x} \) are of the form

\[ \mathbf{x} = \mathbf{a} - \eta \mathbf{b}, \]
where $\eta$ satisfies a non-linear equation

$$\eta = \frac{v f'(z)}{[-f(z) + zf'(z)]},$$

where the right-hand side can be expressed in terms of $\eta$ by substituting $x = a - \eta b$ in all expressions depending on $x$.

5.4.2. Binomial model

In practice asset models are often implemented using discrete distributions of asset returns. In option pricing, the binomial model is one of the most popular models used in practice. For asset allocation the use of similar models allows for numerical optimization of different investment criteria and makes available the techniques used in option pricing for asset allocation. Pliska (1997) provides coverage of the use of discrete models in asset allocation.

For these models we are interested in the case when $X$ has a Bernoulli-type distribution. That is, $X$ takes a value $a$ with probability $p$ and a value $b$ with probability $q = 1 - p$. We can clearly assume without loss of generality that $a \geq b$. In this situation we can compute $H_{a,b,g}(X)$ explicitly as follows:

$$H_{a,b,g}(X) = \alpha + (b - \alpha)^+ - (\alpha - b)^+ + h(p)[(a - \alpha)^+ - (b - \alpha)^+] - g(q)[(\alpha - b)^+ - (\alpha - a)^+].$$

6. Conclusions

In this paper, we propose a new class of risk measure for use in asset allocation. The class of risk measures incorporates concepts developed and applied recently to insurance pricing. The class has properties of risk aversion and diversification. It treats the upside and downside of possible outcomes differently in a similar manner to the concept of “regret”. The class of measures does not satisfy the axioms of expected utility and is therefore not an expected utility risk measure. This means that the class of risk measures proposed can handle cases where risk taking behaviour does not conform to the expected utility axioms.

Acknowledgements

The authors would like to acknowledge the research assistance of Paul Park, financial support of an Australian Research Council Grant, support from an Institute of Actuaries of Australia Research Grant and helpful discussions with Shaun Wang and Philippe Artzner.

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