On the form and risk-sensitivity of zero coupon bonds for a class of interest rate models

Luis H.R. Alvarez*

Institute of Applied Mathematics, University of Turku, FIN-20014 Turku, Finland

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Abstract

We consider the form and the comparative static properties of the price of a zero coupon bond with maturity $T$ for a broad class of interest rate models. We first demonstrate that increased volatility increases the price of a $T$-claim whenever the price is convex as a function of the current short rate. We then present a class of diffusion models (including, for example, the Dothan, the Black–Derman–Toy, and the Merton model of interest rates) for which the positivity of the sign of the relationship between volatility and the price of zero coupon bonds is always unambiguously guaranteed. Consequently, we find that for the considered class of models the price of zero coupon bonds can be completely ordered in terms of the riskiness of the underlying interest rate dynamics. We also show that for the proposed class of interest rate models, increased volatility increases the price of all convex and non-increasing $T$-claims as well. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Zero coupon bonds constitute undoubtedly the most studied class of interest rate derivatives both in theory and in practice (see, for example, Björk, 1997a,b; Duffie and Kan, 1996 and references therein; see also Alvarez, 1998; Cox et al., 1985; Longstaff, 1993). Due to the complexity of the parabolic term structure equation describing the price of a zero coupon bond, the analysis of these derivatives has been thoroughly carried out only in rather simple parametrized cases like the well-known affine term structure model (cf. Alvarez, 1998; Björk, 1997a,b; Duffie and Kan, 1996; Longstaff, 1993). In the presence of more complex interest rate dynamics (for example, general mean reversion), solving the term structure equation explicitly becomes unfeasible and characterizing rigorously the price of a zero coupon bond is, therefore, impossible. Moreover, not much has been done in considering the form and, especially, the comparative static properties of the price of zero coupon bonds for more general interest rate processes.

* Tel.: +358-2-333-5620; fax: +358-2-333-6595.
E-mail address: alvarez@mailhost.utu.fi (L.H.R. Alvarez).

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Given these arguments, we plan to consider in this study both the form and the comparative static properties of the price of zero coupon bonds and other similar path-dependent $T$-claims for a class of interest rate models. In line with previous results on path-independent options (cf. El Karoui et al., 1998; Hobson, 1998) we first prove the qualitatively important result that increased volatility increases the price of all $T$-claims with convex prices in the path-dependent case as well. Having established this result, we then present a broad class of interest rate models subject to a concave (but otherwise general) drift and a linear diffusion coefficient for which the price of a zero coupon bond is always non-increasing and convex. Thus, we are able to demonstrate that the results obtained in models subject to an affine term structure are qualitatively robust and that for the class of interest rate models considered in this study the price of zero coupon bonds can be completely ordered in terms of the volatility (that is the riskiness) of the underlying interest rate dynamics. Furthermore, we also demonstrate that the price of any non-increasing and convex $T$-claim is convex as a function of the current short rate and, therefore, that increased stochastic fluctuations increase their prices as well. Thus, we find that the basic conclusions valid for zero coupon bonds can also be extended for a broader class of contingent $T$-claims.

2. Convexity of prices and increased volatility

In the absence of arbitrage, the term structure is determined by specifying the dynamic behavior of the short rate of interest under the equivalent martingale measure $Q$ (Björk, 1997a,b; Duffie, 1996; Duffie and Kan, 1996). Consider now the case where the interest rate dynamics $\{r(t); t \geq 0\}$ are described under the risk neutral measure $Q$ by the (Itô-) stochastic differential equation

$$d r(t) = \mu(t, r(t)) \, dt + \sigma(t, r(t)) \, d \tilde{W}(t), \quad r(0) := r, \quad (1)$$

where $\tilde{W}(t)$ is $Q$-Brownian motion. In order to guarantee the existence and uniqueness of a solution for the stochastic differential equation (1), we assume that $\mu : \mathbb{R}^2 \mapsto \mathbb{R}$ and $\sigma : \mathbb{R}^2 \mapsto \mathbb{R}$ are given measurable mappings satisfying the Lipschitz-condition

$$|\mu(t, x) - \mu(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq D|x - y|, \quad x, y \in \mathbb{R}, \quad t \in [0, T] \quad (2)$$

for a given known positive constant $D$ and the growth condition

$$|\mu(t, r)| + |\sigma(t, r)| \leq C(1 + |r|) \quad (3)$$

for a given known positive constant $C$ (cf. Øksendal, 1998, p. 66).

Let $\Phi : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}_+$ be a given measurable mapping satisfying the uniform integrability constraint

$$E^Q_{[t, \tau]}[e^{-\int_t^{\tau} r(s) \, ds} \, \Phi(T, r(T))] < \infty$$

for all $(t, \tau) \in [0, T] \times \mathbb{R}$. It is well known that under our assumptions, the price of a contingent claim $X = \Phi(t, r)$ with maturity $T$ is given as

$$p(t, r; T) = E^Q_{[0, T]}[e^{-\int_t^{T} r(s) \, ds} \, \Phi(T, r(T))]. \quad (4)$$

Especially, the price $p(t, r; T)$ satisfies the terminal value problem

$$\frac{\partial p}{\partial t}(t, r; T) + \mu(t, r) \frac{\partial p}{\partial r}(t, r; T) + \frac{1}{2} \sigma^2(t, r) \frac{\partial^2 p}{\partial r^2}(t, r; T) - rp(t, r; T) = 0, \quad p(T, r; T) = \Phi(T, r), \quad (5)$$

where the parabolic partial differential equation (5) is known as the term structure equation. Our principal result characterizing the monotonicity of a zero coupon bond is now stated in the following lemma.
Lemma 1. The price of a zero coupon bond is non-increasing as a function of the current short rate $r$. That is,

$$E_Q^Q[e^{-\int_t^T r(s) \, ds}] \geq E_Q^Q[e^{-\int_t^T r(s) \, ds}]$$

for all $t \in [0, T]$ and $x < y$.

Proof. Denote now as $r_x(s)$, $s \geq t$, the solution of the stochastic differential equation (1) subject to the initial condition $r(t) = x \in \mathbb{R}$. The uniqueness of a solution of (1) then guarantees that $r_x(s) \leq r_y(s)$ a.s. for all $s \in [t, T]$ and $x < y$ (cf. El Karoui et al., 1998). Thus,

$$e^{-\int_t^T r_x(s) \, ds} \geq e^{-\int_t^T r_y(s) \, ds}$$

completing the proof of our lemma.

Since our main objective is to analyze the effect of increased volatility on the price of a $T$-claim and especially on the price of zero coupon bonds, we now assume that the interest rate process $Q_r(t)$ evolves under the risk neutral measure $Q$ according to the diffusion described by the (Itô-) stochastic differential equation

$$dQ_r(t) = \mu(t, \bar{r}(t)) \, dt + \sigma(t, \bar{r}(t)) \, d\tilde{W}(t), \quad \bar{r}(0) := r,$$

where $\sigma : \mathbb{R}^2 \mapsto \mathbb{R}$ is a given measurable mapping satisfying the inequality $\tilde{\sigma}(t, r) \geq \sigma(t, r)$ for all $(t, r) \in [0, T] \times \mathbb{R}$. Again, we assume that $\mu(t, r)$ and $\tilde{\sigma}(t, r)$ satisfy the Lipschitz-condition

$$|\mu(t, x) - \mu(t, y)| + |\tilde{\sigma}(t, x) - \tilde{\sigma}(t, y)| \leq \bar{D}|x - y|, \quad x, y \in \mathbb{R}, \quad t \in [0, T]$$

for a given known positive constant $\bar{D}$ and the growth condition

$$|\mu(t, r)| + |\tilde{\sigma}(t, r)| \leq \bar{C}(1 + |r|)$$

for a given known positive constant $\bar{C}$. The main result of this section is now summarized in the following theorem.

Theorem 1. Assume that $p(t, r; T)$ is convex as a function of the current short rate $r$ and define the functional $\tilde{p}(t, r; T)$ as

$$\tilde{p}(t, r; T) = E_Q^Q[e^{-\int_t^T \tilde{r}(s) \, ds} \Phi(T, \tilde{r}(T))].$$

Then, $p(t, r; T) \leq \tilde{p}(t, r; T)$ for all $(t, r) \in [0, T] \times \mathbb{R}_+$. 

Proof. Assume that $p(t, r; T)$ is convex. Since $p(t, r; T)$ satisfies the partial differential equation (5), applying Itô’s theorem to the mapping $p(t, r; T)$ yields

$$E_Q^Q[e^{-\int_t^T \tilde{r}(s) \, ds} p(T, \tilde{r}(T); T)] = p(t, r; T) + E_Q^Q\left[\int_t^T e^{-\int_t^s \tilde{r}(r) \, dr} \frac{1}{2} \Delta(s, \tilde{r}(s)) \frac{\partial^2 p}{\partial r^2}(s, \tilde{r}(s); T) \, ds\right],$$

where $\Delta(t, r) = (\tilde{\sigma}^2(t, r) - \sigma^2(t, r))$. Since $\Delta(t, r) \geq 0$, the assumed convexity of $p(t, r; T)$ and the boundary condition $p(T, r; T) = \Phi(T, r)$ then implies that

$$\tilde{p}(t, r; T) = E_Q^Q[e^{-\int_t^T \tilde{r}(s) \, ds} \Phi(T, \tilde{r}(T))] \geq p(t, r; T)$$

completing the proof of our theorem.

Theorem 1 demonstrates that the curvature of the price of the $T$-claim is the essential factor determining the sign of the effect of increased volatility on the price of a $T$-claim. If $p(t, r; T)$ is convex as a mapping of the current
short rate $r$, then increased volatility increases the price of a $T$-claim. It is clear that if $p(t, r; T)$ is concave, then it is the opposite argument which holds. However, due to the convexity of the exponential mapping $e^{-x}$, it is difficult to construct examples leading to globally concave prices of $T$-claims. An interesting consequence of Theorem 1 is now summarized in the following corollary.

**Corollary 1.** Assume that $r(t)$ has an affine term structure (that is, when both the drift coefficient $\mu(t, r)$ and the infinitesimal variance coefficient $\sigma^2(t, r)$ are linear in $r$). Then, increased volatility increases the value of zero coupon bonds.

**Proof.** It is well known that if $r(t)$ has an affine term structure, then $p(t, r; T)$ is convex as a mapping of the initial rate $r$ for all maturities $T$ (Alvarez, 1998; Björk, 1997a,b; Cox et al., 1985; Duffie, 1996; Duffie and Kan, 1996; Longstaff, 1993). The result is then a direct consequence of Theorem 1. $\square$

Another interesting result associated to that of Theorem 1 and presenting a lower boundary for the price of zero coupon bonds is now summarized in the following lemma.

**Lemma 2.** Assume that $E_{[t,r]}^Q[r(s)]$ is concave in $r$ for all $t \in [0, T], s \in [t, T]$, and that

$$E_{[t,r]}^Q \int_t^T \sigma^2(s, r(s)) \, ds < \infty$$

for all $t \in [0, T]$. Then,

$$E_{[t,r]}^Q[\tilde{r}(s)] \leq E_{[t,r]}^Q[r(s)] \leq r_0(s), \quad E_{[t,r]}^Q[e^{-\int_t^T r(s) \, ds}] \geq e^{-\int_t^T r_0(s) \, ds},$$

where $r_0(s), s \in [t, T]$, denotes the unique solution of the deterministic differential equation

$$r_0'(s) = \mu(s, r_0(s)), \quad r_0(t) = r.$$

**Proof.** Define now the mapping $u : [0, T] \times \mathbb{R} \mapsto \mathbb{R}$ as $u(t, r) = E_{[t,r]}^Q[r(s)]$ for all $s \in [t, T]$. It is well known that under our assumptions $u(t, r)$ satisfies the terminal value problem

$$\frac{\partial u}{\partial t}(t, r) + \mu(t, r) \frac{\partial u}{\partial r}(t, r) + \frac{1}{2} \sigma^2(t, r) \frac{\partial^2 u}{\partial r^2}(t, r) = 0, \quad u(s, r) = r.$$

By relying now on the same technique as in Theorem 1, we find that the assumed concavity of $u(t, r)$ implies that

$$u(t, r) \geq E_{[t,r]}^Q[\tilde{r}(s)],$$

proving that increased volatility decreases the expected interest rate and, therefore, that $r_0(t) \geq u(t, r)$. Jensen’s inequality for convex mappings then implies that

$$E_{[t,r]}^Q[e^{-\int_t^T r(s) \, ds}] \geq e^{-\int_t^T E_{[t,r]}^Q[r(s)] \, ds},$$

proving the latter inequality of our lemma. $\square$

Lemma 2 shows that in the presence of stochastic fluctuations the price of a zero coupon bond dominates the value of its deterministic counterpart whenever the expected future interest rate $u(t, r)$ is concave as a function of the current rate $r$. 
3. A class of interest rate models

Having shown that convex prices of $T$-claims lead to a positive relationship between stochastic fluctuations and the price of $T$-claims, it is our purpose to now present a class of diffusion models for which the convexity of the price of zero coupon bonds maturing at $T$ is always guaranteed as in the case of an affine term structure. To this end, we now plan to investigate the form and comparative static properties of the price of $T$-claims under the assumption that the interest rate evolves under the risk neutral measure $\mathbb{Q}$ according to the diffusion \( \{r(t); t \geq 0\} \) described by the (Itô-) stochastic differential equation

\[
dr(t) = \mu(t, r(t)) \, dt + (\alpha(t)r(t) + \beta(t)) \, d\hat{W}(t), \quad r(0) := r,
\]

where $\alpha : [0, T] \mapsto \mathbb{R}$ and $\beta : [0, T] \mapsto \mathbb{R}$ are given measurable mappings such that the Lipschitz-condition (2) and the growth condition (3) are satisfied. Our first key result is now stated in the following theorem.

**Theorem 2.** Assume that the short rate of interest evolves according to the diffusion described by the stochastic differential equation (9) and that $\mu(t, r)$ is concave in $r$ for all $(t, r) \in [0, T] \times \mathbb{R}$. Then, the price of a zero coupon bond maturing at $T$ is non-increasing and convex as a function of the initial rate $r$. That is, for all $x, y \in \mathbb{R}$, $t \in [0, T]$ we have $\rho(t, \lambda x + (1 - \lambda) y, T) \leq \lambda \rho(t, x, T) + (1 - \lambda) \rho(t, y, T)$, where

\[
\rho(t, r, T) = E^\mathbb{Q}_{[t, T]}[e^{- \int_t^T r(s) \, ds}],
\]

Moreover, increased volatility increases the price of the zero coupon bond.

**Proof.** The monotonicity of $\rho(t, r, T)$ follows from Lemma 1. Assume now that $\mu(t, r)$ is continuously differentiable with respect to the current short rate $r$ and denote as $r_x(\tau)$, $\tau \geq t$, the solution of the stochastic differential equation (9) subject to the initial condition $r_t = x \in \mathbb{R}$. It is clear that under our assumptions $r_x(\tau)$ can be written in the (Itô-) form

\[
r_x(\tau) = x + \int_t^\tau \mu(s, r_x(s)) \, ds + \int_t^\tau (\alpha(s)r_x(s) + \beta(s)) \, d\hat{W}(s).
\]

Especially, our assumptions imply that $r_x(\tau)$ constitutes a continuously differentiable flow in $x$ (cf. Protter, 1990, Theorem V. 38 and 39). Denote now $Y(\tau) = \partial r_x(\tau)/\partial x$. It is well known that (cf. Protter, 1990, Theorem V. 39)

\[
Y(\tau) = 1 + \int_t^\tau \mu_r(s, r_x(s))Y(s) \, ds + \int_t^\tau \alpha(s)Y(s) \, d\hat{W}(s),
\]

where $\mu_r(t, r) = \partial \mu(t, r)/\partial r$. A simple application of Itô’s theorem then implies that the solution of the stochastic differential equation (11) can be written as

\[
Y(\tau) = \frac{\partial r_x(\tau)}{\partial x} = \exp \left( \int_t^\tau \mu_r(s, r_x(s)) \, ds \right) Z(\tau),
\]

where

\[
Z(\tau) = \exp \left( \int_t^\tau \alpha(s) \, d\hat{W}(s) - \frac{1}{2} \int_t^\tau \alpha^2(s) \, ds \right)
\]

is a positive martingale independent of the current state $x$. The assumed concavity of the drift $\mu(t, r)$ then implies that $\mu_r(t, r)$ is non-increasing in $r$ and, therefore, that $\mu_r(s, r_x(s)) \geq \mu_r(s, r_x(s))$ for all $x \leq y$ and $s \in [t, \tau]$. Consequently, we find that $\partial r_x(\tau)/\partial x$ is non-increasing in $x$, proving the alleged concavity of the solution $r_x(\tau)$ as a function of the current short rate $x$ in the continuously differentiable case.

Assume now that $\mu(t, r)$ is concave but not necessarily continuously differentiable. It is now possible to construct a sequence of continuously differentiable mappings $\mu_n(t, r)$ converging uniformly on compacts to $\mu(t, r)$ so that
\( \frac{\partial \mu_t}{\partial r}(t, r) / \partial r \) decreases to \( \mu_t(t, r) \) as \( n \to \infty \) (a mollification of \( \mu(t, r) \); cf. Protter, 1990, Theorem IV. 47). The alleged concavity of the solution \( r_t(t) \) as a function of the current short rate \( x \) then follows from the proof of the continuously differentiable case and monotonic convergence.

The concavity of \( r_t(t) \) as a function of the current short rate \( x \) now implies that \( \lambda r_t(t) + (1 - \lambda) r_y(t) \leq r_{x + (1 - \lambda) y}(t) \) for all \( \tau \in [t, T], x, r, y, \in \mathbb{R} \), and \( \lambda \in [0, 1] \). Thus, the monotonicity and convexity of the exponential function implies that

\[
\rho(t, \lambda x + (1 - \lambda) y; T) = E^Q [e^{-\int_t^T r_{x + (1 - \lambda) y}(s) ds}] \leq E^Q [e^{-\lambda \int_t^T r_x(s) ds - (1 - \lambda) \int_t^T r_y(s) ds}]
\]

\[
\leq \lambda E^Q [e^{-\int_t^T r_x(s) ds}] + (1 - \lambda) E^Q [e^{-\int_t^T r_y(s) ds}]
\]

\[
= \lambda \rho(t, x; T) + (1 - \lambda) \rho(t, y; T).
\]

The positivity of the sign of the relationship between volatility and the price of the zero coupon bond follows from Theorem 1.

Theorem 2 summarizes our result characterizing both the form and the comparative static properties of the prices of zero coupon bonds when the underlying interest rate dynamics are described by a stochastic differential equation of the form (9). Especially, Theorem 2 demonstrates that increased volatility increases the price of zero coupon bonds for the Dothan model \( \mu(t, r) = ar, \sigma(t, r) = br \), where \( a \) and \( b \) are constants; cf. Björk, 1997a, p. 78), the Black–Derman–Toy model \( \mu(t, r) = a(t)r, \sigma(t, r) = b(t)r \), where \( a(t) \) and \( b(t) \) are known sufficiently smooth mappings; cf. Björk, 1997a, p. 78), the Merton model \( \mu(t, r) = ar(\bar{r} - r), \sigma(t, r) = br \), where \( a, \bar{r}, \) and \( b \) are known positive constants; cf. Merton, 1975) and the stochastic flexible accelerator model \( \mu(t, r) = a(r - \bar{r}), \sigma(t, r) = br \), where \( a, \bar{r}, \) and \( b \) are known positive constants; a modified Vasičec-model, cf. Björk, 1997a, p. 78) of interest rates. It is worth emphasizing that the results of Theorem 2 are considerably strong since the conditions of Theorem 2 are satisfied by most models subject to mean reversion and a linear diffusion coefficient. An interesting corollary of Lemma 2 and Theorem 2 is now summarized in the following corollary.

**Corollary 2.** Assume that the conditions of Theorem 2 are satisfied, and that

\[
E_{t, r} \int_t^T \left( \alpha(s) r(s) + \beta(s) \right)^2 ds < \infty
\]

for all \( (t, r) \in [0, T] \times \mathbb{R} \). Then,

\[
\tilde{\rho}(t, r; T) \geq \rho(t, r; T) \geq e^{-\int_t^T r_0(s) ds},
\]

where the deterministic process \( r_0(t) \) is defined as in Lemma 1, and

\[
\tilde{\rho}(t, r; T) = E^Q_{t, r} [e^{-\int_t^T \tilde{r}(s) ds}]
\]

denotes the price of a zero coupon bond written on the riskier process \( \tilde{r}(t) \) defined as in (6).

**Proof.** The result is a straightforward consequence of Lemma 2 and Theorem 2.

Corollary 2 states a set of conditions under which the prices of zero coupon bonds can be completely ordered in terms of the riskiness of the underlying interest rate process (9). Our second important result characterizing the form and comparative static properties of prices for a class of \( T \)-claims is now summarized in the following theorem.

**Theorem 3.** Assume that the short rate of interest evolves according to the diffusion described by the stochastic differential equation (9), that \( \mu_t(t, r) \) is concave in \( r \) for all \( (t, r) \in [0, T] \times \mathbb{R} \), and that \( \Phi : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}_+ \) is non-increasing and convex in \( r \). Then, the price of a contingent claim \( X = \Phi(T, r(T)) \) maturing at \( T \) is
non-increasing and convex as a function of the initial short rate \( r \). That is, for all \( x, y \in \mathbb{R}, \lambda \in [0, 1], \) and \( t \in [0, T] \) we have the inequality \( p(t, \lambda x + (1 - \lambda) y; T) \leq \lambda p(t, x; T) + (1 - \lambda) p(t, y; T) \). Moreover, increased volatility increases the price of the \( T \)-claim.

**Proof.** We know from the proof of Theorem 2 that
\[
e^{-\int_t^T r_{x+(1-\lambda)y}(s) \, ds} \leq \lambda e^{-\int_t^T r_x(s) \, ds} + (1 - \lambda) e^{-\int_t^T r_y(s) \, ds}
\]
aalmost surely for all \( \tau \in [t, T], x, y, \in \mathbb{R}, \) and \( \lambda \in [0, 1] \). Moreover, since \( \Phi(t, r) \) is non-increasing and convex in \( r \), we also find that
\[
\Phi(\tau, r_{x+(1-\lambda)y}(\tau)) \leq \lambda \Phi(\tau, r_x(\tau)) + (1 - \lambda) \Phi(\tau, r_y(\tau))
\]
aalmost surely for all \( \tau \in [t, T], x, y, \in \mathbb{R}, \) and \( \lambda \in [0, 1] \). Since the product of two non-negative, non-increasing, and convex mappings is also non-negative, non-increasing, and convex, we observe that
\[
e^{-\int_t^T r_{x+(1-\lambda)y}(s) \, ds} \Phi(\tau, r_{x+(1-\lambda)y}(\tau)) \leq \lambda e^{-\int_t^T r_x(s) \, ds} \Phi(\tau, r_x(\tau)) + (1 - \lambda) e^{-\int_t^T r_y(s) \, ds} \Phi(\tau, r_y(\tau))
\]
aalmost surely for all \( \tau \in [t, T], x, y, \in \mathbb{R}, \) and \( \lambda \in [0, 1] \). Since
\[
p(t, x; T) = E^Q[e^{-\int_t^T r_s(s) \, ds} \Phi(T, r_s(T))]
\]
we find that \( p(t, \lambda x + (1 - \lambda) y; T) \leq \lambda p(t, x; T) + (1 - \lambda) p(t, y; T) \) for all \( \tau \in [t, T], x, y, \in \mathbb{R}, \) and \( \lambda \in [0, 1] \). The positivity of the sign of the relationship between volatility and the price of the zero coupon bond follows from Theorem 1.

Theorem 3 establishes that the results of Theorem 2 are valid also for the prices of a class of \( T \)-claims whenever the claim is non-increasing and convex as a function of the short rate of interest at maturity. An interesting consequence of this result is that if \( S > T \), \( \Lambda : \mathbb{R}_+ \mapsto \mathbb{R}_+ \) is an increasing and convex mapping, and
\[
\Phi(T, r(T)) = \Lambda(p(T, r(T); S))
\]
then \( p(t, r; T) \) is convex in \( r \) and increased volatility will increase its value. In other words, increasing and convex contingent contracts maturing at \( T \) and written on zero coupon bonds maturing at a later date \( S \) end up yielding a positive relationship between stochasticity and the value of the claim.

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