Optimal procurement policies under price-dependent demand

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Abstract

We study a periodic-review inventory model where, in addition to the procurement quantity, price is also a decision variable. We develop a model where demand in each period is a random variable having a price- and, possibly, period-dependent probability distribution, with the expected demand decreasing in price. The model includes price limits and fixed ordering costs in addition to unit procurement holding and shortage costs. We study the optimal policies which jointly maximize the discounted expected profit over a finite planning horizon. We characterize the form of the optimal procurement policy under a general price–demand relationship and give a sufficient condition for it to be \((s_n, S_n)\) type. We also discuss some special cases and extensions to the basic model, including the infinite horizon problem. © 2000 Elsevier Science B.V. All rights reserved.

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Notation

\(n\) period index \((n = 1\) corresponds to the last period\)
\(N\) total number of periods in the planning horizon
\(i_n\) beginning inventory level before ordering in period \(n\)
\(q_n\) beginning inventory level after ordering in period \(n\)
\(p_n\) price in period \(n\)
\(P_f\) price floor
\(P_c\) price ceiling
\(c\) unit procurement cost
\(r\) unit shortage cost
\(h\) unit holding cost
\(v\) unit salvage value at the end of the planning horizon
\(K_n\) fixed ordering cost in period \(n\)
1. Introduction

Under increased competition, inventory-based businesses are forced to better coordinate their procurement and marketing decisions, to avoid carrying excessive stock when sales are low or shortages when they are high. An effective means of such coordination is to conduct the inventory control and pricing decisions jointly. The main task in doing so is to determine the optimal inventory policy, given the price–demand relationship that is expected to prevail in the market place in the short term.

In addressing the above issue, we are concerned in this paper with a single-item, periodic-review inventory system where the vendor, who enjoys a degree of monopoly power in the market place, is in a position to influence demand by its pricing decisions. It is thus confronted with simultaneous pricing and procurement quantity decisions, which would jointly maximize the present value of expected profit over a planning horizon. We will assume that the ordering policy does not change the demand pattern during the planning horizon.

Ordering cost, shortage cost and temporal increases in the procurement cost are among the important factors that force the decision maker to carry inventories. Holding cost has the opposite effect. Thus, in the absence of pricing, the main decision problem is to determine the optimal inventory levels to strike a balance between these opposing factors, under a given demand forecast, cost structure, and a set of operating conditions.

There are critical differences between a fixed-price, periodic-review inventory model [1] and a model that also includes pricing decisions. The economical interpretations of these differences relate to the various ways that price plays into the decision problem. First, price is a decision variable that determines the revenue per unit sold. Second, price is a factor that influences the demand, thus the period-ending inventory levels. In addition, when backlogging occurs, apart from the amount of backorders, the backlogging policy must also account for the price that applies to backorders and the timing of revenue collection. For instance, backorders could be sold at the current price in advance, or at a future price set at the time of delivery.

Therefore, when the decision maker confronts the pricing and inventory decisions simultaneously, in addition to the two opposing cost-related effects that we noted above, price-related factors must also be taken into account. Mainly, there will be a trade-off between the high-price low-demand and low-price high-demand scenarios, in terms of discounted total revenue. This revenue trade-off, however, will not be
independent of the above described cost trade-off, since the demand levels (i.e., inventory levels) will be affected by the pricing decisions. Unfortunately, this complex relationship between the cost and revenue trade-offs does not allow the model to simplify into separate pricing and procurement decisions.

Various versions of the procurement-and-pricing model have been studied in the literature. In his pioneering work, Whitin [2] proposed a link between price theory and inventory control. He noted that such a model would have stronger managerial implications, as compared to fixed-price models. Later, Mills [3,4] and Karlin and Carr [5] established a conceptual framework for a general inventory model, where price is a decision variable. Subsequent studies concentrated mostly on the characterization of the optimal solution for some special cases of the general model [6–13]. Thomas [14] provided some numerical examples to demonstrate the nature of the decision problem under the presence of fixed ordering costs. More recently, Gallego and Van Ryzin [15] studied the continuous-time version of the problem, and Petruzzi [16] investigated the approximate solutions under a learning approach. Also, under a dynamic model with Bayesian learning about the demand distribution, Subrahmanyan and Shoemaker [17] studied a number of numerical examples which provide insights about the sensitivity of optimal prices and inventory levels to changes in the unit procurement cost, price elasticity of demand, form of the demand distribution and the expected demand function.

The existing analytical models impose rather restrictive assumptions on the form of the expected demand function (e.g., concavity assumptions in [9,11,12]), demand distribution (e.g., multiplicative demand in [9], additive demand in [9,11]), or the cost structure (e.g., parameter restrictions and no fixed ordering cost in [11,10]) in analyzing the optimal procurement policy. In this paper, by relaxing some of the more limiting modeling assumptions, we seek to characterize the fundamental properties of the optimal procurement policy and the resulting expected profit in a more general setting (i.e., general demand distribution, linear cost structure with the addition of set-up cost). We also address the sensitivity of the optimal procurement policy to the underlying price-dependent demand uncertainty.

In what follows, we describe the price–demand relationship in Section 2, develop the basic model in Section 3, and characterize its solution in Section 4. The rest of the paper is devoted to special cases and extensions. We consider the infinite-horizon problem in Section 5, the model with no fixed ordering costs in Section 6, the non-stationary extensions of the basic model in Section 7, and the special case with deterministic demand in Section 8. Section 9 is devoted to some concluding remarks. Sections 4 and 5 include numerical examples. Proofs are given in the appendix.

2. Demand uncertainty

It has been a common practice in demand modeling to express random demand as a combination of an expected demand function, which exhibits some form of price-dependency, and a random term, which is price-independent. This approach conveniently isolates the effects of price and uncertainty, while retaining mathematical tractability. However, it has some shortcomings.

Under the additive model [3,5,9,11], we have \( X(p) = \bar{X}(p) + \varepsilon \) where \( \bar{X}(p) \) is a decreasing function of \( p \) and \( \varepsilon \) is a random variable with \( E[\varepsilon] = 0 \). The additive model can also be interpreted as a homoscedastic regression model where \( \bar{X}(p) \) represents the regression function and \( \varepsilon \) is the error term. One implication of this model is that while the expected demand is a function of the price, the demand variance is price-independent (demand distribution shifts as price varies). Also, the model allows for negative demand, unless the price values are bounded from above.

The multiplicative model [5,6,8] is another commonly used form where \( X(p) = c*p \bar{X}(p) \) with \( E[c] = 1 \). This model implies the restriction that the demand equals to the product of its expected value and a random term. As a result, the coefficient of variation of demand equals that of \( \varepsilon \), which is constant in price, and demand variance decreases at a rate faster than the expected value and it approaches to zero at high prices.
In addition to these, there are models that represent random demand by a mixture of additive and multiplicative terms \([10,12,13]\), such as \(X(p) = \bar{X}(p)a_1(e) + a_2(e)\), or \(X(p) = a_1(p) + ea_2(p)\), where \(a_1\) and \(a_2\) are differentiable functions.

Additional simplification can be achieved by assuming that the price-demand relationship is perfectly predictable. This leads to the deterministic model, \(X(p) = \bar{X}(p)\), which serves as a first-order approximation, and which has been utilized as a benchmark in the literature \([3,5,9,12]\). It has been reported in these studies that the form of demand uncertainty, whether it is additive or multiplicative, plays a critical role in determining optimal prices. Optimal prices are lower than their deterministic counterparts under the additive demand model, while they are found to be greater under the multiplicative demand model. Additional findings about the form of demand uncertainty, as it impacts optimal prices and inventory levels, are reported in \([18]\) p. 617.

In this study, we represent the demand by a continuous random variable, distributed over the range \([L(p), U(p)]\) with a known density function \(f(x; p)\). \(L\) and \(U\) are differentiable functions which represent the bounds on demand, where \(0 \leq L(p) < X(p) < U(p) < \infty\) for all \(p\). For convenience, we first assume stationarity, whereby the demand density is given by \(f\) in all periods. We then show in Section 8 that the results we obtain for the finite-horizon problem can be extended to non-stationary (period-dependent) demand distributions.

There is empirical evidence that consumers react to price in purchasing. Other things being equal, a lower price facilitates demand in a “fair” market. Therefore, it is natural to assume that the probability that demand is less than a given level \(x\), \(F(x; p) = P[X(p) \leq x]\), is a non-decreasing function of price. That is,

\[
\frac{\partial F(x; p)}{\partial p} \geq 0 \quad \forall x \in (L(p), U(p)). \tag{1}
\]

It follows from Eq. (1) that \(F(x; p_1) \leq F(x; p_2)\) for \(p_1 < p_2\). Thus, price induces a stochastic ordering of demand distributions.

The expected demand is given by

\[
\bar{X}(p) = \int_{L(p)}^{U(p)} xf(x; p) \, dx = \int_{0}^{\infty} [1 - F(x; p)] \, dx. \tag{2}
\]

It also follows from Eqs. (1) and (2) that \(\bar{X}(p)\) is a decreasing function of \(p\):

\[
\frac{\partial F(x; p)}{\partial p} \geq 0 \Rightarrow \frac{d\bar{X}(p)}{dp} = -\int_{0}^{\infty} \frac{\partial F(x; p)}{\partial p} \, dx < 0.
\]

3. Mathematical model and assumptions

Consider a periodic-review inventory system with \(N\) periods where the decision variables are \(q_n\) and \(p_n\). The review periods are linked by period-ending inventory levels such that the leftovers are transferred fully to the next period and shortages are lost. That is, \(i_n = [q_{n+1} - X(p_{n+1})]^+\) for \(0 \leq n \leq N - 1\), and \(i_N \geq 0\) is a given parameter. At the beginning of a period, the vendor decides how much to order \((q_n - i_n)\) and what price to charge until the next decision point. There is a fixed cost of ordering \((\bar{X})\), but no cost is assumed for pricing. Therefore, it is for the vendor’s benefit to reconsider pricing at each decision epoch.

We assume that the vendor has full information about the inventory and procurement costs, and the demand distributions in all periods of the planning horizon. At the beginning of a review period, say period \(n\),
given the inventory position, the vendor is to determine the procurement quantity and the price to maximize the expected n-period profit which represents the expected value of the sum of the current period’s profit and the discounted optimal expected profit to be obtained profit during the remaining periods.

Let \( P^r_n \) and \( P^w_n \) be the price floor and the price ceiling, respectively, in period \( n \), such that the interval \([P^r_n, P^w_n]\) represents the range of allowable prices. Then, the sequences \( \{P^l_1, P^l_2, \ldots, P^l_N\} \) and \( \{P^u_1, P^u_2, \ldots, P^u_N\} \) establish limiting-price profiles during the planning horizon. In regulated markets, these profiles represent the temporal rules of regulation. Also, in a case where a manufacturer might issue price limits for its dealers (price-maintenance), or provide “suggested retail prices”, the allowable price ranges could be represented by the limiting-price profiles.

In the basic model, without loss of generality, we assume uniform profiles where the feasible price values in any period are between \( P^r_n \) and \( P^w_n \). If there is no price regulation or other constraint on prices, then the price limits may be taken as 0 and \( p_\infty \), respectively, where \( p_\infty \), referred to as the “null price” in the literature, is the highest possible price level at which there will be no demand, that is, \( P[X(p) \leq 0] = 1 \) for all \( p \geq p_\infty \). Allowing \( p_\infty = \infty \) leads to detailed technical considerations (see [8,9]) which we choose to avoid in this paper.

We assume that inventory costs are proportional to period-ending inventory levels. The cost and discount-factor parameters may be allowed to change from period to period. However, as in the case of the demand distribution, we keep them time-invariant for convenience in developing the basic model (see Section 8 for non-stationary extensions of the basic model). We take \( P_n > c \) so that it is possible to make profit. In addition, the leftovers at the end of the planning horizon are assumed to be salvaged at a discount price, that is, \( v \leq c \).

Under these assumptions, the n-period optimal expected profit, as a function of \( i_n \), is obtained from

\[
\Pi^*_n(i_n) = \max \{\Pi_n(i_n, p_n); p_n \in [P^r_n, P^w_n]\},
\]

and

\[
\Pi_n(i_n, p_n) = \max \{M_n(i_n, p_n, q_n) - \mathcal{H} \delta(q_n - i_n); q_n \geq i_n\},
\]

where \( M_n \) is the expected value of the n-period pseudo-profit function which is defined for \( n \geq 1 \) as

\[
M_n(i_n, p_n, q_n) = -c(q_n - i_n) + \left\{ \begin{align*}
\alpha \Pi^*_{n-1}(0) + p_n q_n - r(X(p_n) - q_n), & \quad 0 \leq q_n \leq X(p_n), \\
\alpha \Pi^*_{n-1}(q_n - X(p_n)) + p_n X(p_n) - h(q_n - X(p_n)), & \quad X(p_n) \leq q_n
\end{align*} \right.
\]

with \( \Pi^*_0(i_0) = v[i_0]^+ \). This leads to

\[
M_n(i_n, p_n, q_n) = (p_n + r - c)q_n - rX(p_n) - (p_n + r + h)\Theta(p_n, q_n) + \alpha i_n + \alpha \Pi^*_{n-1}(0)[1 - F(q_n, p_n)] \\
+ \alpha \int_0^{q_n - X(p_n)} \Pi^*_{n-1}(x)f(q_n - x; p_n) \, dx,
\]

where \( \Theta(p_n, q_n) = E[q_n - X(p_n)]^+ \) is the expected value of leftovers at the end of period \( n \). It is seen that \( \Theta(p_n, q_n) \) is a convex increasing function in \( q_n \) and an increasing function in \( p_n \); its additional properties are given in [19].

Therefore, the overall decision problem is described by

\[
\Pi^*_n(i_n) = \max \{\Pi_n^*(i_n, q_n) - \mathcal{H} \delta(q_n - i_n); q_n \geq i_n\} \quad (4)
\]

\[
\Pi^*_n(i_n, q_n) = \max \{\Pi_n(i_n, p_n, q_n); p_n \in [P^r_n, P^w_n]\} = \Pi_n(i_n, p^*_n(q_n), q_n), \quad (5)
\]

for all \( n \geq 1 \), where \( p^*_n(q_n) \) represents the optimal price for a given \( q_n \) in period \( n \). The objective is to determine the optimal decision variables \( p^*_n \) and \( q^*_n \) for all \( n \), which jointly maximize \( \Pi_N \) for a given \( i_N \). Note that if \( \Pi^*_n(i_N) \leq 0 \) then the optimal decision would be not to do business.
4. Model solution

In order to determine the optimal procurement policy for period \( n \), we need to characterize the form of \( \bar{M}_n^*(i_n, q_n) \). First, we note from Eqs. (3) and (5) that \( \bar{M}_n^* \) satisfies

\[
\bar{M}_n^*(i_n, q_n) = \bar{M}_n^*(0, q_n) + c i_n,
\]

so that \( \bar{M}_n^* \) is additively separable in \( i_n \) and \( q_n \). Hence, it suffices to characterize the form of \( \bar{M}_n^*(0, q_n) \). For that purpose, we need to solve the maximization problem:

\[
\bar{M}_n^*(0, q_n) = \max_{q_n} \left\{ (p_n + r - c)q_n - rX(p_n) - (p_n + r + h)\Theta(p_n, q_n) + \alpha \Pi_n^{* - 1}(0)[1 - F(q_n; p_n)]
\right. \\
+ \left. \alpha \right\}
\]

It turns out that the solution methods that have been developed for the fixed price (i.e., \( P_r = P_u \)) versions of the model [20–23] are not directly applicable for the solution of the problem at hand. Instead, we will follow a new solution approach which follows the inductive setting described in [21].

4.1. Form of the optimal procurement policy

The dynamic programming problem represented by Eq. (6) is worked out in the appendix. The results are summarized below in a theorem and a corollary after the following definitions.

**Definition.** Let \( G(i, p, q) = (p + r - c)q - rX(p_n) = (p_n + r + h - \alpha c)\Theta(p_n, q_n) + ci, \) and \( G^*(0, q) = \max\{ G(0, p, q); p \in [P_r, P_u] \} \); then, we define the critical inventory levels \( 0 \leq s_n^w < s_y < S_y < S_y < s_n^w < S_n \) as shown in Fig. 1.

Note that the generic function \( G \) represents the expected profit in a single-period, lost-sales model where the unit salvage value is \( c \) (i.e., the leftovers are returned to the supplier at the original cost). It will play a critical role in the characterization of \( \bar{M}_n^* \), and in establishing bounds on the optimal control parameters.

The following theorem describes the form of \( \bar{M}_n^* \) in general:

**Theorem 1.** There exist an even integer \( k_n \) and critical inventory levels, \( s_n^1 < s_n^2 < \cdots < s_n^{k_n - 1} = S_n < s_n^k \), defined by

\[
S_n = \arg \max \{ M_n^*(0, q); 0 \leq q < \infty \},
\]

\[
s_n^1 = \min \{ q: M_n^*(0, q) \geq M_n^*(0, S_n - \mathcal{K}) \},
\]

\[
s_n^j = \min \{ q: q > s_n^{j-1}, M_n^*(0, q) = M_n^*(0, S_n - \mathcal{K}) \} \quad \text{for} \ j = 2, 3, \ldots, k_n,
\]

such that if \( M_n^*(0, q_1) \) is unimodal, then for \( n \geq 2 \):

(a) \( M_n^*(0, q_n) = G^*(0, q_n) + \alpha \Pi_n^{* - 1}(0, s_n^{j-1}) \), \quad \forall q_n \in [0, s_n^{j-1}],

(b) \( M_n^*(0, q_n) \leq G^*(0, q_n) + \alpha \Pi_n^{* - 1}(0, s_n^{j-1}) + \alpha \mathcal{K} \), \quad \forall q_n \in [s_n^{j-1}, \infty),

(c) \( M_n^*(0, q_n) \geq G^*(0, q_n) + \alpha \Pi_n^{* - 1}(0, s_n^{j-1}) \), \quad \forall q_n \in [s_n^{j-1}, \infty),

(d) \( \forall q_n \in [S_n, s_n^{j-1}], M_n^*(0, q_n) \geq M_n^*(0, q_n) - \alpha \mathcal{K} \), \quad \text{for any} \ q_n \in (q_n, \infty),

(e) \( M_n^*(0, 0) < \infty \),

(f) \( \lim_{q_n \to \infty} M_n^*(0, q_n) = - \infty \).
Corollary 1. For \( n \geq 2 \), \( s_1^n < s_1^n < s_1^{n-1} < s'_y < s''_y \), and \( s'_y < S_n < S''_y \); also, if \( s_y < s_1^n \), then \( s_y < s_1^n \).

Regarding the condition of unimodality of \( M^*(0, q_1) \) in Theorem 1, there are analytical difficulties in proving this property, except for some special cases \([3,5,8,12,19,24]\). For instance, it is reported in \([19]\) that \( M^*(0, q_1) \) is unimodal if (1) demand is deterministic, (2) demand is additive, \( \varepsilon \) has a uniform distribution and \( \bar{X}(p) \) is linear, or (3) demand is multiplicative, \( \varepsilon \) has an exponential distribution and \( \bar{X}(p) \) is linear. The discussions of unimodality and sufficient conditions can be found in the above-cited references.

Since \( G \) is a special case of \( M^*_1 \) (with \( v = c \)), under the unimodality assumption, \( G^* \) is also a unimodal function with a maximizer at \( S_y \) and a reorder point at \( s_y \) (\( < S_y \)). Thus, it follows from (a) and Corollary 1 \( (s_1^{n-1} < S_y) \) that \( M^*_n(0, q_n) \) is an increasing function of \( q_n \) on \([0, s_1^{n-1}]\). Furthermore, (a) implies that the optimal pricing decision, \( p_n(q_n) \), for \( q_n \in [0, s_1^{n-1}] \) is given by \( \text{argmax}\{ G(0, p, q_n); p \in [P_\gamma, P_\alpha] \} \), the maximizing price under the generic single-period model. (Note that if \( q_n \leq s_1^{n-1} \), there will be an order placed in period \( n - 1 \); hence, the inventory carried over to period \( n - 1 \) would worth \( \infty \), and in this case the pricing decision is based only on \( G \).)

It is implied by (b) and (c) that \( M^*_n(0, q_n) \) is bounded by two unimodal functions which are at most \( 2\kappa \) apart on \([s_1^{n-1}, s_1^{n-2}]\). Hence, (b) and (c) establish bounds on \( M^*_n(0, q_n) \) based on the value of the optimal \((n-1)\)-period profit, provided that an order takes place in period \( n - 1 \). In this case, the contribution of the current period’s profit is limited to between \( G^*(0, q_n) \) and \( G^*(0, q_n) - 2\kappa \).
It follows from (a) that $M_n^*(0, 0) = G^*(0, 0) + \alpha M_{n-1}^*(0, s_{n-1}^1)$; hence, properties (b) and (c) can be rewritten independent of $M_{n-1}^*(0, s_{n-1}^1)$ as

\[
\begin{align*}
(b') & \quad M_n^*(0, q_n) - M_n^*(0, 0) \leq G^*(0, q_n) - G^*(0, 0) + \alpha' \lambda, \quad \forall q_n \in [s_{n-1}^1, \infty), \\
(c') & \quad M_n^*(0, q_n) - M_n^*(0, 0) \geq G^*(0, q_n) - G^*(0, 0), \quad \forall q_n \in [s_{n-1}^1, s_{n-1}^{k_n-1}].
\end{align*}
\]

Thus, under the optimal solution, the increase in expected $n$-period profit, upon raising the stock level from 0 to $q_n$, must be greater than or equal to the change in one-period expected profit (with $v = c$) under a similar stock movement (i.e., $G^*(0, q_n) - G^*(0, 0)$). This increase, however, is limited by $G^*(0, q_n) - G^*(0, 0) + \alpha' \lambda$.

Property (b) extends the upper bound function $G^*(0, q_n) - \alpha M_{n-1}^*(0, s_{n-1}^1) + \alpha \lambda$ over $[s_{n-1}^{k_n-1}, \infty)$. On the other hand, it follows from (d) that $M_n^*(0, q_n)$ is $\alpha \lambda$-decreasing ([22]) over $[S_n, s_{n-1}^{k_n-1}]$, and the functional value of $M_n^*$, evaluated at any point beyond $s_{n-1}^{k_n-1}$, is strictly below the global maximum of $M_n^*(0, q_n)$.

A general expected $n$-period pseudo-profit function, as characterized by Theorem 1, is depicted in Fig. 2. It is seen that there could be more than one reorder point, order-up-to level policy exists [21].

The possible presence of multiple order-up-to levels poses serious difficulties in characterizing and computing the optimal procurement policies. These difficulties, however, can be circumvented if the beginning inventory level before ordering in an arbitrary period $n$ is less than or equal to $s_n^k$. Then, since $M_n^*(0, 0)$ is finite and $M_n^*(0, q_n)$ tends to decline as $q_n$ tends to infinity, the optimal procurement policy can be characterized as provided that $i_n \leq s_n^k$:

\[
q_n^* = \begin{cases} 
S_n & \text{if } i_n \in O_n, \\
i_n & \text{if } i_n \in O_n
\end{cases}
\]

for all $n$, where $O_n$ the “reorder region”, is the union of reorder intervals, given by $[0, s_n^1] \cup [s_n^2, s_n^3] \cup \cdots \cup [s_n^{k_n-2}, s_n^{k_n-1}]$ and $O_n$ is the complement of $O_n$ with respect to $[0, s_n^k]$. Note that under the unimodality condition imposed on $M_n^*$, we have $k_1 = 2$ and $q_n^*$ is determined by an $(s_1, S_1)$ policy.

In implementing the above optimal procurement policy, we need to compute $s_n^k$ for each period, which can be a computational burden. Instead, a stronger but more useful condition can be established by referring to Corollary 1, where we have $s_n^k \leq s_n$. That is, we replace the condition of the above optimal policy ($i_n \leq s_n^k$) by
which involves the computation of $s_n^q$ once and in advance by using the problem primitives. Also, it is seen that the condition $i_n \leq s_n^q$ is satisfied if $S_n < s_n^q$ for all $n$ and $i_n \leq S_n$.

The fundamental difference between the $(s_n, S_n)$ policies defined for the fixed-price models [20,21] and the policy we define in Eq. (8) is that, at each decision epoch, the vendor has to make a pricing decision whether or not it decides to place an order. Thus, the vendor needs to determine the optimal price, $p_n(i_n)$, at the beginning of each review period, given the observed value of $i_n$. This requires the preparation of price tables for the vendor to read the optimal prices from. These tables can be prepared simultaneously during the computation of reorder points and the order-up-to level in each period.

4.2. A sufficient condition for a single reorder point

Properties (b) and (c) in Theorem 1 indicate that $M_\ast(0, q_n)$ is $x$-increasing over $[s_n^{1 - 1}, s_n^q]$. Theoretically, this allows for more than one reorder point. However, operating a system under multiple reorder points could be a burden in practice. Determination of the critical levels $s_n^1, s_n^2, \ldots, s_n^q$ for all periods with sufficient accuracy could be difficult. It is, therefore, important to know under what conditions there exists a single reorder point (i.e., $k_n = 2$).

It follows from property (a) in Theorem 1 that $M_\ast(0, q_n)$ is increasing over $[0, s_n^{1 - 1}]$; thus, if $s_n^1 \leq s_n^{1 - 1}$, that is if $M_\ast(0, s_n^{1 - 1}) \geq M_\ast(0, s_n^1)$, then $k_n = 2$ and $s_n^1$ is the only reorder point. However, this is not a convenient sufficient condition, since, under a general demand distribution, the value of $s_n^{1 - 1}$, or $M_\ast(0, s_n^{1 - 1})$, is not analytically measurable with sufficient accuracy.

Another possibility is to show that $M_\ast(0, q_n)$ is increasing over $[s_n^q, s_n^{q - 1}]$, implying $k_n = 2$ under Theorem 1. ($s_n^{k_n - 1} \leq s_n^q$ for all $n$; see Corollary 1.) To this end, we assume that $M_\ast(0, q_n - 1)$ is increasing over $[s_n^{q - 1}, s_n^q]$, and investigate the sufficient conditions under which $M_\ast(0, q_n) \leq M_\ast(0, q_n')$ for $s_n^{q - 1} \leq q_n < q_n' \leq s_n^q$ with arbitrary $q_n$ and $q_n'$. This leads to the following result:

**Corollary 2.** $M_\ast(0, q_n)$ is non-decreasing in $q_n$ over $[s_n^{1 - 1}, s_n^q]$ for $n \geq 1$, that is there exists a single reorder point ($k_n = 2$), if $s_n^q \leq S_1$ and for all $q_n \in [s_n^q, s_n^q]$:

$$F(q_n; p_n) = \frac{p_n + r - c}{p_n + r + h - \alpha c}, \quad \forall p_n \in \{p : p + h - \alpha c > 0, P_r \leq p \leq P_u\}. \quad (9)$$

The RHS in Eq. (9) is a concave increasing function of $p_n$, and $F(q_n, p_n)$ is assumed to be a non-decreasing function of $p_n$ for all $q_n$ (see Section 2).

In view of Eq. (9), we can make the following observations. If the vendor administers a higher price, he is able to increase $F(q_n, p_n)$, the likelihood that there will be no shortage at level $q_n$ to a desired level, while the relative weight of unit underage cost rises (relative weight of unit overage cost declines). Hence, the vendor is inclined to increase the stock level to above $q_n$ and this continues until a break-even point is reached. This is verified by the fact that $\frac{\partial G(0, p, q)}{\partial q} \geq 0$ under Eq. (9). Thus, based on the demand distribution and unit costs that prevail in the market, Eq. (9) reflects the vendor’s capability to increase the expected current period profit by increasing the stock level to above $s_n^q$. In other words, Eq. (9) attributes a higher monopoly power to the vendor than the vendor would have in its absence. In this interpretation, we utilized the myopic notion that the higher the monopoly power, the less responsive is the demand to changes in price. That is, probability of satisfying the demand fully, $F(q_n, p_n)$, does not respond strongly to an increase in price, since the increase in $F(q_n, p_n)$ is limited by the RHS in Eq. (9).

Under Theorem 1 and Corollary 2, the optimal procurement policy is defined by

$$q_n^\ast = i_n + (S_n - i_n)\delta(s_n^1 - i_n), \quad (10)$$
provided that \( i_n \leq s_n^* \) is satisfied for \( n > 1 \). It also follows that \( s_n' \leq S_n \) for \( n \geq 1 \), such that \( s_n' \) is a lower bound on order-up-to levels.

Note that in order to have \( k_n = 2 \), \( \bar{M}_n^*(0, q_n) \) need not be increasing over \([s_n^*, s_n']\). There could be other conditions leading to the same result. Our experience with a number of numerical examples suggests that cases with \( k_n > 2 \) would be rare in practice.

4.3. Sufficient conditions for the optimality of \((s_n, S_n)\) policies

If, in property (d), the range for \( q_n \) were established as \([S_n, \infty)\), then \( \bar{M}_n^*(0, q_n) \) would be characterized as a \( \mathcal{K} \)-decreasing function over \([S_n, \infty)\), \( S_n \) would be the only order-up-to level, and there would be no need for the condition \( i_n \leq s_n^* \) in Eq. (8) or Eq. (10). This is precisely the case in the development of the optimality of \((s_n, S_n)\) policies under the fixed-price \((P_f = P_u)\) model [21].

In the following corollary, we investigate the sufficient conditions under which \( M_n^* \) assumes a desirable shape over \([S_n, \infty)\) to yield a single order-up-to level.

**Corollary 3.** \( M_n^*(0, q_n) \geq \bar{M}_n^*(0, q_n) - \alpha \mathcal{K} \) for all \( q_n \) and \( q_n' \) with \( S_n \leq q_n < q_n' \) if \( s_1 \leq S_n \) and

\[
(p_n + r + h - \alpha c)F(q_n; p_n) \geq p_n + r - c, \quad \forall p_n \in [P_f, P_u] \text{ for } n \geq 1.
\]

(11)

Note that for large \( q \) values, \( F(q; p) \) tends to 1 (particularly, \( F(q; p) = 1 \) when \( q \geq U(p) \)), and this complies well with condition Eq. (11). That is, \( M_n^*(0, q_n) \) is \( \mathcal{K} \)-decreasing in the limit as \( q_n \) tends to infinity. Also, it is intuitive that at considerably large inventory levels, the vendor would tend to reduce the price substantially, even below the unit procurement cost, in order to deplete inventories. In this regard, it is seen in Eq. (11) that the condition tends to hold when \( c \) and \( h \) get larger and \( r \) gets smaller (it holds at all \( q \) levels when \( p + r - c \leq 0 \leq p + r + h - \alpha c \), that is, \( \alpha c - h - r \leq p \leq c - r \)). In general, the condition tends to hold when \( c \) and \( h \) get larger or \( r \) gets smaller.

Thus, under Theorem 1 and Corollary 3, the optimal procurement policy is characterized by

\[
q_n^* = \begin{cases} 
S_n & \text{if } i_n \in O_n, \\
i_n & \text{if } i_n \in \bar{O}_n \text{ or } i_n > s_n^*. 
\end{cases}
\]

Also, under both Corollaries 2 and 3, it follows from Theorem 1 that \( k_n = 2 \) for all \( n \) and \( q_n^* \) is determined by an \((s_n, S_n)\) policy: \( q_n^* = i_n + (S_n - i_n)\delta(s_n^1 - i_n) \).

4.4. Numerical example

To demonstrate our findings, we consider a 5-period problem defined by the base parameter set \( c = 0.5, r = 1.5, h = 0.4, v = 0.1, \mathcal{K} = 5, \alpha = 0.95 \); the expected demand function \( X(p) = ae^{-bp} \) with \( a = 150 \) and \( b = 0.2 \); the price limits \( P_f = 0.1 \) and \( P_u = 10 \); and the additive-uniform demand distribution \( F(x; p) = 0.5(x - \bar{X}(p) + \lambda)/\lambda \) for \( -\lambda \leq x - \bar{X}(p) \leq \lambda \) with \( \lambda = 20 \). \( \lambda = 0 \) corresponds to the deterministic demand case.) For sensitivity analyses, we used multiple parameter values as \( h = 0.02, 0.1, 0.4; \mathcal{K} = 1, 5, 10; b = 0.05, 0.1, 0.2; \text{ and } \lambda = 0, 5, 10, 20 \).

Using a computer program, we computed the expected pseudo-profit functions \( \bar{M}_1^*, \bar{M}_2^*, \ldots, \bar{M}_5^* \) over the \( q_n \) range of \([0, 250]\) with a step size of 1. The resulting optimal values of the control parameters and the critical functional values are shown in Tables 1 and 2. Also, the pseudo-profit functions computed for \( \lambda = 0, \mathcal{K} = 10 \), the deterministic demand case, and \( \lambda = 10, \mathcal{K} = 10 \) are plotted in Figs. 3 and 4.
Table 1
Critical inventory levels, prices and pseudo-profit values under the 5-period example problem

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</table>

We observe that the optimal control values are quite sensitive to the price sensitivity of demand (note that $b$ is the price elasticity multiplier of the expected demand). The more sensitive the expected demand to price, the less is the expected profit. As for the holding cost, smaller $h$ values tend to yield considerably higher order-up-to levels.

Comparing the critical inventory levels listed in Tables 1 and 2, we see that the conditions given in Corollary 1 hold in each case. We also note that $s^w \leq s^l$ in all cases except $h = 0.1$ and $h = 0.02$, for which the reorder-region order-up-to level policy will be optimal.

Table 1 shows that the optimal prices evaluated at respective order-up-to levels are the same in all periods. Consequently, if an order is to be placed in each period, then the optimal prices in successive periods will be constant. We also observe that demand uncertainty has considerable impact on the parameters of the optimal procurement policy. As the demand variance is decreased (i.e., $\lambda$ is decreased), the order-up-to and reorder levels both tend to be lower, and the vendor foresees higher expected profits.

On the other hand, on comparing the deterministic and probabilistic pseudo-profit curves in Fig. 3, we find that the deterministic values are greater than the corresponding probabilistic values, especially, about the $q_n$ mid-range, which includes respective order-up-to levels.

In Fig. 4, we plot $M^*(0, q_n) - M^*(0, 0)$ vs. $q_n$ together with the boundary functions. It is seen that the properties listed in Theorem 1 and Corollary 1 hold. The boundary functions established in Eq. (7) also satisfy the conditions (b’) and (c’). We also observe that the pseudo-profit curves exhibit considerable slope changes across the $q_n$ mid-range.

The best price trajectories $p_n(q_n)$, evaluated for $n = 1, 2, \ldots, 5$, are plotted in Fig. 5. The figure shows that the optimal pricing decision at a given inventory level is not necessarily trivial. We observe that $p_n(q_n)$ is generally decreasing in $q_n$, but it can exhibit mild increases, or stagnate (become “sticky”) over various $q_n$ ranges where it does not respond to changes in stock levels. Also, on comparing the deterministic and probabilistic optimal prices, we observe that it is not necessarily true that the deterministic prices are lower than the probabilistic prices or vice versa (cf. [4,5]).

Intuitively, price should be lower at higher stock levels, so as to facilitate higher demand to deplete the excess inventories. However, the vendor would still tend to increase the price for two basic reasons: to shrink the demand so that there will be leftovers for future periods, or to realize a higher profit per unit sold. If the fixed ordering cost in a future period is to be saved by satisfying that period’s demand by carying a part of the current inventory forward, then both of these reasons will prevail. Once the total cost of inventory holding and the differential cost of procurement breaks even with the savings in the fixed ordering cost, however, it would not be profitable to consider carrying stock into future periods. As $n$ gets larger, there are more periods ahead to consider; thus, there could be a recurrence of this break-even behaviour at higher $q_n$ levels. This can be seen in Fig. 5 by comparing $p_5(q_5)$ with $p_2(q_2)$. The passage from one pricing regime to another, at a break-even point, need not be smooth and there could be sudden changes in $p_n(q_n)$.
Fig. 3. Expected pseudo-profit curves, $\hat{M}^*_n(0, q_n)$ vs. $q_n$, that are obtained for the 5-period example problem in Section 5.4 ($c = 0.5, r = 1.5, h = 0.02, b = 0.2, \lambda^* = 10, \lambda = 10.0$). The thin curves represent the deterministic case where $\lambda = 0$. 
Fig. 4. Combined representation of the curves in Fig. 3. The thin curves represent $M^*_n(0, q) - M^*_n(0, 0)$ vs. $q$, for $n \geq 1$ and the thick curves represent the boundary functions $G^*(0, q) - G^*(0, 0)$ and $G^*(0, q) - G^*(0, 0) + \mathcal{X}$.

5. Infinite-horizon model

In this section, we study the infinite-horizon version of the model. We drop the period indices from the notation and rewrite the expected pseudo-profit function from Eq. (6) as

$$
\bar{M}^*(0, q) = \max \left\{ G(0, p, q) + x \bar{M}^*(0, s^1) + \int_0^{q-L(p)} \left[ \bar{M}^*(0, x) - \bar{M}^*(0, s^1) \right]^+ f(q - x; p) \, dx : p \in [P_\ell, P_u] \right\}
$$

$$
= G(0, p_q, q) + x \bar{M}^*(0, s^1) + \int_0^{q-L(p_q)} \left[ \bar{M}^*(0, x) - \bar{M}^*(0, s^1) \right]^+ f(q - x; p_q) \, dx,
$$

(12)

where $p_q \equiv p(q)$, the maximizing price at $q$.

The solution for $\bar{M}^*(0, q)$ involves the integral in Eq. (12) which represents the convolution of the density function with a filtering function based on $\bar{M}^*(0, q)$ (i.e., the function $\bar{M}^*(0, q)$ is filtered over the level $\bar{M}^*(0, s^1)$). Because of this, we need to solve the integral Eq. (12) recursively. In each stage of the solution, $\bar{M}^*(0, q)$ is established over a non-overlapping interval, and the solution obtained is employed in the next stage.
Fig. 5. Optimal price trajectories that are obtained for the 5-period example problem shown in Fig. 3.
It follows from Eq. (12) that for $0 \leq q \leq s^1$ we have $\tilde{M}^*(0, q) = G^*(0, q) + \varpi \tilde{M}^*(0, s^1)$, which is an increasing function of $q$ over $[0, s^1]$ where $s^1 < S_q$. Evaluating $\tilde{M}^*(0, q)$ at $s^1$, we obtain $\tilde{M}^*(0, s^1) = G^*(0, s^1)/(1 - \varpi)$, which implies

$$\tilde{M}^*_1(0, q) = G^*(0, q) + \frac{\varpi}{1 - \varpi} G^*(0, s^1),$$

(13)

where the index in roman numeral denotes the stage number. Successive stages I, II, III, ... correspond to intervals $[0, s^1], [s^1, s^2], [s^2, s^3], ...$, respectively.

In the next stage, we have $s^1 \leq q \leq s^2$ for which the effective integration range starts at $s^1$ in Eq. (12) and we have

$$\tilde{M}^*_2(0, q) = G(0, p_q, q) + \varpi \tilde{M}^*_1(0, s^1) + \varpi \int_{s^1}^{q - L(p_q)} [\tilde{M}^*_1(0, x) - \tilde{M}^*_1(0, s^1)] f(q - x; p_q) \, dx$$

$$= G(0, p_q, q) + \varpi [1 - F(q - s^1; p_q)] \tilde{M}^*_1(0, s^1) + \varpi \int_{s^1}^{q - L(p_q)} \tilde{M}^*_1(0, x) f(q - x; p_q) \, dx,$$

(14)

which is a renewal equation [25] with solution

$$\tilde{M}^*_2(0, q) = \max \left\{ G(0, p, q) + G^*(0, s^1) \left[ \frac{\varpi}{1 - \varpi} - R_q(q - s^1; p) \right] + \varpi \int_{s^1}^{q - L(p)} G(0, p, x) dR_q(q - x; p); \ p \in [P_l, P_u] \right\},$$

where $R_q(x; p) = \sum_{i=1}^{\infty} \alpha^i F^{(i)}(x; p)$ and $F^{(i)}$ is the $i$-fold convolution of $F$ with itself. (Derivation is given in the appendix.)

Having established $\tilde{M}^*_2(0, q)$ over $[0, s^2]$, we check whether $s^2 < s^3$ where $s^2$ is a lower bound on $s^3$ (see Corollary 1). If $s^2 < s^3$, then $k = 2$ and we stop. In this case, there will be one reorder point. If $s^2 < s^3$, however, then we need to consider the $q$ range beyond $s^2$ and pass to the next stage.

Evaluating Eq. (12) for $s^2 \leq q \leq s^3$, we obtain

$$\tilde{M}^*_3(0, q) = G(0, p_q, q) + \varpi \tilde{M}^*_2(0, s^1) + \varpi \int_{s^1}^{s^2} [\tilde{M}^*_2(0, x) - \tilde{M}^*_2(0, s^1)] f(q - x; p_q)$$

$$= G(0, p_q, q) + \frac{\varpi}{1 - \varpi} G^*(0, s^1) [1 + F(q - s^2; p_q) - F(q - s^1; p_q)] + \varpi \int_{s^1}^{s^2} \tilde{M}^*_2(0, x) f(q - x; p_q) \, dx.$$

Since $\tilde{M}^*_2$ inside the integral is obtained earlier, it can be substituted in the above to solve for $\tilde{M}^*_3$.

The fourth stage will involve another renewal equation. For $s^3 \leq q \leq s^4$, we have

$$\tilde{M}^*_4(0, q) = G(0, p_q, q) + \varpi \tilde{M}^*_3(0, s^1) + \varpi \int_{s^1}^{q - L(p_q)} [\tilde{M}^*_3(0, x) - \tilde{M}^*_3(0, s^1)] f(q - x; p_q)$$

$$+ \varpi \int_{s^1}^{q - L(p_q)} \tilde{M}^*_3(0, x) f(q - x; p_q)$$
\[
G(0, p_q, q) + \frac{\alpha}{1 - \alpha} G^*(0, s^1)[1 + F(q - s^2; \ p_q) - F(q - s^1; \ p_q) - F(q - s^3; \ p_q)] \\
+ \alpha \int_{s^1}^{s^2} M^*_1(0, x)f(q - x; \ p_q) dx + \alpha \int_{s^1}^{q-\lambda(p)} M^*_1(0, x)f(q - x; \ p_q) dx,
\]

which is a renewal equation on \(M^*_1\). The solution for this equation can be obtained by following a similar procedure to the one used in solving for \(M^*_n(0, q)\).

If \(s^1 \leq s^*_n\), then we stop. Otherwise, we follow the same solution process to establish \(M^*_n\) in the next two stages, and check the stopping criterion.

This method works in principle. However, there is a computational difficulty due to the fact that critical inventory levels \(s^1, s^2, s^3, \ldots\) are unknown in advance. They have to be computed simultaneously as we establish \(M^*_n\) numerically.

One approach is to start with the assumption that \(s^1 = s_q\). Based on this initial value of \(s^1\), \(M^*_n\) can be established over \([0, s^2]\) and the value of \(s^2\) can be found by incrementing the value of \(q\) until \(M^*_n(0, q)\) equals \(M^*_n(0, s^1)\). Note that \(s^2\) will be the first point greater than \(s^1\) satisfying the above equality. Likewise, the process can be continued to include \(s^3\) and \(s^4\) and so on. Once the stopping condition \((s^*_n \leq s^k)\) is satisfied, using the current solution of \(M^*_n\), \(s^1\) can be updated. If it is tolerably close to its previous value, the procedure can be terminated. Otherwise, based on its updated value, a new \(M^*_n\) will be established over \([0, s^k]\).

It is beyond the scope of this paper to investigate the convergence of the numerical method proposed above. To demonstrate its merits, however, we use it to obtain the solution for an infinite-horizon problem defined by the parameter set \(c = 0.5, r = 0.25, h = 0.3, \nu = 0.1, \kappa = 8, \alpha = 0.7\); the expected demand function \(\bar{X}(p) = 150 \exp(-0.5p)\); the price limits \(P_r = 0.1\) and \(P_a = 4\); and the multiplicative-exponential demand distribution \(F(x; \ p) = 1 - \exp(-(x/\bar{X}(p)))\), \(x \geq 0\), under which the discounted renewal function is given by \(R_d(x; \ p) = 2 [1 - \exp(-(1-\alpha)x/\bar{X}(p))] / (1 - \alpha)\).

We obtained the solution in 10 iterations. The successive \(s^1\) values were 28.20, 35.25, 31.26, 33.40, 32.15, 32.87, 32.44, 32.70, 32.55 and 32.64 until we reached a tolerance of 0.1 units. In this example problem, it was found that \(k = 2\); hence, it was not necessary to pass beyond the second stage in computations.

In order to demonstrate the transition from \(M^*_n\) to \(\bar{M}^*_n\), we solved a 15-period problem under the same problem data, and compared the resultant expected pseudo-profit functions with the function we obtained by solving the infinite horizon problem. Fig. 6 depicts \(\bar{M}^*_n(0, q_n)\) functions for \(n = 1, 2, \ldots, 15\) and \(\bar{M}^*\), which become indistinguishable from the pseudo-profit functions for large \(n\), as predicted by the theory.

6. No-fixed-ordering-cost

In order to establish a link between this paper and the earlier studies reported in the literature, here we take a brief look at the model with \(\kappa = 0\).

Following a similar inductive approach to the one we used previously, we suppose that \(\bar{M}^*_n(0, q_1)\) is unimodal and \(\Pi^*_n(i_{n-1})\) is determined by a single-parameter policy:

\[
\Pi^*_n(i_{n-1}) = ci_{n-1} + \begin{cases} 
\bar{M}^*_n(0, S_{n-1}), & i_{n-1} \leq S_{n-1}, \\
\bar{M}^*_n(0, i_{n-1}), & S_{n-1} < i_{n-1}.
\end{cases}
\]
Fig. 6. Expected pseudo-profit curves, $\overline{M}^\ast_n(0, q_n)$ vs. $q_n$, that are evaluated for the 15-period problem specified in Section 6. The curve at the top represents the solution for the infinite horizon model with the same parameter set. (It coincides with the expected pseudo-profit curves at large $n$.)

Under this setting, it follows from Eq. (6) that

$$\overline{M}^\ast_n(0, q_n) = \max \left\{ G(0, p_n, q_n) + \alpha \overline{M}^\ast_{n-1}(0, S_{n-1}) \right\}$$

$$+ \int_{S_{n-1}}^{q_n - L(p_n)} \left[ \overline{M}^\ast_{n-1}(0, x) - \overline{M}^\ast_{n-1}(0, S_{n-1}) \right] f(q_n - x; p_n) \, dx; \, p_n \in [P_L, P_u] \right\}.$$  (15)

It can be seen that for $q_n \leq S_{n-1}$, the integral drops out and we have

$$\overline{M}^\ast_n(0, q_n) = G^\ast(0, q_n) + \alpha \overline{M}^\ast_{n-1}(0, S_{n-1}),$$

where $G^\ast(0, q_n)$ is a unimodal function. For $S_{n-1} < q_n$, however, we note that the integral in Eq. (15) is negative-valued. Based on these observations, we establish $\overline{M}^\ast_n(0, q_n)$ in two alternative ways, as shown in Fig. 7. Considering Eq. (15) and Fig. 7, we discover that

$$\min \{S_g, S_{n-1}\} \leq S_n \leq S_g,$$  (16)

for $n \geq 2$, and the configuration shown in Fig. 7a can only occur for $n = 2$.

The recursive ordering in Eq. (16) implies that if $S_g \leq S_1$, then $S_n = S_g$ for all $n \geq 2$. It is clear that if $v = c$, then $\overline{M}_1 \equiv G$, and $S_1 = S_g$ by definition. But, it is more plausible, in general, that $v < c$, in which case $S_1$ will be different from $S_g$.\n
Intuitively, when the price is fixed, $S_g$ must be greater than $S_1$. Having the option of salvaging the leftovers at a higher return, the decision maker will be encouraged to procure up to higher levels. But, under the impact of pricing, there could be a trade-off between small-quantity high-price and large-quantity low-price scenarios. Demand uncertainty at a given level will then be contingent on the pricing decision. For this reason, under a general price-demand relationship, it is theoretically possible that $S_g < S_1$.

Fig. 7b shows that the expected profit level will be higher if $S_{n-1}$ shifts towards $S_g$ and the contribution of $G$ is more than the contribution of the negative valued integral in Eq. (15). Thus, overcoming the transient behaviour represented by Eq. (16), there will be a propensity, under the optimal solution, for $S_n$ to approach $S_g$ as the number of periods is increased.

Sufficient conditions for the optimality of a single-critical-number policy are given in [11]. In principle, these conditions can be utilized to verify the form of the optimal procurement policy, but they are difficult to interpret in terms of their managerial implications. Under some restrictions, Thowsen [11] shows that the single-parameter policy will be optimal if the demand is additive with a PF$_2$ (Pólya frequency function of type 2) distributed random term and a linear expected demand function. Also, sufficient conditions for the existence and uniqueness of the optimal decision variables are given in [9,10]. The characterization portrayed in these studies are based on taking derivatives. The drawback of that approach is that it can not be immediately extended to the model with fixed ordering costs, where differentiability is restricted to sub-regions.
7. Non-stationary extensions

In addition to being dependent on price, it is possible that the demand distribution exhibits periodic shifts during the planning horizon due to seasonality or changing market conditions. The natural way of incorporating such time-variability in our model would be to represent the demand in period \( n \) by a random variable \( X_d(p_n) \) with distribution \( F_d(x; p_n) \). The generic function, now denoted by \( G_n \), and the critical parameters defined for \( G_n^*(0, q_n) \), \( s_n^* \leq \cdots \leq s_{n-1}^* \) would also be period-dependent.

By virtue of the fact that the demand distribution depends on price and the price can be set at the beginning of each period, extensions to time-dependent demand are immediate. It can be seen, going through the proofs in the appendix, that this does not alter the arguments and derivations, provided that \( G_n^*(0, q_n) \) is unimodal and the condition \( s_n^* \leq s_{n-1}^* \) holds for all \( n > 1 \). This condition parallels the fixed-price version of the non-stationary demand model \([21,23]\) where it is assumed that \( S_{q_n}^* \leq S_{q_{n-1}}^* \) for \( n > 1 \). It can also be seen that if \( S_{q_n}^* \leq S_{q_{n-1}}^* \), then \( K_n = 2 \).

Similarly, we can let the cost parameters and the discount factor depend on the period (i.e., \( c_n, r_n, h_n, K_n, x_n \)), without any difficulty, provided that \( K_n \geq x_{n-1} \) \( K_{n-1} (x_0 = 1) \), and \( M_1^* \) is unimodal for \( n > 1 \) (which implies unimodality of \( G_n^* \)).

8. Deterministic demand

Under deterministic demand, \( X_d(p) = X_d(p) \) for all \( n \). For technical reasons, we will assume that \( X_d(p) \) is \( o(1/p) \) as \( p \to \infty \) and as \( p \to 0^+ \). Under this assumption, the riskless revenue function \( R_d(p) = pX_d(p) \) starts growing at 0 when \( p = 0 \), reaches a maximum level, and declines down to 0 as \( p \) tends to infinity. We further assume that \( R_d(p) \) is a pseudoconcave function of \( p \) on \((0, \infty)\) so that there is a unique breakeven point between low-price-high-demand and high-price-low-demand situations in terms of revenue. It is shown in [19] that if \( X_d(p) \) is a convex or concave decreasing function, then \( R_d(p) \) is pseudoconcave on \((0, \infty)\). A detailed discussion of the properties of the riskless revenue function and related functions can be seen in [7,19].

Using \( X_d(p) = X_d(p) \) in Eq. (3) we obtain

\[
M_d(i_n, p_n, q_n) = G_n(i_n, p_n, q_n) + \alpha[M_{n-1}^*([q_n - X_d(p_n)]^+) - c[q_n - X_d(p_n)]^+]
\]

where

\[
G_n(i_n, p_n, q_n) = (p_n + r - c)q_n - rX_d(p_n) - (p_n + r + h - \alpha)[q_n - X_d(p_n)]^+ + ci_n.
\]  

(17)

This representation leads to:

**Corollary 4.** Under deterministic demand, \( M_1^*(0, q_1) \) is quasiconcave, and \( M_n^*(0, q_n) \) is quasi-\( K \)-concave for \( n \geq 2 \), where:

(i) \( M_n^*(0, q_n) \) is increasing over \( q_n \in [0, X_d(P_0^*]) \), \( n \geq 2 \), where \( P_0^* \) is the maximizer of \((p_n - c)X_d(p_n)\) over \([P_n, P_0^*] \).

(ii) \( M_n^*(0, q_n) \geq G_{n-1}^*(0, q_n) + \alpha M_{n-1}^*(0, s_{n-1}) \), \( M_n^*(0, q_n) \leq G_{n-1}^*(0, q_n) + \alpha M_{n-1}^*(0, s_{n-1}) + \alpha K \), for \( q_n > X_d(P_0^*) \) and \( n \geq 2 \).

(iii) The optimal inventory level is determined by the \((s_n, S_n)\) policy: \( q_n^* = i_n + (S_n - i_n)\delta(s_n - i_n) \).

This result was also obtained by Thomas [26], through a price-dependent lot-sizing approach. He demonstrated that there exists a positive integer \( m \), where \( m + 1 \) is referred to as the planning horizon, such
that for all \( n \), \( S_n = X_1(p_n^*) + X_{n-1}(p_{n-1}^*) + \cdots + X_{n-m}(p_{n-m}^*) \), where \( p_{n-j}^* \) is the maximizer of \((p_{n-j} - e_{n-j})X_n-j(p_{n-j})\), with

\[
    e_{n-j} = \left[ c + h\left(1 + \sum_{k=1}^{j-1} (2^{n-2k-1} 2^{n-2k} \cdots 2^{n-j})\right)\right]/(2^{n-2j-1} 2^{n-2j} \cdots 2^{n-j}),
\]

for \( 1 \leq j \leq m \) and \( p_n^* = p_n^c \).

Thus, since \( e_{n-(j+1)} > e_{n-j} \geq c \) for \( 1 \leq j \leq m \), it follows from Theorem A.1 in the appendix that \( p_n^* = P_n^c \) and \( p_n^* = P_{n-j}^c \) for \( 1 \leq j \leq m \); in particular, if \( X_n(\cdot) = X_{n-j}^*(\cdot) \) for \( 1 \leq j \leq m \), then \( p_n^* = P_n^c \leq p_{n-1}^* \leq p_{n-2}^* \leq \cdots \leq p_{n-m}^* \).

9. Conclusions

We established that a reorder-region, order-up-to-level procurement policy is optimal for the decision problem of interest in this paper, given that \( M_n^*(0, q_n) \) is unimodal, and \( i_n \leq s_n^* \) for \( n \geq 2 \). Under this policy, the vendor administers the best price in each period, whether or not an order is placed. We have also provided upper and lower bounds on the reorder and order-up-to levels, which should expedite the computational work, and sufficient conditions under which a reorder-point, order-up-to-level policy is optimal.

Under the special case with deterministic demand, we have shown that the optimal ordering policy is determined by an \((s, S)\) type policy, provided that the revenue function \( R(p) \) is unimodal. For the infinite horizon stationary model, we have provided a solution method that could be used to obtain the optimal solution numerically.

Furthermore, in the absence of a fixed ordering cost, the order-up-to levels are bounded from above by \( S^* \), and they are ordered in a non-decreasing fashion in \( n \) (see Eq. (16)). With \( \lambda' > 0 \), this ordering does not hold, however, and \( S^* \) is not necessarily an upper bound for order-up-to levels. For instance, we have obtained \( S_1 < S_2 < S_3 < S_4 < S_5 \) for the example problem with \( h = 0.1 \) (see Tables 1 and 2).

One of the contributions of the paper is the solution method which is based on a non-derivative approach. Through this approach, we were able to develop distribution-independent and non-stationary results. The existing solution methods predominantly utilize the derivative approach, and impose restrictive assumptions on the demand distribution, the expected demand function, or the parameter set.

In terms of additional research that could follow the present effort, one interesting area is approximations. As a stationary approximation, we formulated the infinite-horizon model in Section 5, and proposed an algorithm for it, based on the recursive solution of a sequence of renewal equations. It would be interesting to study the underlying convergence problems, both in terms of the proposed algorithm and the optimal control parameter values.

We have demonstrated that the optimal pricing trajectories, \( p_n(q_n), n \geq 1 \), can be complicated enough to make the evaluation and administration of the optimal pricing strategy impractical. A simpler strategy, though not optimal, might be desirable in applications. One possibility is to approximate the best price by utilizing the single-period model characterized by \( G_n \). That is, \( p_n(q_n) = \arg\max\{G_n(0, p, q_n) : p \in [P_n, P_n]\} \), for \( n > 1 \). The function \( M_n(0, p_n(q_n), q_n) \), as an approximation for \( M_n^*(0, q_n) \), would then be used recursively for the determination of \( M_{n+1}^* \). This myopic approximation to the pricing policy worked very well in a preliminary investigation of its performance. We feel that a detailed study of its virtues and limitations is justified. It would also be interesting to investigate optimal procurement policies under this and other suboptimal pricing strategies.
Appendix A

Proof of Theorem 1. Under the unimodality assumption about $\bar{M}^*(0, q_1)$, it is sufficient to show, inductively, that $\bar{M}^*(0, q_n)$ satisfies conditions (a)-(f), given that $\bar{M}^*_n(0, q_n-1)$ does, since using the same line of arguments in the proof, it can be trivially shown that $\bar{M}^*_2(0, q_2)$ satisfies these conditions also. In what follows, we have $q_n \leq s_{n-1}^H$ unless specified otherwise.

Defining the “reorder region” $O_{n-1}$ for period $n-1$ by

$$O_{n-1} = \{ q: \bar{M}^*_n(0, q) \leq \bar{M}^*_n(0, S_{n-1}) - \mathcal{K}, 0 \leq q \leq s_{n-1}^{k_{n-1}} \}$$

it follows under the inductive assumption about $\bar{M}^*_n$ that $\bar{\Pi}^*_n(0) = \bar{M}^*_n(0, s_{n-1}^H)$ and

$$\bar{\Pi}^*_n(i_{n-1}) = c i_{n-1} + \begin{cases} \bar{M}^*_n(0, s_{n-1}^H), & i_{n-1} \in O_{n-1}, \\ \bar{M}^*_n(0, i_{n-1}), & i_{n-1} \in \overline{O}_{n-1}, \end{cases} \quad (A.1)$$

where the region $\overline{O}_{n-1}$ is complementary to region $O_{n-1}$ with respect to $[0, s_{n-1}^{k_{n-1}}]$. Note that $O_{n-1}$ is the union of $k_{n-1}/2$ reorder intervals.

We assume that $\mathcal{K}$ is not exceedingly large so that $s_{n-1}^{k_{n-1}} > 0$. This assumption does not have any critical influence on the results; it only decreases the number of terms to carry in the analysis and simplifies the mathematical representation.

By considering Eq. (A.1), the integral in Eq. (3) can be rewritten as

$$\int_0^{q_{n-L(p_n)}} \bar{\Pi}^*_n(x)f(q_n - x; p_n) \, dx = c \Theta(p_n, q_n) + \int_{x \in O_{n-1}[0, q_{n-L(p_n)}]} \bar{M}^*_n(0, s_{n-1}^H) f(q_n - x; p_n) \, dx$$

$$+ \int_{x \in O_{n-1}[0, q_{n-L(p_n)}]} \bar{M}^*_n(0, x) f(q_n - x; p_n) \, dx$$

$$= c \Theta(p_n, q_n) + \bar{M}^*_n(0, s_{n-1}^H) F(q_n; p_n)$$

$$+ \int_{x \in O_{n-1}[0, q_{n-L(p_n)}]} [\bar{M}^*_n(0, x) - \bar{M}^*_n(0, s_{n-1}^H)] f(q_n - x; p_n) \, dx$$

$$+ \int_0^{q_{n-L(p_n)}} [\bar{M}^*_n(0, x) - \bar{M}^*_n(0, s_{n-1}^H)]^+ f(q_n - x; p_n) \, dx,$$

where the integral with respect to $x \in O_{n-1}[0, q_{n-L(p_n)}]$ represents the sum of integrals over the composite $x$ range $O_{n-1} \cap [0, q_{n-L(p_n)}]$. Same is true for $\overline{O}_{n-1}[0, q_{n-L(p_n)}].$

Using this result in Eq. (3) we obtain

$$\bar{M}_n(i_n, p_n, q_n) = (p_n + r - c)q_n - r \bar{X}(p_n) - (p_n + r + h - \kappa c) \Theta(p_n, q_n) + ci_n + \kappa \bar{M}^*_n(0, s_{n-1}^H)$$

$$+ \kappa \int_0^{q_{n-L(p_n)}} [\bar{M}^*_n(0, x) - \bar{M}^*_n(0, s_{n-1}^H)]^+ f(q_n - x; p_n) \, dx$$

$$= G(i_n, p_n, q_n) + \kappa \bar{M}^*_n(0, s_{n-1}^H) + \kappa \int_0^{q_{n-L(p_n)}} [\bar{M}^*_n(0, x) - \bar{M}^*_n(0, s_{n-1}^H)]^+ f(q_n - x; p_n) \, dx,$$
which can be substituted in Eq. (5) to obtain

\[
\bar{M}_s^*(0, q_n) = \max\left\{ G(0, p_n, q_n) + z\bar{M}_n^{-1}(0, s_n^1) \right\}_{q_n - L(p_n)} + z \int_0^{q_n - L(p_n)} \left[ \bar{M}_n^{-1}(0, x) - \bar{M}_n^{-1}(0, s_n^1) \right]^+ f(q_n - x; p_n) \, dx; \, p_n \in \left[ P_r, P_u \right] \}
\]

(A.2)

where it is understood that \( q_n - L(p_n) \leq s_n^1 \), which is also implied by \( q_n \leq s_n^1 \).

It follows that for \( q_n < s_n^1 \) the integrand in Eq. (A.2) vanishes and we have:

\[
\bar{M}_s^*(0, q_n) = G^*(0, q_n) + z\bar{M}_n^{-1}(0, s_n^1). \]

(A.3)

This establishes (a). Since \( G \) is a special case of \( \bar{M}_1 \) (with \( v = c \)), we consider \( G^* \) to be a unimodal function with a maximizer at \( S \). Thus, under the inductive assumption that \( s_n^1 < S_N \) we conclude from Eq. (A.3) that \( \bar{M}_s^*(0, q_n) \) is an increasing function of \( q_n \) on \([0, s_n^1]\). Later, we shall demonstrate that in fact \( s_n^1 < S_N \) for all \( n \geq 2 \) (see the proof of Corollary 1).

Next, we consider the \( q_n \) range of \([s_n^1, s_n^1]\). Evaluating the maximand in Eq. (A.2) at \( p_N \), where \( p_N = \text{argmax}\{G(0, p_n, q_n); p_n \in \left[ P_r, P_u \right]\} \), we obtain

\[
\bar{M}_s^*(0, q_n) \geq G^*(0, q_n) + z\bar{M}_n^{-1}(0, s_n^1) + z \int_0^{q_n - L(p_N)} \left[ \bar{M}_n^{-1}(0, x) - \bar{M}_n^{-1}(0, s_n^1) \right]^+ f(q_n - x; p_N) \, dx.
\]

(A.4)

We note that for all \( q_n \in [s_n^1, s_n^1] \),

\[
0 \leq \int_0^{q_n - L(p_N)} \left[ \bar{M}_n^{-1}(0, x) - \bar{M}_n^{-1}(0, s_n^1) \right]^+ f(q_n - x; p_N) \, dx \leq \mathcal{K} F(q_n; p_N) \leq \mathcal{K},
\]

for all \( p_N \in \left[ P_r, P_u \right] \). Thus, it follows from Eq. (A.4) that

\[
\bar{M}_s^*(0, q_n) \geq G^*(0, q_n) + z\bar{M}_n^{-1}(0, s_n^1), \quad \forall q_n \in [s_n^1, s_n^1],
\]

which proves (c).

For the proof of (b), we need to extend the \( q_n \) range beyond \( s_n^1 \); hence, we cannot use Eq. (A.2) as a starting point. We rather start at Eq. (3).

Since, under the inductive assumption, \( \bar{H}_n^{-1}(x) \leq cx + \bar{M}_n^{-1}(0, S_n - 1) \) for all \( x \in [0, \infty) \), the integral in Eq. (3) satisfies

\[
\int_0^{q_n - L(p_N)} \bar{H}_n^{-1}(x)f(q_n - x; p_N) \, dx \leq c\Theta(p_n, q_n) + \bar{M}_n^{-1}(0, S_n - 1)F(q_n; p_N).
\]

(A.5)

Therefore, it follows from Eq. (3) that

\[
\bar{M}_s^*(0, q_n) \leq \max\{G(0, p_n, q_n) + z\bar{M}_n^{-1}(0, s_n^1)[1 - F(q_n; p_n)] \}
\]

\[
+ z\bar{M}_n^{-1}(0, s_n^1) + \mathcal{K} F(q_n; p_N); \, p_N \in \left[ P_r, P_u \right]) \}
\]

\[
\leq G^*(0, q_n) + z\bar{M}_n^{-1}(0, s_n^1) + z\mathcal{K} \max\{F(q_n; p_N); \, p_N \in \left[ P_r, P_u \right]\},
\]

which implies (b).
Under properties (b) and (c) we have
\[ G^*(0, q_n) + zM_{n-1}^*(0, s_{n-1}^1) \leq M_n^*(0, q_n) \leq G^*(0, q_n) + zM_{n-1}^*(0, s_{n-1}^1) + z\mathcal{K}, \] (A.6)
which indicates that \( M_n^*(0, q_n) \) is bounded by two unimodal functions that are at most \( z\mathcal{K} \) distance (vertically) apart on \( q_n \in [s_{n-1}^1, s_{n-1}^1] \).

Next, we consider the region \([S_p, s_{n-1}^1]\). Let \( q'_n \in (q_n, \infty) \) be an arbitrary level for any given \( q_n \in [S_p, s_{n-1}^1] \). It follows from Eq. (A.5) that
\[
\int_0^{q_n - L(p)} \tilde{H}_{n-1}^*(x)f(q'_n - x; p_n) \, dx - c\Theta(p_n, q'_n) - M_{n-1}^*(0, s_{n-1}^1)F(q'_n, p_n) \leq \mathcal{K}F(q'_n, p_n) \leq \mathcal{K},
\] (A.7)
for all \( p_n \in [P_r, P_u] \). Therefore, by defining \( p' \) with \( M_n^*(0, q'_n) = M_n(0, p', q'_n) \), considering Eqs. (A.4) and (A.7), and recalling that \( G^* \) is non-increasing on \([S_p, \infty)\) we can proceed to prove (d) as follows:
\[
M_n^*(0, q_n) \geq G^*(0, q_n) + zM_{n-1}^*(0, s_{n-1}^1) + \int_0^{q_n - L(p)} \tilde{H}_{n-1}^*(x)f(q'_n - x; p_n) \, dx
\]
\[
\geq G^*(0, q'_n) + zM_{n-1}^*(0, s_{n-1}^1)
\]
\[
\geq G(0, p', q'_n) + zM_{n-1}^*(0, s_{n-1}^1)
\]
\[
\geq G(0, p', q'_n) + zM_{n-1}^*(0, s_{n-1}^1) + \int_0^{q_n - L(p')} \tilde{H}_{n-1}^*(x)f(q'_n - x; p') \, dx - c\Theta(p', q'_n)
\]
\[
= M_{n-1}^*(0, s_{n-1}^1)(F(q'_n, p') - \mathcal{K})
\]
which implies that \( M_n^*(0, q_n) \) is \( z\mathcal{K} \)-decreasing over \([S_p, s_{n-1}^1]\), and \( M_n^*(0, s_{n-1}^1) \geq M_n^*(0, q'_n) - z\mathcal{K} \) for all \( q'_n \in (s_{n-1}^1, \infty) \).

So far, we have shown that properties (a)–(d) hold. For properties (e) and (f), we establish \( M_n^*(0, q_n) \) both at \( q_n = 0 \) and as \( q_n \) tends to infinity. It follows from Eq. (A.3) that
\[ M_n^*(0, 0) = G^*(0, 0) + zM_{n-1}^*(0, s_{n-1}^1) = -r\overline{X}(P_o) + zM_{n-1}^*(0, s_{n-1}^1). \]

Thus, it is clear that \( M_n^*(0, q_n) \) has a finite support at \( q_n = 0 \).

On the other hand, it has been shown in [10] that \( \lim_{q_n \to \infty} M_n^*(0, q_1) = -\infty \). To establish a similar result for all \( n \), we let \( \lim_{q_n \to \infty} M_{n-1}^*(0, q_{n-1}) = -\infty \), then we obtain
\[
\lim_{q_n \to \infty} M_n^*(0, q_n) = \max_{q_n \to \infty} \left\{ G(0, p_m, q_n) + zM_{n-1}^*(0, s_{n-1}^1)[1 - F(q_n, p_m)] + \int_0^{q_n - L(p)} \tilde{H}_{n-1}^*(x)f(q_n - x; p_n) \, dx : p_n \in [P_r, P_u] \right\} = -\infty.
\]
Thus, \( \lim_{q_n \to \infty} M_n(0, q_n) = -\infty \) for all \( n \). This completes the proof of Theorem 1. \( \square \)

**Proof of Corollary 1.** Evaluating Eq. (A.6) at \( q_n = S_p \) we obtain
\[ G^*(0, S_p) + zM_{n-1}^*(0, s_{n-1}^1) \leq M_n^*(0, S_p) \leq M_n^*(0, S_p). \]
Similarly, for $q_n = S_m$, we have
\[
\tilde{M}_n^*(0, S_m) \leq G^*(0, S_m) + \alpha \tilde{M}_{n-1}^*(0, s_{n-1}^1) + \alpha \mathcal{K} \leq G^*(0, S_q) + \alpha \tilde{M}_{n-1}^*(0, s_{n-1}^1) + \alpha \mathcal{K}.
\]
Combining the above inequalities we get
\[
G^*(0, S_q) + \alpha \tilde{M}_{n-1}^*(0, s_{n-1}^1) \leq \tilde{M}_n^*(0, S_m) \leq G^*(0, S_q) + \alpha \tilde{M}_{n-1}^*(0, s_{n-1}^1) + \alpha \mathcal{K}.
\]
(A.8)

Thus, considering the two boundary functions in Eq. (A.6) which can be represented by a pair of unimodal functions $\alpha \mathcal{K}$ distance apart, and by indicating the critical inventory levels $s_q^m < s_q < s_q' < s_q < s_q^u < s_q'$ as in Fig. 1, it can be seen that $s_q^m < s_q^1 < s_q^1 < s_q' < s_q^u < s_q'$, and $s_q' < s_q < s_q^u$ for $n > 1$. Moreover, since $G^*$ is an increasing function over $[0, S_q]$, it follows from the proof of properties (a)–(c) in Theorem 1 that if $s_q < s_q^1$, then $s_q < s_q^1$ for all $n > 1$.

**Proof of Corollary 2.** To initiate the inductive proof, we assume that $s_q^1 \leq S_1$ and $\tilde{M}_n^*(0, q_n)$ is increasing over $[0, s_q^1]$. Since under property (a) $\tilde{M}_n^*(0, q_n)$ is increasing over $[0, s_n^1]$, it follows from Eq. (A.2) that
\[
\tilde{M}_n^*(0, q_n) = \max \left\{ G(0, p_n, q_n) + \alpha \tilde{M}_{n-1}^*(0, s_{n-1}^1) \right\}
\]
\[
+ \alpha \int_{L(p_n)} [\tilde{M}_{n-1}^*(0, q_n - x) - \tilde{M}_{n-1}^*(0, s_{n-1}^1)] f(x; p_n) \, dx.
\]
Note that $\tilde{M}_{n-1}^*(0, q_n - x)$ is increasing in $q_n$ over the range of interest and $[\tilde{M}_{n-1}^*(0, q_n - x) - \tilde{M}_{n-1}^*(0, s_{n-1}^1)]$ is positive over the $x$ range of $[q_n - s_{n-1}^1, q_n^1 - s_{n-1}^1]$. Thus, we can proceed as follows:
\[
\tilde{M}_n^*(0, q_n) = G(0, p_n(q_n), q_n) + \alpha \tilde{M}_{n-1}^*(0, s_{n-1}^1)
\]
\[
+ \alpha \int_{L(p_n)} [\tilde{M}_{n-1}^*(0, q_n - x) - \tilde{M}_{n-1}^*(0, s_{n-1}^1)] f(x; p_n(q_n)) \, dx
\]
\[
\leq G(0, p_n(q_n), q_n) + \alpha \tilde{M}_{n-1}^*(0, s_{n-1}^1)
\]
\[
+ \alpha \int_{L(p_n)} [\tilde{M}_{n-1}^*(0, q_n^1 - x) - \tilde{M}_{n-1}^*(0, s_{n-1}^1)] f(x; p_n(q_n)) \, dx
\]
\[
\leq G(0, p_n(q_n), q_n) + \alpha \tilde{M}_{n-1}^*(0, s_{n-1}^1)
\]
\[
+ \alpha \int_{L(p_n(q_n))} [\tilde{M}_{n-1}^*(0, q_n^1 - x) - \tilde{M}_{n-1}^*(0, s_{n-1}^1)] f(x; p_n(q_n)) \, dx
\]
\[
= G(0, p_n(q_n), q_n^1) + \alpha \tilde{M}_{n-1}^*(0, s_{n-1}^1) + G(0, p_n(q_n), q_n) - G(0, p_n(q_n), q_n)
\]
\[
+ \alpha \int_{L(p_n(q_n))} [\tilde{M}_{n-1}^*(0, q_n^1 - x) - \tilde{M}_{n-1}^*(0, s_{n-1}^1)] f(x; p_n(q_n)) \, dx
\]
\[
\leq \tilde{M}_n^*(0, q_n) + G(0, p_n(q_n), q_n) - G(0, p_n(q_n), q_n).
\]
Hence, if $G(0, p_n(q_n), q_n) \leq G(0, p_n(q_n), q_n)$, then $\tilde{M}_n^*(0, q_n) \leq \tilde{M}_n^*(0, q_n)$ which implies that $\tilde{M}_n^*(0, q_n)$ is non-decreasing over $[s_{n-1}^1, s_q^1]$, where $s_q^1 \leq S_q$. 

It follows from the definition of \( G \) (see Section 4.1) that \( G(0, p_n(q_n), q_n') - G(0, p_n(q_n), q_n) = (q_n' - q_n)(p_n(q_n) + r - c) - (p_n(q_n) + r + h - xc)[\Theta(p_n(q_n), q_n') - \Theta(p_n(q_n), q_n)] \). Thus, we have

\[
G(0, p_n(q_n), q_n') \geq G(0, p_n(q_n), q_n) \iff \frac{\Theta(p_n(q_n), q_n') - \Theta(p_n(q_n), q_n)}{q_n' - q_n} \leq \frac{p_n(q_n) + r - c}{p_n(q_n) + r + h - xc} \tag{A.9}
\]

provided that \( p_n(q_n) + r + h - xc > 0 \). In fact, it can be shown through standard derivative analysis that \( p_n(q_n) \) satisfies \( \hat{G}(0, p, q_n) \hat{p}|_{p_n(q_n)} < 0 \), which implies that \( p_n(q_n) + h - xc > 0 \), and this in turn yields \( p_n(q_n) + r + h - xc > 0 \).

Eq. (A.9) can be verified exactly only if \( p_n(q_n) \) is known for all \( q_n \in [s_{n-1}^1, s_n^1] \) which, in general, analytically intractible. For this reason, we need to strengthen Eq. (A.9) in order to establish a verifiable sufficient condition.

Since \( \Theta(p_n, q_n) \) is a continuous, convex increasing function in \( q_n \) for all \( p_n \) and \( \hat{\Theta}(p_n, q_n) \hat{q}_n = F(q_n; p_n) \), the result in the Corollary is implied by Eq. (9) and Corollary 1.

**Proof of Corollary 3.** To initiate the inductive proof, we assume that \( s_1 \leq S_p \), which ensures that \( M_n^*(0, q_1) \) is \( \mathcal{K} \)-decreasing over \( [S_p, \infty) \). Then, we let \( S_q < q < q' \) where it was determined earlier that \( s_{n-1}^1 < S_q < s_n^1 \). Furthermore, we assume that \( M_n^{*}_{n-1}(0, q_{n-1}) \) is \( \mathcal{K} \)-decreasing over \( [S_p, \infty) \). Then, it follows from Eqs. (3) and (5) that

\[
M_n^*(0, q_n) = \max \left\{ G(0, p_n, q_n) + \alpha M_{n-1}^*(0, s_{n-1}^1) \right. \\
+ \int_{S_q}^{s_{n-1}^1} [M_{n-1}^*(0, q_n - x) - \mathcal{K} - \bar{M}_{n-1}^*(0, s_{n-1}^1)] f(x; p_n) \, dx \\
+ \left. \int_{S_q}^{s_{n-1}^1} [M_{n-1}^*(0, q_n - x) - \mathcal{K} - \bar{M}_{n-1}^*(0, s_{n-1}^1)]^+ f(x; p_n) \, dx : p_n \in [P_c, P_u] \right\}.
\]

Having this representation, we can proceed as follows:

\[
M_n^*(0, q_n) \geq \max \left\{ G(0, p_n, q_n') + \alpha M_{n-1}^*(0, s_{n-1}^1) \right. \\
+ \int_{S_q}^{s_{n-1}^1} [M_{n-1}^*(0, q_n' - x) - \mathcal{K} - \bar{M}_{n-1}^*(0, s_{n-1}^1)] f(x; p_n) \, dx \\
+ \left. \int_{S_q}^{s_{n-1}^1} [M_{n-1}^*(0, q_n' - x) - \mathcal{K} - \bar{M}_{n-1}^*(0, s_{n-1}^1)]^+ f(x; p_n) \, dx : p_n \in [P_c, P_u] \right\} \\
= \max \{ M_n(0, p_n, q_n') + G(0, p_n, q_n) - G(0, p_n, q_n) \\
+ \alpha \mathcal{K} [1 - F(q_n' - s_{n-1}^1; p_n) : p_n \in [P_c, P_u]] - \alpha \mathcal{K} \\
\geq M_n^*(0, q_n) + G(0, p_n(q_n'), q_n) - G(0, p_n(q_n'), q_n) - \alpha \mathcal{K}.
\]
which implies that
\[ M^*_n(0, q_n) \geq M^*_n(0, q'_n) - \mathcal{K}, \tag{A.10} \]
if \( G(0, p_n(q'_n), q_n) - G(0, p_n(q'_n), q'_n) \geq 0 \). Note that Eq. (A.10) characterizes \( M^*_n \) as \( \mathcal{K} \)-decreasing (which implies that it is \( \mathcal{K} \)-decreasing) over \([S_p, \infty] \), and this leads to the desired result.

**Derivation of** \( M^*_n(0, q) \). The solution of the renewal equation Eq. (14) is ([25]):
\[
M^*_n(0, q) = G(0, p_q, q) + \int_{q-L(p)}^{\infty} \left[ 1 - F(q - s^1; p_q) \right] M^*(0, s^1) dR_s(q - x; p_q) + \int_{q-L(p)}^{\infty} \left[ 1 - F(x - s^1; p_q) \right] dR_s(q - x; p_q), \tag{A.11}
\]
where \( R_s \) is determined by
\[ R_s(x; p) = \int_{L(p)}^{s} F(x - u; p) dR_s(u; p). \]

We evaluate \( R_s(x; p) \) at \( x = q - s^1 \) and \( p = p_q \),
\[ R_s(q - s^1; p_q) = \int_{q-L(p)}^{\infty} \left[ 1 - F(q - s^1; p_q) \right] dR_s(q - x; p_q), \]
which can be substituted in Eq. (A.11). Thus, we obtain
\[
M^*_n(0, q) = G(0, p_q, q) + \int_{q-L(p)}^{\infty} G(0, p_q, x) dR_s(q - x; p_q)
+ \int_{q-L(p)}^{\infty} \left[ 1 - F(q - s^1; p_q) + R_s(q - s^1; p_q) + F(q - s^1; p_q) - R_s(q - s^1; p_q)/z \right] dR_s(q - x; p_q)
+ \int_{q-L(p)}^{\infty} G(0, p_q, x) dR_s(q - x; p_q) + G^*(0, s^1) \left[ \frac{z}{1 - z} - R_s(q - s^1; p_q) \right].
\]

This implies the desired result.

**Proof of Corollary 4.** Before we present the induction proof, we will establish the following property:

**Property 1.** \( \forall a, b \) with \( a > b \), we have \( P_a \leq P_b \) where \( P_a = \arg\sup\{ (p + a)X(p); p \in (0, \infty) \} \) and \( P_b = \arg\sup\{ (p + b)X(p); p \in (0, \infty) \} \).

**Proof of Property 1.** Let \( A(p) = (p + a)X(p) \) and \( B(p) = (p + b)X(p) \) for \( a > b \). It is shown in [19] that if \( R(p) \) is pseudoconcave on \((0, \infty)\), which is our basic assumption, then \( A(p) \) and \( B(p) \) are pseudoconcave functions of \( p \) as well. In Fig. 8, we demonstrate \( A(p) \) and \( B(p) \) under an hypothetical \( X(p) \) function.
If \( P_a, P_b \in (0, \infty) \), then they must satisfy the first-order conditions \( A'(P_a) = 0 \) and \( B'(P_b) = 0 \). We rewrite \( A(p) = A(p) = B(p) + (a - b)X(p) \), which leads to \( A'(p) = B'(p) + (a - b)X(p) \). Evaluating the last equation at \( p = P_a \) we get \( A'(P_a) = B'(P_a) + (a - b)X'(P_a) \), which implies
\[
B'(P_a) = (p - b)X'(P_a) > 0.
\] (A.12)

That is, \( B(p) \) is increasing at \( p = P_a \). Since \( B(p) \) is a pseudoconcave function we deduce that \( P_a \leq P_b \). If \( P_a \) and \( P_b \) are both non-interior point solutions, then \( P_a = P_b = 0 \) since \( A(p) \geq B(p) \) for all \( p \). If \( P_a = 0 \) and \( P_b \in (0, \infty) \), then \( P_a \leq P_b \). Finally, if \( P_a \in (0, \infty) \), then from Eq. (A.12) we conclude that \( P_a \leq P_b \). Hence, \( P_a \leq P_b \) in any case.

We identify \( P_c^{*} \) as the maximizer of \((p_n - c)X_{n}(p_{n})\) over \([P_c, P_u]\). That is, \( P_c^{*} = \min \{ \max \{ P_c^{n}, P_c \}, P_u \} \) where \( P_c^{n} \) is the maximizer of \((p_n - c)X_{n}(p_{n})\) over \((0, \infty)\). Similarly, \( P_h^{n}, P_h^{c} \) and \( P_r^{c} \) are the maximizers of \((p_n + h - v)X_{n}(p_{n}), (p_n + h - \alpha)X_{n}(p_{n}) \) and \((p_n - r - \alpha)X_{n}(p_{n})\) over \([P_c, P_u]\), respectively. Since \(-c \leq h - v, -c \leq h - \alpha \) and \(-r - \alpha \leq -c \), it follows from Property 1 that \( P_c^{*} \geq P_h^{n}, P_c^{*} \geq P_h^{c} \) and \( P_r^{c} \geq P_r^{c} \). That is, \( X_{n}(P_c^{*}), X_{n}(P_h^{n}), X_{n}(P_h^{c}) \leq X_{n}(P_h^{c}) \) and \( X_{n}(P_r^{c}) \leq X_{n}(P_r^{c}) \).

Now we can establish the proof of Corollary 4. It is shown in [19] that \( G_n^{*}(0, q_n) \) is quasiconcave with \( S_n = X_{n}(P_c^{*}) \). Having this in mind, assume that \( \tilde{M}_{n-1}^{*}(0, q_{n-1}) \) satisfies (i) and (ii) in Theorem 2. Thus, there exists single reorder and order-up-to levels for period \( n - 1 \). That is, \( k_{n-1} = 2 \). Using this result, we obtain
\[
\tilde{M}_n(i_{n}, p_{n}, q_{n}) = G_n(i_{n}, p_{n}, q_{n}) + x_{\tilde{M}_n^{*}}^{n-1}(0, s_{n-1}), [q_n - X_n(p_n)]^{+} < s_{n-1}, [q_n - X_n(p_n)]^{+}, s_{n-1} \leq [q_n - X_n(p_n)]^{+}.
\] (A.13)

First, we represent the \( n \)-period pseudo-profit function as \( \tilde{M}_n^{*}(0, q_{n}) = \max \{ \tilde{M}_n^{(1)}(0, q_{n}), \tilde{M}_n^{(2)}(0, q_{n}) \} \), where
\[
\tilde{M}_n^{(1)}(0, q_{n}) = \max \{ \tilde{M}_n^{(0)}(0, p_{n}, q_{n}), \tilde{p}_n < p_{n} < p_{u} \},
\]
\[
\tilde{M}_n^{(2)}(0, q_{n}) = \max \{ \tilde{M}_n^{(0)}(0, p_{n}, q_{n}), \tilde{p}_n < p_{n} < p_{u} \},
\]
and \( \tilde{p}_n \) is defined by \( X_n(\tilde{p}_n) = \min \{ q_n, X_n(P_c) \} \). That is, \( \tilde{p}_n = P_r \) if \( q_n > X_n(P_r) \), \( \tilde{p}_n = P_u \) if \( q_n < X_n(P_u) \); otherwise, \( \tilde{p}_n \) is the unique solution of \( X_n(\tilde{p}_n) = q_n \). Note that if \( p_{n} \in [P_c, \tilde{p}_n) \), then \( q_n < X_n(p_{n}) \) and if \( p_{n} \in (\tilde{p}_n, P_u] \), then \( q_n \geq X_n(p_{n}) \). Therefore, \( \tilde{M}_n^{(1)} \) and \( \tilde{M}_n^{(2)} \) represent the \( n \)-period pseudo-profit values.
under the best pricing policy that results in no leftovers and no shortages, respectively, at the end of period \( n \) for a given \( q_n \). In what follows, we work with these complementary subproblems to analyse \( \tilde{M}_n^* (0, q_n) \) on various \( q_n \) ranges and then combine our findings to complete the proof.

It follows from Eqs. (A.13) and (17) that, \( \tilde{M}_n^1 (0, q_n) = z \tilde{M}_{n-1}^* (0, s_{n-1}) - c q_n + \max \{ p_n q_n - r(X_n(p_n) - q_n) : P_c \leq p_n \leq \tilde{p}_n \} \), where the maximand is an increasing function of \( p_n \). Hence, we obtain

\[
\tilde{M}_n^* (0, q_n) = \begin{cases} 
    \tilde{M}_n (0, P_w, q_n), & 0 \leq q_n \leq X_n(P_w), \\
    \tilde{M}_n^{(2)} (0, q_n), & X_n(P_w) < q_n.
\end{cases}
\] 

(A.14)

Note that \( \tilde{M}_n (0, P_w, q_n) \) is a linear increasing function of \( q_n \) and if \( P_u \to \infty \), then \( X_n(P_u) \to 0 \) and we have \( \tilde{M}_n^* (0, q_n) = \tilde{M}_n^{(2)} (0, q_n) \) for \( q_n \in [0, \infty) \).

Next, consider the \( q_n \) range of \( [X_n(P_u), X_n(P_c)] \), where we have established that \( \tilde{M}_n^* (0, q_n) = \tilde{M}_n^{(2)} (0, q_n) \). We further divide \( \tilde{M}_n^* (0, q_n) = \max \{ \tilde{M}_n^{(1)} (0, q_n), \tilde{M}_n^{(2)} (0, q_n) \} \), where

\[
\tilde{M}_n^{(1)} (0, q_n) = \max \{ \tilde{M}_n (0, p_n, q_n) : \tilde{p}_n \leq p_n \leq \tilde{p}_n \} = \tilde{M}_{n-1}^* (0, s_{n-1}) + \max \{ G_n (0, p_n, q_n) : \tilde{p}_n \leq p_n \leq \tilde{p}_n \},
\]

(A.15)

\[
\tilde{M}_n^{(2)} (0, q_n) = \max \{ \tilde{M}_n (0, p_n, q_n) : \tilde{p}_n \leq p_n \leq P_u \} = \max \{ \tilde{M}_{n-1}^* (0, q_n - X_n(X_n(P_u))) + G_n (0, p_n, q_n) : \tilde{p}_n \leq p_n \leq P_u \}.
\]

(A.16)

and \( X_n(\hat{p}_n) = \max \{ \min \{ q_n - s_{n-1}, X_n(P_u) \} \} \). That is, \( \hat{p}_n = P_u \) if \( q_n \leq X_n(P_u) + s_{n-1} \) and \( \hat{p}_n = P_c \) if \( X_n(P_u) + s_{n-1} \leq q_n \), otherwise \( \hat{p}_n \) is the unique solution of \( X_n(\hat{p}_n) = q_n - s_{n-1} \). Note that \( q_n - X_n(p_n) \leq s_{n-1} \) when \( p_n < \tilde{p}_n \) and \( s_{n-1} \leq q_n - X_n(p_n) \) when \( \tilde{p}_n < p_n \).

We shall demonstrate that \( \tilde{M}_n^{(1)} \) and \( \tilde{M}_n^{(2)} \) are both increasing on \( [X_n(P_u), X_n(P_c)] \) which implies that \( \tilde{M}_n^* \) is also increasing over the same \( q_n \) range.

Considering the maximand in Eq. (A.15) and Fig. 8 with \( a = h - xc \). We observe that the maximizing price is \( \tilde{p}_n \) for \( X_n(P_u) \leq q_n \leq X_n(P_{hc}) \). Because, over this \( q_n \) range we have \( P_{hc} \leq \tilde{p}_n \) and \( (p_n + a - xc)X_n(p_n) \) is decreasing on \( [\tilde{p}_n, P_u] \). For \( X_n (P_{hc}) \leq q_n \leq X_n (P_{hc}) + s_{n-1} \), we have \( \tilde{p}_n \leq P_{hc} \leq \tilde{p}_n \). Thus, the maximizer in Eq. (A.15) is \( P_{hc} \). For \( X_n (P_{hc}) + s_{n-1} \leq q_n \leq X_n (P_c) + s_{n-1} \), we have \( \tilde{p}_n \leq P_{hc} \) and the maximizer is \( \tilde{p}_n \). Finally, for \( X_n (P_c) + s_{n-1} \leq q_n \) we have \( \tilde{p}_n = P_c \), and the function \( \tilde{M}_n^* (0, q_n) \) is not defined over the \( q_n \) range of \( n[X_n (P_c) + s_{n-1}, \infty) \).

Summarizing our findings we have

\[
\tilde{M}_n^{(1)} (0, q_n) = \tilde{M}_{n-1}^* (0, S_{n-1}) - X_n(p_n) \geq q_n \leq X_n (P_{hc}) \]

\[
= \begin{cases} 
    (\tilde{p}_n - c)q_n + (P_{hc} + h - xc)(P_{hc}) - (c + h - xc)q_n, & X_n (P_{hc}) \leq q_n \leq X_n (P_{hc}) + s_{n-1}, \\
    (\tilde{p}_n - c)q_n - (c + h - xc)S_{n-1}, & X_n (P_{hc}) + s_{n-1} \leq q_n \leq X_n (P_c) + s_{n-1}.
\end{cases}
\]

(A.17)

For \( q_n = X_n(\hat{p}_n) \), it can be shown that \( (\tilde{p}_n - c)q_n \) is a pseudoconcave function of \( q_n \) [19], where

\[
\frac{d}{dq_n} (\tilde{p}_n - c)q_n = \frac{d}{d\tilde{p}_n} ((\tilde{p}_n - c)X_n(\hat{p}_n)) \frac{d\hat{p}_n}{dq_n} 
\]

Note that \( d\hat{p}_n/dq_n < 0 \), and increasing \( q_n \) means tracing \( p_n \) backwards in Fig. 8. Thus, \( (\tilde{p}_n - c)q_n \) is increasing on \( [X_n (P_u), X_n (P_c)] \) and it is decreasing on \( [X_n (P_{hc}), X_n (P_c)] \). Likewise, \( (\tilde{p}_n - c)q_n \) is decreasing on \( [X_n (P_{hc}) + s_{n-1}, X_n (P_c) + s_{n-1}] \), since \( \tilde{p}_n \leq P_{hc} \leq P_c \) over this \( q_n \) range. It, therefore, results from Eq. (A.17) that \( \tilde{M}_n^{(1)} \) is increasing on \( [X_n (P_u), X_n (P_c)] \) and decreasing on \( [X_n (P_{hc}), X_n (P_c) + s_{n-1}] \).
Next, we shall analyse $M^{(22)}_n$. Note that for $q_n \leq X_n(P_u) + s_{n-1}$, we have $\hat{p}_n = P_u$ and $M^{(22)}_n = M^{(21)}_n$. Thus, we need to consider $M^{(22)}_n(0, q_n)$ only on $[X_n(P_u) + s_{n-1}, \infty)$. 

We identify $q_n$ arbitrarily with $X_n(P_u) + s_{n-1} \leq q_n \leq X_n(P_u)$ to show that $M^{(22)}_n(0, q_n) < M^{(22)}_n(0, q'_n)$. Let $p$ be the maximizer in Eq. (A.16) which satisfies $\hat{p}_n \leq p \leq P_u$. Then, define $\hat{p}'_n$ by $q_n - s_{n-1} = X_n(\hat{p}'_n)$, and $p'$ by $q_n - X_n(p) = q_n - X_n(p')$. Thus, we have

$$\hat{p}'_n = q_n - X_n(p') = q_n - X_n(p) \geq s_{n-1} = q_n - X_n(\hat{p}_n),$$

which implies $P_u \leq \hat{p}'_n \leq p' < p \leq P_u$. Considering the order of these critical prices and observing Fig. 8 we write

$$M^{(22)}_n(0, q_n) = zM^{* -1}_n(0, q_n - X_n(p)) - (c + h - \alpha c)q_n + (p + h - \alpha c)X_n(p)$$

$$= zM^{* -1}_n(0, q_n - X_n(p)) - (c + h - \alpha c)[q_n - X_n(p)] + (p - c)X_n(p)$$

$$= zM^{* -1}_n(0, q'_n - X_n(p')) - (c + h - \alpha c)[q'_n - X_n(p') + (p - c)X_n(p)$$

$$< zM^{* -1}_n(0, q_n - X_n(p)) - (c + h - \alpha c)[q_n - X_n(p')] + (p' - c)X_n(p')$$

$$\leq \max\{zM^{* -1}_n(0, q_n - X_n(p)) - (c + h - \alpha c)q_n + (p + h - \alpha c)X_n(p)\} \leq \hat{p}_n \leq p_n \leq P_u$$

$$= M^{(22)}_n(0, q_n),$$

which indicates that $M^{(22)}_n$ is an increasing function of $q_n$ over $[X_n(P_u) + s_{n-1}, X_n(P_u)]$.

We have so far shown that $M^{(21)}_n$ and $M^{(22)}_n$ are increasing in $q_n$ over $[X_n(P_u), X_n(P_u)]$; thus, $M^{(21)}_n ( = M^{*}_n)$ is also increasing in $q_n$ over the same range.

The next $q_n$ region will be $[X_n(P_u), X_n(P_u) + s_{n-1}]$. We have $M^{* -1}_n(0, q_n - X_n(p_n)) \leq M^{* -1}_n(0, S_{n-1})$ for all feasible $p_n$ and $q_n$ values. In addition, since $P_{n+1} < \hat{p}_n$, $(p_n + h - \alpha c)X_n(p_n)$ is decreasing over $[\hat{p}_n, P_u]$. Hence, we have

$$M^{(22)}_n(0, q_n) \leq \max\{zM^{* -1}_n(0, S_{n-1}) - (c + h - \alpha c)q_n + (p_n + h - \alpha c)X_n(p_n)\} \leq \hat{p}_n \leq p_n \leq P_u$$

$$\leq \max\{zM^{* -1}_n(0, S_{n-1}) - (c + h - \alpha c)q_n + (p_n + h - \alpha c)X_n(p_n)\} \leq \hat{p}_n \leq P_u$$

$$= M^{(21)}_n(0, q_n) + \alpha \mathcal{K}_{n-1},$$

which implies that $M^{(22)}_n(0, q_n)$, given by Eq. (A.6), is bounded by $M^{(21)}_n(0, q_n)$ and $M^{(21)}_n(0, q_n) + \alpha \mathcal{K}_{n-1}$, where $M^{(21)}_n$ is decreasing (see Eq. (A.17)). Hence, $\mathcal{M}^{(2)}(0, q_n)$ is an $\alpha \mathcal{K}_{n-1}$-decreasing function over $[X_n(P_u), X_n(P_u) + s_{n-1}]$.

Consider now $X_n(P_u) + s_{n-1} \leq q_n$. In this range, $\hat{p}'_n \leq \hat{p}_n \leq P_{n+1}$. Hence, $(p_n + h - \alpha c)X_n(p_n)$ is increasing over $[\hat{p}_n, \hat{p}'_n]$. In addition, under the inductive assumption, $M^{* -1}_n(0, q_n - X_n(p_n))$ is $\mathcal{K}_{n-1}$-decreasing in $q_n$ for $X_n(P_u) + s_{n-1} \leq q_n$ with $\hat{p}_n \leq P_u$. Hence, we have

$$M^{(22)}_n(0, q_n) = \max\{zM^{* -1}_n(0, q_n - X_n(p_n)) - (c + h - \alpha c)q_n + (p_n + h - \alpha c)X_n(p_n)\} \leq \hat{p}_n \leq p_n \leq P_u$$

$$= \max\{zM^{* -1}_n(0, q_n - X_n(p_n)) - (c + h - \alpha c)q_n + (p_n + h - \alpha c)X_n(p_n)\} \leq \hat{p}_n \leq p_n \leq P_u$$

$$\geq \max\{zM^{* -1}_n(0, q_n - X_n(p_n)) - (c + h - \alpha c)q_n + (p_n + h - \alpha c)X_n(p_n)\} \leq p_n \leq P_u$$

$$\geq \max\{zM^{* -1}_n(0, q_n - X_n(p_n)) - (c + h - \alpha c)q_n + (p_n + h - \alpha c)X_n(p_n)\} \leq \hat{p}_n \leq p_n \leq P_u$$

$$\geq \max\{zM^{* -1}_n(0, q_n - X_n(p_n)) - (c + h - \alpha c)q_n + (p_n + h - \alpha c)X_n(p_n)\} \leq p_n \leq P_u$$

$$= M^{(22)}_n(0, q_n) - \alpha \mathcal{K}_{n-1},$$
which implies that $M_n^{(2)}$ is $\propto K_{n-1}$-decreasing. Since $M_n^{(21)}(0, q_n)$ is decreasing over $[X(q_n P_{hc}^n) + s_{n-1}, X_q(P_h) + s_{n-1}]$ (see Eq. (A.17)), $M_n^{(2)}(0, q_n)$ is $\propto K_{n-1}$-decreasing over $[X_q(P_h) + s_{n-1}, \infty)$.

Finally, to complete the continuity we need to consider $q_n = X_q(P_h) + s_{n-1}$. Since $M_n^{(21)}(0, S_{n-1})$ and $(P_h^n + h - xc)X_n(P_h^n)$ are the global maxima of their respective functions, and $M_n^{(21)}$ is decreasing over $[X_q(P_h^n), X_q(P_h^n) + s_{n-1}]$, we obtain from Eq. (A.17) that

$$M_n^{(2)}(0, q_n) \geq M_n^{(21)}(0, q_n) = \alpha(M_n^{*}(0, S_{n-1}) - K_{n-1}) - (c + h - xc)q_n + (P_h^n + h - xc)X_n(P_h^n)$$

for all $q_n > q_n = X_q(P_h^n) + s_{n-1}$.

Therefore, the proof follows by combining the results for the entire $q_n$ range.

References

[16] N.C. Petrozzi, Learning models for pricing and inventory control under uncertainty, Ph.D. Dissertation, Department of Management, Purdue University, West Lafayette, IN, 1995.