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Optimal ordering, discounting, and pricing in the single-period problem

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Abstract

The single-period problem (SPP), also known as the newsboy or newsvendor problem, is to find the order quantity which maximizes the expected profit in a single-period probabilistic demand framework. Previous extensions to the SPP include, in separate models, the simultaneous determination of the optimal price and quantity when demand is price-dependent, and the determination of the optimal order quantity when progressive discounts with preset prices are used to sell excess inventory. In this paper, we extend the SPP to the case in which demand is price-dependent and multiple discounts with prices under the control of the newsvendor are used to sell excess inventory. First, we develop two algorithms for determining the optimal number of discounts under fixed discounting cost for a given order quantity and realization of demand. Then, we identify the optimal order quantity before any demand is realized. We also analyze the joint determination of the order quantity and initial price. We illustrate the models and provide some insights using numerical examples. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Inventory management; Production management; Operations management

1. Introduction

The classical single-period problem (SPP) is to find a product's order quantity which maximizes the expected profit in a probabilistic demand framework. The SPP model assumes that if any inventory remains at the end of the period, one discount is used to sell it or it is disposed of [1]. If the order quantity is smaller than the realized demand, the newsvendor, hereafter NV, forgoes some profit. If the order quantity is larger than the realized demand, the NV losses some money because he/she has to discount the remaining inventory to a price below cost. The SPP is reflective of many real-life situations and is often used to aid decision making in the fashion and sporting industries, both at the manufacturing and retail levels [2]. The SPP can also be used in managing capacity and evaluating advanced booking of orders in service industries such as airlines, hotels, etc. [3].

Several researchers have suggested SPP extensions in which demand is price dependent [4–10]. Whitin [4] assumed that the expected demand is a function of price and using incremental analysis, derived the necessary optimality condition. He then

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provided closed-form expressions for the optimal price, which is used to find the optimal order quantity for a demand with a rectangular distribution. Mills [5] also assumed demand to be a random variable with an expected value that is decreasing in price and with constant variance. Mills derived the necessary optimality conditions and provided further analysis for the case of demand with rectangular distribution.

Lau and Lau (LL) [6] introduced a model in which the NV has the option of decreasing price in order to increase demand. LL analyzed two cases for demand:

(a) *Case A*: Demand is given by a simple homoscedastic regression model $x = a - bP + \varepsilon$, where a and b are constants, x is the quantity demanded, P is unit price, and ε is normally distributed. The above equation implies a normally distributed demand with an expected value which decreases linearly with unit price.

(b) *Case B*: Demand distribution is constructed using a combination of statistical data analysis and experts' subjective estimates. The 'method of moments' was used to fit the four-parameter beta distribution to estimate demand.

For case A, LL showed that the expected profit is unimodal and thus the golden section method can be used for maximization. For case B, there is no guarantee the expected profit is unimodal. Thus, LL developed a search procedure for identifying local maximums. LL also solved the problem under the objective of maximizing the probability of achieving a target profit and considered both zero and positive shortage cost cases. For zero shortage cost and demand given by case A, LL derived closed-form solutions for the optimal order quantity and optimal price. For zero shortage cost and demand given by case B, LL developed a procedure for computing the probability of achieving a target profit and used a search procedure for finding a good solution. For positive shortage cost and demand given by cases A or B, the probability of achieving a target profit may not be unimodal. LL developed procedures for computing the probability of achieving a target profit and identifying a good solution.

Polatoglu [7] also considered the simultaneous pricing and procurement decisions. Polatoglu identified few special cases of the demand process addressed in the literature: (i) an additive model in which the demand at price P is $x(P) = \mu(P) + \varepsilon$, where $\mu(P)$ is the mean demand as a function of price, and ε is a random variable with a known distribution and $E[\varepsilon] = 0$, (ii) a multiplicative model in which $x(P) = \mu(P)\varepsilon$ where $E[\varepsilon] = 1$, (iii) a riskless model in which $X(P) = \mu(P)$. Polatoglu analyzed the SPP under general demand uncertainty to reveal the fundamental properties of the model independent of the demand pattern. Polatoglu assumed an initial inventory of I, $\mu(P)$ is a monotone decreasing function of P on $(0, \infty)$, and a fixed ordering cost of k. For linear expected demand, ($\mu(P) = a - bP$, where $a, b > 0$) Polatoglu proved the unimodality of the expected profit for uniformly distributed additive demand and exponentially distributed multiplicative demand.

Khouja [8] solved an SPP in which multiple discounts are used to sell excess inventory. In this model, retailers progressively increase the discount until all excess inventory is sold. The product is initially offered at the regular price P_0. After some time, if any inventory remains the price is reduced to $P_1, P_0 > P_1$. In general, the prices are P_i, $i = 0, 1, \ldots, n$, where $P_i > P_{i+1}$. The amount demanded at each P_i is assumed to be a multiple t_i, $i = 1, \ldots, n$ of the demand at the regular price P_0. Khouja solved the problem under two objectives: (a) maximizing the expected profit, and (b) maximizing the probability of achieving a target profit. Khouja showed that the expected profit is concave and derived the sufficient optimality condition for the order quantity. For maximizing the probability of achieving a target profit, Khouja provided closed-form expression for the optimal order quantity. Khouja [10] developed an algorithm for identifying the optimal order quantity for the multi-discount SPP when the supplier offers the NV an all-units quantity discount. Khouja and Mehrez [9] provided a solution algorithm to the multi-product multi-discount constrained SPP.

The above models may not capture some actual problems facing many NVs. While NV may consider the demand–price relationship is determining the order quantity, he/she still faces the problem of what to do with excess inventory when the order quantity exceeds the realized demand. Most retailers, for example, do not use a single discount to sell excess inventory as assumed in the classical SPP. The assumption of multiple discounts proposed by Khouja [8] contributes a step toward solving this problem. However, Khouja's model is limited in that it assumes that the discount prices are preset and are not part of the decisions of the NV. The model also assumes that the quantity sold at each discount price is a given multiple of the quantity sold at the initial price without any assumptions about demand–price relationship. Finally, the model assumes that discounting a product does not incur a fixed cost, whereas many retailers incur a fixed discounting cost resulting from the need to advertise the discount and markdown the discounted items. In this paper, we extend the SPP to the case in which:

1. demand is price dependent,
2. multiple discount prices are used to sell excess inventory,
3. the discount prices used to sell excess inventory are under the control of the NV, and
4. there is a positive setup cost associated with discounting a product due to the costs of advertising and marking down the discounted items.

The resulting problem is composed of two smaller problems. In the first problem, for a given realization of demand, a given demand–price relationship, and a given order quantity, the NV must determine the optimal discounting scheme. This problem will be referred to as the discounting problem. In the second problem, the NV must determine the order quantity which maximizes the expected profit prior to any demand being realized. This problem will be referred to as the order quantity problem.

In the next section, we review the classical SSP. In Section 3, we analyze the discounting problem and develop two algorithms for determining the optimal discounting scheme under two different assumptions about the behavior of the NV. In Section 4, we solve the optimal order quantity problem. In Section 5, we analyze the joint quantity and initial pricing decisions. We conclude in Section 6.

2. Basic results and problem motivation

Define the following notation:

x = quantity demanded, a random variable,
$f(x)$ = the probability density function of x,
$F(x)$ = the cumulative distribution function of x,
P = per unit selling price,
C = per unit cost,
V = per unit salvage value,
S = per unit shortage penalty cost,
C_o = $C - V$, per unit overage cost,
C_u = $P - C + S$, per unit underage cost, and
Q = order quantity, a decision variable.

The profit per period is

$$\pi = \begin{cases} (P-C)Q - S(x-Q) & \text{if } x \geqslant Q, \\ Px + V(Q-x) - CQ & \text{if } x < Q. \end{cases} \tag{1}$$

Simplifying and taking the expected value of π gives the following expected profit:

$$E(\pi) = (P+S-C)\int_Q^\infty Qf(x)\,\mathrm{d}x - S\int_Q^\infty xf(x)\,\mathrm{d}x + (P-V)\int_0^Q xf(x)\,\mathrm{d}x - (C-V)\int_0^Q Qf(x)\,\mathrm{d}x. \tag{2}$$

Let the superscript * denote optimality. Using Leibniz's rule to obtain the first and second derivatives of $E(\pi)$ shows that it is concave, and thus, the sufficient optimality condition is to set the first derivative to zero which yields the well-known fractile formula:

$$F(Q^*) = \frac{P+S-C}{P+S-V}. \tag{3}$$

Suppose the NV uses multiple discounts to sell excess inventory. In this case, the SPP decomposes into two problems: the discounting problem and the order quantity problem. The sequence of events in the SPP is as follows: (1) The NV sets the initial price at which to offer the product (P_0) based on the competitive situation of the firm, (2) The NV determines the order quantity (Q), (3) The NV finds out the quantity demanded at P_0, and (4) The NV determines the discounting scheme to use. In order to determine the optimal order quantity, the discounting problem must be solved first. In analyzing the discounting problem, we assume:

1. The relationship between demand and price is linear and is given by

   $$P = W - bx, \tag{4}$$

   where b is a positive constant known to the NV from historical data and W is a random variable with a known distribution but whose actual realization becomes known only after ordering. The demand function in Eq. (4) is the classical function used in economics. This assumption implies that the NV knows how demand changes with price (i.e. demand elasticity), but does not know the initial level of demand (x_0) when the product is offered at an initial price of P_0 (he/she knows the expected value $E(x_0) = \mu_0$). After the NV orders Q units and the selling period begins, x_0, and thus W, become known. Fig. 1 shows the expected demand as a function of price.
2. The discounts are equally spaced in terms of price on the domain [0, P_0]. While this assumption restricts the options available to the NV, it significantly simplifies the analysis and allows us to focus on the effects of progressive discounting on the SPP. Let

h = $1, \dots, n$, the number of prices to use to sell the product (excluding 0), a decision variable,
P_i = the ith selling price, and
F = the fixed cost of discounting which includes the cost of advertising and marking down the discounted items.

The prices charged under different values of h are as shown in Table 1. Note that for any h, the order quantity may not be large enough to offer the product at lower prices. Let x_0 be the quantity

Table 1
Prices for different discounting schemes on [0, P_0]

Number of prices, h	Prices used, P_i
1	P_0, 0
2	P_0, $P_0/2$, 0
3	P_0, $2P_0/3$, $P_0/3$, 0
4	P_0, $3P_0/4$, $P_0/2$, $P_0/4$, 0
....	....
h	P_0, $(h-1)P_0/h$, $(h-2)P_0/h, \dots, P_0/h$, 0
....	....
n	P_0, $(n-1)P_0/n$, $(n-2)P_0/n, \dots, P_0/n$, 0

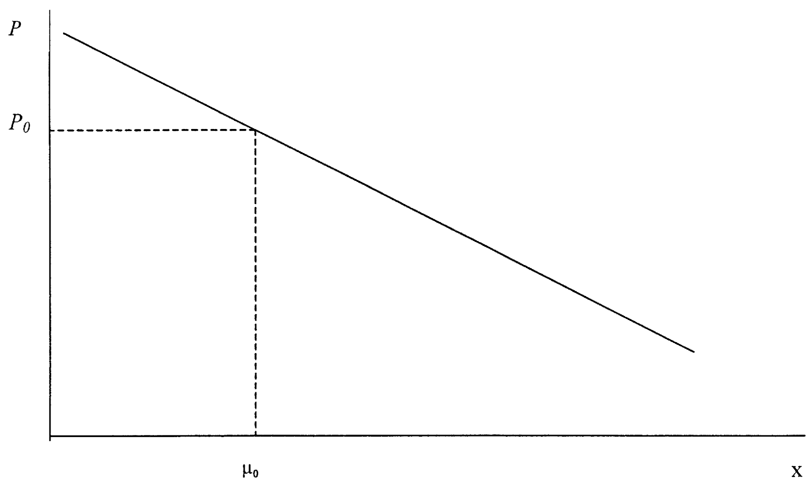

Fig. 1. Expected demand as a function of price.

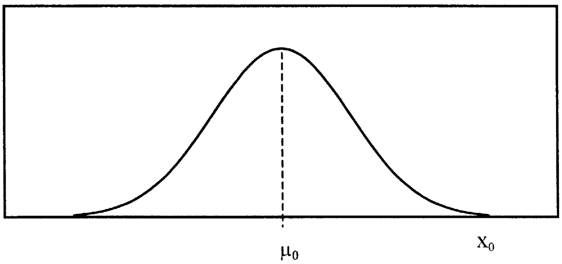

Fig. 2. Demand distribution when the item is offered at unit price P_0.

Table 2
Prices for different discounting schemes on $[P_a, P_0]$

Number of prices, h	Prices used, P_i
2	P_0, P_a
3	$P_0, (P_0 + P_a)/2, P_a$
4	$P_0, (2P_0 + P_a)/3, (P_0 + 2P_a)/3, P_a$
5	$P_0, (3P_0 + P_a)/4, (P_0 + P_a)/2, (P_0 + 3P_a)/4, P_a$
....	....
h	$P_0, ((h-1)P_0 + P_a)/h, ((h-2)P_0 + 2P_a)/h, \ldots, (P_0 + (h-1)P_a)/h, P_a$
....	....
n	$P_0, ((n-1)P_0 + P_a)/n, ((n-2)P_0 + 2P_a)/n, \ldots, (P_0 + (n-1)P_a)/n, P_a$

demanded at an initial price of P_0. From (4)

$$P_0 = W - bx_0. \tag{5}$$

Since W is a random variable with a known distribution, x_0 is a linear function of a random variable and is also a random variable with known distribution. Fig. 2 shows the distribution of x_0 for normally distributed W.

The discounting scheme in Table 1 assumes that unit prices are set at equal intervals on the range $[0, P_0]$. On the other hand, the classical SPP assumes that whatever remaining inventory can be sold at a price (i. e. salvage value) of V. Let P_a be the price at which all of Q will be sold. Then, the assumption that all remaining inventory can be sold at a price of V implies that $V \leqslant P_a$. Therefore, an alternative discounting range is $[P_a, P_0]$ with prices as shown in Table 2.

We use the discount range of $[0, P_0]$ because it allows for the option of not discounting at all which may be profitable when there is a small quantity left and the fixed cost of discounting is large. In addition, any discounting scheme shown in Table 2 for a given value of h may be approximated using larger values of h in Table 1. For example, when $P_a = V = P_0/2$, using four prices on the range $[P_a, P_0]$ may be approximated by using six prices on the range $[0, P_0]$ (of which the last 3 will not be used since all inventory is sold prior to reaching those prices).

3. The discounting problem

The NV starts by offering the product at a price of P_0, which we, for now, assume to be based on competitive considerations. The NV waits until sales start slowing down which indicates that demand at the current price is almost fully satisfied. The NV then plan to discount the product soon. Therefore, the NV at this phase of the problem has perfect knowledge of demand (i.e. the realization of W or x_0 is known). In this section, we analyze the problem of deciding on the discount prices. The NV can follow one of two policies in discounting: A Blind policy, denoted by B, and a Revenue maximizing policy, denoted by M.

Blind policy (B): The NV keeps discounting the product until all inventory is sold or a price of zero is reached (i.e. items discarded). Note that the NV in this case is not concerned with whether the remaining inventory is worth discounting in terms of covering the fixed discounting cost. This policy can be justified on the basis that the NV deals with a large number of products and does not have completely accurate and timely inventory records, which makes it easiest to keep discounting until all inventory is sold. For a given Q, the NV must determine the value of h which maximizes revenue. Let P_a be the price at which all of Q will be sold. Then, for example, if the NV offers the product at only one price (P_0), the revenue is given by

$$R_B(h=1) = \begin{cases} P_0 Q & \text{if } P_a \geqslant P_0 \\ P_0 x_0 & \text{if } P_a < P_0. \end{cases} \tag{6}$$

If the NV offers the product at four prices ($P_0, 3P_0/4, 2P_0/4,$ and $P_0/4$) then, using (5), the incremental quantity demanded at each discount price is $P_0/4b$ and the revenue is given by

$$R_B(h=4)=\begin{cases} P_0Q & \text{if } P_a \geqslant P_0,\\ P_0x_0+(3P_0/4)(Q-x_0)-F & \text{if } 3P_0/4 \leqslant P_a < P_0,\\ P_0x_0+(3P_0/4)(P_0/4b)+(P_0/2)(Q-P_0/4b-x_0)-2F & \text{if } 2P_0/4 \leqslant P_a < 3P_0/4,\\ P_0x_0+(3P_0/4)(P_0/4b)+(P_0/2)(P_0/4b) & \\ \quad +(P_0/4)(Q-P_0/2b-x_0)-3F & \text{if } P_0/4 \leqslant P_a < 2P_0/4,\\ P_0x_0+(3P_0/4)(P_0/4b)+(P_0/2)(P_0/4b) & \\ \quad +(P_0/4)(P_0/4b)-3F & \text{if } P_a < P_0/4. \end{cases}$$

Simplifying gives

$$R_B(h=4)=\begin{cases} P_0Q & \text{if } P_a \geqslant P_0,\\ (3P_0/4)Q+(P_0/4)x_0-F & \text{if } 3P_0/4 \leqslant P_a < P_0,\\ (2P_0/4)Q+(2/4)P_0x_0+P_0^2/16b-2F & \text{if } 2P_0/4 \leqslant P_a < 3P_0/4,\\ (P_0/4)Q+(3P_0/4)x_0+3P_0^2/16b-3F & \text{if } P_0/4 \leqslant P_a < 2P_0/4,\\ P_0x_0+6P_0^2/16b-3F & \text{if } P_a < P_0/4. \end{cases} \tag{7}$$

In general, for any number of prices h, the revenue is given by

$$R_B=\begin{cases} P_0Q & \text{if } P_a \geqslant P_0,\\ \dfrac{j}{h}P_0x_0+\dfrac{h-j}{h}P_0Q+\dfrac{P_0^2}{bh^2}\displaystyle\sum_{i=0}^{j-1} i-jF & \text{if } \dfrac{h-j}{h}P_0 \leqslant P_a < \dfrac{h-j+1}{h}P_0,\\ P_0x_0+\dfrac{P_0^2}{bh^2}\displaystyle\sum_{i=0}^{h-1} i-(h-1)F & \text{if } P_a < \dfrac{P_0}{h}. \end{cases} \tag{8}$$

Obviously, if $P_a > P_0$ (or equivalently $Q < x_0$) then no discounts are needed and $h^* = 1$. Define k as

$$k=\max\left\{h\colon \frac{h-1}{h}\,P_0 \leqslant P_a < P_0\right\}.$$

In other words, k is the largest number of prices which satisfies $[(k-1)/k]P_0 \leqslant P_a$. Lemmas 1 and 2 are introduced to develop an algorithm which can efficiently identify h^*.

Lemma 1. *The optimal number of prices at which to offer the product satisfies $h^* \geqslant k$.*

Proof. The proof of Lemma 1 is provided in the appendix.

Lemma 2. *For any number of prices v such that $v > k$ and for any integer j, if*

$$\frac{v-j}{v}P_0 \leqslant P_a < \frac{v-j+1}{v}P_0 \textit{ and}$$

$$\frac{v+1-j}{v+1}P_0 \leqslant P_a < \frac{v-j+2}{v+1}P_0,$$

then the revenue from using $h = v + 1$ is greater than or equal to the revenue from using $h = v$ (i.e. $R_B(v+1) \geqslant R_B(v)$).

Proof. The proof of Lemma 2 is provided in the appendix.

Algorithm 1: Blind policy

Step 1: Compute P_a using (4). If $P_a \geqslant P_0$ then stop, all of Q will be sold at the initial price P_0.

Step 2: Find the largest h for which $[(h-1)/h]$ $P_0 \leqslant P_a < P_0$. Denote this h as k.

If $k = n$ then stop k is optimal. (by Lemma 1)
Set $i = n$

Step 3: Find the largest integer, j, for which $[(i-j)/i]P_0 \leqslant P_a < [(i-j+1)/i]P_0$.

If $[(i-j-1)/(i-1)]P_0 \leqslant P_a < [(i-j)/(i-1)]P_0$, then $h = i-1$ is suboptimal. (by Lemma 2)

Step 4: Set $i = i-1$.

If $i > k+1$ go to Step 3.

Compute revenue using (8) for $h = k$ and all $h > k$ not found to be suboptimal.

Select h with the largest revenue.

Steps 1 and 2 are performed once. Step 2 requires more computations than Step 1 and is bounded by n. In the worst case, Step 3 is performed n times and each of these iterations is bounded by n. Finally, Step 4 is bounded by n. Therefore, the running time of the algorithm is $O(n^2)$.

Revenue maximizing policy (Policy M): Here, the last discount (j) for a given number of prices (h) is offered only if the additional revenue from it is greater than or equal to the fixed cost of discounting (F). Thus, if $(h-j)P_0/h \leqslant P_a < (h-j+1)P_0/h$ then discount j is used only if the additional revenue it generates from selling the additional units demanded between prices $(h-j)P_0/h$ and P_a is greater than or equal to F. To simplify the analysis, we introduce the following realistic assumption:

Assumption 1. The largest number of prices the NV may use (n), is such that the incremental revenue from selling all units demanded between prices $2P_0/n$ and P_0/n is greater than F.

In other words, the NV will not use prices so close to each other to the point where the incremental revenue of a discount is insufficient to cover the fixed cost of the discount. From (5) the quantities sold at $2P_0/n$ and P_0/n are $(W - P_0/n)/b$ and $(W - 2P_0/n)/b$, respectively. Thus, the incremental units sold at $P = P_0/n$ are P_0/nb and the incremental revenue is $(P_0/nb)P_0/n = P_0^2/bn^2$. Assumption 1 can be restated as

$$\frac{P_0^2}{bn^2} \geqslant F. \tag{9}$$

In order to derive the revenue function under this policy, we need to introduce a new policy, denoted C. Suppose that discount j is not used unless the full quantity demanded at that discount price is available. When the quantity available at discount j is small, the fixed cost of discounting will be greater than the incremental revenue from selling this small quantity. Thus, discounting will reduce revenue and $R_C > R_B$. When the quantity available at discount j is large (but insufficient to satisfy all of the incremental demand at that price), the fixed cost of discounting will be smaller than the incremental revenue from selling this large quantity. Thus, discounting will increase revenue and $R_C < R_B$. To obtain R_C, the revenue from the last discount is eliminated from the expression for R_B and the coefficient of F is reduced by one. For example, the revenue functions for $h = 2$ and 4 in policy C are:

$$R_C(h=2) = \begin{cases} P_0 Q & \text{if } P_a \geqslant P_0, \\ P_0 x_0 & \text{if } P_0/2 \leqslant P_a < P_0, \\ P_0 x_0 + P_0^2/4b - F & \text{if } P_a < P_0/2. \end{cases} \tag{10}$$

and

$$R_C(h=4) = \begin{cases} P_0 Q & \text{if } P_a \geqslant P_0, \\ P_0 x_0 & \text{if } 3P_0/4 \leqslant P_a < P_0, \\ P_0 x_0 + 3P_0^2/16b - F & \text{if } 2P_0/4 \leqslant P_a < 3P_0/4, \\ P_0 x_0 + 5P_0^2/16b - 2F & \text{if } P_0/4 \leqslant P_a < 2P_0/4, \\ P_0 x_0 + 6P_0^2/16b - 3F & \text{if } P_a < P_0/4. \end{cases} \tag{11}$$

In general, for any h, R_C is given by

$$R_C = \begin{cases} P_0 Q & \text{if } P_a \geqslant P_0, \\ P_0 x_0 & \text{if } \dfrac{h-1}{h} P_0 \leqslant P_a < P_0, \\ P_0 x_0 + \dfrac{P_0^2}{bh^2} \displaystyle\sum_{i=h-j+1}^{h-1} i - (j-1)F & \text{if } \dfrac{h-j}{h} P_0 \leqslant P_a < \dfrac{h-j+1}{h} P_0, \\ P_0 x_0 + \dfrac{P_0^2}{bh^2} \displaystyle\sum_{i=1}^{h-1} i - (h-1)F & \text{if } P_a < \dfrac{P_0}{h}. \end{cases} \tag{12}$$

From (8) and (12), the difference in revenue between the Blind policy and policy C is

$$R_B - R_C = \begin{cases} 0 & \text{if } P_0 \geqslant P_a, \\ \dfrac{h-1}{h} P_0 (Q - x_0) - F & \text{if } \dfrac{h-1}{h} P_0 \leqslant P_a < P_0, \\ \dfrac{h-j}{h} P_0 (Q - x_0) + \dfrac{P_0^2}{bh^2} \left[\displaystyle\sum_{i=0}^{j-1} i - \sum_{i=h-j+1}^{h-1} i \right] - F & \text{if } \dfrac{h-j}{h} P_0 \leqslant P_a < \dfrac{h-j+1}{h} P_0, \\ 0 & \text{if } P_a < \dfrac{P_0}{h}. \end{cases} \tag{13}$$

Let R_M be the revenue of the Revenue Maximizing policy, $(R_B - R_C)$ will be used to find h which maximizes R_M. For a given Q and h, if $(R_B - R_C) > 0$ then the last discount should be used and $R_M = R_B$. If $(R_B - R_C) < 0$ then the last discount should not be used and $R_M = R_C$. Because under the Revenue Maximizing policy the last discount may or may not be used we must find both h^* and j^*. Note that h and h^* are used to denote the number of prices and the optimal number of prices, respectively, for both the B and M policies. Using the definition of k introduced before Lemma 1, Lemma 3 introduces important properties of R_M.

Lemma 3. *The optimal number of prices at which to offer the product satisfies $h^* \geqslant k$ or $h^* = 1$.*

Proof. The proof of Lemma 3 is provided in the appendix.

Algorithm 2: Revenue maximizing policy

Step 1. Compute P_a using (4). If $P_a \geqslant P_0$ then stop, all of Q will be sold at the initial price P_0.

Step 2. Find the largest h for which $[(h-1)/h] P_0 \leqslant P_a < P_0$. Denote this h as k.

If $k = n$ and $[(k-1)/k] P_0 (Q - x_0) > F$ then stop $h^* = k$ and $j^* = 1$.

If $k = n$ and $[(k-1)/k] P_0 (Q - x_0) < F$ then stop $h^* = k$ and $j^* = 0$.

If $k < n$ and $[(k-1)/k] P_0 (Q - x_0) > F$ then $R_M(k)$ is given by (8).

If $k < n$ and $[(k-1)/k] P_0 (Q - x_0) < F$ then $R_M(k)$ is given by (12).

Set $h = k + 1$

Step 3. Find the largest integer, j, for which $[(h-j)/h] P_0 \leqslant P_a < [(h-j+1)/h] P_0$.

If

$$\frac{h-j}{h} P_0 (Q - x_0) + \frac{P_0^2}{bh^2} \left[\sum_{i=0}^{j-1} i - \sum_{i=h-j+1}^{h-1} i \right] > F$$

then $R_M(h)$ is given by (8) otherwise $R_M(h)$ is given by (12).

Step 4. Set $h = h + 1$.

If $h < n$ go to Step 3.

Compute $R_M(h)$ using the function constructed in Steps 2 and 3 for all $h \geqslant k$.

Select h with the largest revenue.

Similar to Algorithm 1, Steps 1 and 2 are performed once. Step 2 requires more computations and is bounded by n. In the worst case, Step 3 is performed n times and each of these iterations is bounded by n. Finally, Step 4 is bounded by n. Therefore, the running time of the algorithm is $O(n^2)$.

Numerical Example 1. Consider a product with demand function $P = W - 0.01x$, where W is normally distributed with a mean of 110 and a standard deviation of 10. This implies an increase of 100 units in demand for each \$1 drop in price. The fixed cost of discounting is $F = \$800$. Suppose the NV ordered $Q = 10{,}750$ units and is using the Blind policy. The NV offered the product at an initial price of $P_0 = \$20$ and found that the realization for W was 120 or equivalently $x_0 = 10{,}000$ units The question facing the NV is how many prices to use to sell all of Q? The NV limits the number of possible prices at which to offer the product to $n = 7$ which satisfies assumption 1. Thus, the choice of h must to be made from $h = 1, 2, \dots, 7$. Using (4), the price at which all of the 10,750 units will be sold is $P_a = \$12.5$. Algorithm 1 shows that one of the values of $h = 2, 5$, or 7 is optimal. Using (8) with $Q = 10{,}750$ gives $R_B(h = 2) = \$206{,}700$, $R_B(h = 5) = \$209{,}000$, and $R_B(h = 7) = \$208{,}620$ and thus, $h^* = 5$, which corresponds to discounts of 20% from the original price and unit prices of \$20.00, \$16.00, \$12.00, \$8.00, and \$4.00. The last unit in $Q = 10{,}750$ will be sold for \$12.00. If the value of b is changed to $b = 0.02$, which implies a 50 unit increase in demand for each \$1 drop in unit price, and the realization of W is changed to $W = 220$, then the realized demand at $P_0 = \$20$ remains $x_0 = 10{,}000$. However, the optimal number of prices decreases to $h^* = 4$ with $R_B(h = 4) = \$205{,}100$. Because the additional quantity sold at each discount is smaller with the new b, then it is optimal to use smaller number of prices and larger discounts. If the value of F is changed to $F = \$3{,}200$ then it is optimal to use $h^* = 2$ for which $R_B(h = 2) = \$204{,}300$. Because the cost of discounting is higher, it is optimal to use a smaller number of prices and larger discounts.

Now suppose the NV is using a Revenue Maximizing policy. The application of Algorithm 2 for $Q = 10{,}750$ and $P_a = \$12.50$ results in the following revenue function:

$$R_M = \begin{cases} 0.5P_0x_0 + 0.5P_0Q - F = \$206{,}700 & \text{if } h = 2, \\ P_0x_0 + 2P_0^2/bh^2 - F = \$208{,}089 & \text{if } h = 3, \\ 2P_0x_0/4 + 2P_0Q/4 + P_0^2/bh^2 - 2F = \$208{,}400 & \text{if } h = 4, \\ 2P_0x_0/5 + 3P_0Q/5 + P_0^2/bh^2 - 2F = \$209{,}000 & \text{if } h = 5, \\ 3P_0x_0/6 + 3P_0Q/6 + 3P_0^2/bh^2 - 3F = \$208{,}433 & \text{if } h = 6, \\ 3P_0x_0/7 + 4P_0Q/7 + 3P_0^2/bh^2 - 3F = \$208{,}620 & \text{if } h = 7. \end{cases} \tag{14}$$

Thus, $h^* = 5, j^* = 2, R_M^* = R_B^* = \$209{,}000$ and the Blind and the Revenue Maximizing policy give the same results. For $Q = 10{,}680$, $P_a = \$13.2$ and the Blind policy gives $h^* = 5$, and $R_B^* = \$208{,}160$ whereas the Revenue Maximizing policy gives $h^* = 6$, $j^* = 2$, and $R_M^* = \$208{,}400$ which corresponds to discounts of 16.67% from the initial price.

4. The order quantity problem

We start by assuming that the initial selling price is preset with value that depends on the competitive situation of the firm. We focus our attention on the Blind policy since dealing with the Revenue maximizing policy is complex and will distract us from our purpose of analyzing the effects of multiple discounts on the optimal order quantity and initial price. For the order quantity problem, the NV only knows the distribution (i.e. the mean, standard deviation and distributional form) of demand at different prices. From (8), the expected profit for a given h is

$$E(\pi(h)) = \int_Q^\infty P_0 Q f(x_0)\,\mathrm{d}x_0 + \sum_{j=1}^{h-1} \int_{Q-jP_0/hb}^{Q-(j-1)P_0/hb} \left[\frac{j}{h}P_0 x_0 + \frac{h-j}{h}P_0 Q + \frac{P_0^2}{bh^2}\sum_{i=0}^{j-1} i - jF\right] f(x_0)\,\mathrm{d}x_0 + \int_0^{Q-(h-1)P_0/hb} \left[P_0 x_0 + \frac{P_0^2}{bh^2}\sum_{i=0}^{h-1} i - (h-1)F\right] f(x_0)\,\mathrm{d}x_0 - CQ. \tag{15}$$

We simplify (15) and find Q^* for normally and uniformly distributed demand.

4.1. Uniform demand distribution

Suppose x_0 is uniformly distributed on the range $[\alpha, \beta]$, then Eq. (15) gives

$$E(\pi(h)) = \frac{1}{\beta-\alpha}\left[\frac{P_0}{2h}\left(-hQ^2 + 2hQ\beta - h\alpha^2 - \frac{P_0^2 S_1}{h^2 b^2}\right) + \frac{P_0^2 S_2}{h^2 b}(Q-\alpha) - (h-1)F\left(Q - \alpha + \frac{P_0 S_3}{h(h-1)b}\right) - CQ(\beta-\alpha)\right], \tag{16}$$

where

$$S_1 = \sum_{i=0}^{h-1} i^2, \quad S_2 = \sum_{i=0}^{h-1} i, \quad \text{and} \quad S_3 = \sum_{i=0}^{h-2} i.$$

Lemma 4 provides proof of concavity of $E(\pi(h))$. Setting $\mathrm{d}E(\pi(h))/\mathrm{d}Q = 0$ yields Q_h^* for a given h:

$$Q_h^* = \beta + \frac{P_0}{h^2 b}\sum_{i=0}^{h-1} i - \frac{(h-1)F + C(\beta-\alpha)}{P_0}. \tag{17}$$

Lemma 4. *The expected profit function is concave in Q for uniformly distributed demand.*

Proof. The proof of Lemma 4 is provided in the appendix.

Thus, to obtain the optimal order quantity (Q^*), the optimal order quantities for all values of h (i.e. Q_h^*, $h = 1, \dots, n$) are computed using (17) and then $E(\pi(h))$ for each Q_h^* is computed using (16). By enumerating over all $h = 1, 2, \dots, n$ we identify the Q_h^* that maximizes $E(\pi(h))$.

4.2. Normal demand distribution

Suppose x_0 is normally distributed with mean μ_0 and standard deviation σ. From (15), the revenue for a given h is

$$E(\pi(h)) = \int_Q^{+\infty} P_0 Q f(x_0)\,\mathrm{d}x_0 + \sum_{j=1}^{h-1} \int_{Q-jP_0/hb}^{Q-(j-1)P_0/hb} \left(\frac{jP_0 x_0}{h} + \frac{(h-j)P_0 Q}{h} + \frac{P_0^2}{bh^2}S_2 - jF\right) f(x_0)\,\mathrm{d}x_0 + \int_{-\infty}^{Q-(h-1)P_0/hb} \left(P_0 x_0 + \frac{P_0^2}{bh^2}S_2 - (h-1)F\right) f(x_0)\,\mathrm{d}x_0. \tag{18}$$

Our analysis indicates that $E(\pi(h))$ is concave for different values of h. However, we are able to prove that Q_h^* is in the concave region of $E(\pi(h))$ only under the conditions stated in Lemma 5. Define P_k as the highest price for which $P_k \leqslant C$ for a given h.

Lemma 5. *If $F + C - P_{k+1} \geqslant k(P_0 - C)$ and $kP_0^2 \leqslant b^2 h^2 \sigma^2$, then for normally distributed demand $\mathrm{d}E(\pi(h))/\mathrm{d}Q = 0$ is a sufficient condition for optimality.*

Proof. The proof of Lemma 5 is provided in the appendix.

Setting $\mathrm{d}E(\pi(h))/\mathrm{d}Q = 0$, gives the sufficient optimality condition:

$$\frac{P_0}{h}\left[h - \sum_{i=0}^{h-1} F\left(Q - \frac{iP_0}{hb}\right)\right] - F\sum_{i=0}^{h-2} f\left(Q - \frac{iP_0}{hb}\right) - C = 0, \tag{19}$$

Table 3
Optimal order quantities and expected profit for different discounting schemes and uniform demand distribution

h	Q_h^*	$E(\pi(h))$
1	10000	\$90,000.00
2	10460	\$93,879.00
3	10587	\$94,741.93
4*	10630	\$94,804.75
5	10640	\$94,544.00
6	10633	\$94,123.15
7	10617	\$93,613.39

Table 4
Optimal order quantities and expected profit for different discounting schemes and normal demand distribution

h	Q^*	$E(\pi(h))$
1	10000	\$91,999.97
2	10459	\$95,466.63
3	10582	\$96,550.64
4	10622	\$96,939.17
5*	10631	\$97,043.67
6	10623	\$97,007.84
7	10607	\$96,894.11

which can be solved using a simple search procedure. Now $Q_h^*, h = 1, \dots, n$ and their corresponding $E(\pi(h))$ are computed and the best Q_h^* is ordered. Evaluating $E(\pi(h))$ requires numerical integration.

Numerical Example 2. Reconsider Example 1 with W uniformly distributed on [100, 140] which implies that x_0 is uniformly distributed with $\alpha = 8{,}000$ and $\beta = 12{,}000$ for $P_0 = \$20$. Using Eqs. (16) and (17) gives the results in Table 3. Thus, the optimal number of prices is $h^* = 4, Q^* = 10{,}630$ and $E(\pi(h)) = \$94{,}804.75$. After the order is placed and demand becomes known, the application of Algorithm 1 may result in a new value of h^* at the discounting problem. For smaller cost of discounting of $F = \$200$, the solution changes to $h^* = 7$, $Q^* = 10{,}797$ and $E(\pi(h)) = \$96{,}692.67$.

Suppose W is normally distributed with a mean of $E(W) = 120$ and standard deviation $\sigma(W) = 10$ which implies that x_0 is normally distributed with mean $\mu_0 = 10{,}000$ and standard deviation $\sigma = 1{,}000$ for $P_0 = \$20$. Using Eqs. (18) and (19) gives the results in Table 4. Thus, the optimal number of prices is $h^* = 5, Q^* = 10{,}631$, and $E(\pi(h)) = \$97{,}043.67$.

5. Optimal initial price

In this case, initial unit offering price (P_0) is a decision variable and the NV must determine the optimal values of both Q and P_0. We analyze the case of uniformly distributed demand (i.e. $x_0 \sim \mathrm{U}[\alpha, \beta]$) and provide closed-form solution for it. Using $P_0 = C$ with the expressions for α and β gives the minimum and maximum possible demands, $(W_1 - C)/b$ and $(W_2 - C)/b$, respectively, if the product is offered at cost. Let

$$a_1 = \frac{1}{b^2}\left(\frac{S_1}{2h^3} + \frac{1}{2} - \frac{S_2}{h^2}\right), \tag{20}$$

$$a_2 = \frac{1}{b}\left(Q - \frac{W_1}{b}\right)\left(1 - \frac{S_2}{h^2}\right), \tag{21}$$

$$a_3 = \frac{1}{2}(Q^2 + W_1^2) + \frac{F}{b}\left(\frac{S_3}{h} + h - 1\right) - \frac{QW_2}{b} \tag{22}$$

and

$$a_4 = (h-1)F\left(\frac{W_1}{b} - Q\right) - \frac{CQ}{b}(W_2 - W_1). \tag{23}$$

Substituting from (20)–(23) into (16) gives

$$E(\pi(h)) = \frac{b}{W_2 - W_1}\left[a_4 - a_3 P_0 - a_2 P_0^2 - a_1 P_0^3\right]. \tag{24}$$

The derivative of $E(\pi(h))$ with respect to P_0 is

$$\frac{\partial E(\pi(h))}{\partial P_0} = \frac{-b}{W_2 - W_1}\left[a_3 + 2a_2 P_0 + 3a_1 P_0^2\right] \tag{25}$$

with roots at

$$P_{01}, P_{02} = \frac{-2a_2 \pm \sqrt{a_2^2 - 12 a_1 a_3}}{6a_1}. \tag{26}$$

Substituting different values of h into (20) shows that $a_1 > 0$. If $a_2 > 0$ then (26) gives one negative root and one positive root independent of the value of a_3. If $a_2 < 0$ and $a_3 < 0$, (26) also gives one negative root and one positive root. If $a_2 < 0$ and $a_3 > 0$, (26) gives two positive roots. Eqs. (17) and (26) can now be used in an iterative fashion until convergence to obtain the optimal quantity and price.

For normally distributed demand, we are unable to provide closed-form expressions for Q_h^* and P_h^*. For Q_h to be optimal, it must satisfy (19). For P_h to be optimal it must satisfy $\mathrm{d}E(\pi(h))/\mathrm{d}P = 0$, where $E(\pi(h))$ is given by Eq. (18). Therefore, this case would require solving two nonlinear equations until convergence for finding Q_h^* and $P_h^*, h = 1, \ldots, n$. Numerical integration can then be used to evaluate the corresponding $E(\pi(h))$ for each h to identify h^*.

Numerical Example 3. Using the data of Example 1, Eqs. (17) and (26) converge to the P_{0h}^*'s and Q_h^*'s, $h = 1, \ldots, 15$ shown in Table 5 after 20 iterations (actually convergence occurs in less than 20 iterations depending on the degree of precision required).

From Table 5, the optimal order quantity is $Q^* = 8575.4$, the optimal initial offering price is $P_0^* = \$90.23$, the optimal number of prices is $h^* = 12$ (which may change after demand becomes known), and the optimal expected profit is $E(\pi(h)) = \$451{,}780$. Increasing the discounting cost to $F = \$3{,}000$ gives $Q^* = 8353$, $P_0^* = \$85.82$, and $h^* = 6$, for which $E(\pi(h)) = \$411{,}530$.

Table 5
Optimal order quantities and initial prices for $h = 1, \ldots, 15$

h	P_{0h}^* (\$)	Q_h^*	$E(\pi(h))$ (\$)
1	64.762	6953.8	285,583
2	75.757	7799.9	363,068
3	80.950	8103.7	398,843
4	83.933	8260.8	418,744
5	85.846	8356.4	430,933
6	87.158	8420.4	438,795
7	88.101	8465.9	443,988
8	88.801	8499.7	447,419
9	89.330	8525.7	449,627
10	89.737	8546.0	450,952
11	90.053	8562.3	451,618
12	90.298	8575.4	451,780
13	90.489	8586.2	451,548
14	90.635	8595.0	451,004
15	90.746	8602.4	450,206

Reconsider Example 1 with uniformly distributed W on [40, 80] which implies x_0 is uniformly distributed with $\alpha = 500$ and $\beta = 1{,}500$ at $P_0 = \$20$. The demand function is $P = W - 0.04x$, which implies a 25 units increase in demand for each \$1 drop in price. Eqs. (17) and (26) converge to $Q^* = 1{,}014$, $P_0^* = \$45.72$, and $h^* = 4$, for which $E(\pi(h)) = \$18{,}296$. Increasing the discounting cost to $F = \$1{,}500$ gives $Q^* = 975$, $P_0^* = \$43.58$, $h^* = 3$, for which $E(\pi(h)) = \$16{,}368$. Decreasing the discounting cost to $F = \$300$ results in $Q^* = 1{,}057$, $P_0^* = \$48.40$, and $h^* = 6$, for which $E(\pi(h)) = \$21{,}130$.

6. Discussion and conclusion

In this paper, we extended the classical SPP to the case in which

1. demand is price dependent,
2. multiple discount prices are used to sell excess inventory,
3. the discount prices are under the control of the NV, and
4. there is a positive setup cost associated with discounting the product.

The resulting SPP is composed of two problems. In the first problem, after demand becomes known, the NV must determine the optimal discounting scheme. In the second problem, the NV must determine the order quantity which maximizes the expected profit prior to any demand being realized. Two algorithms were developed for solving the discounting problem under two different assumptions about the behavior of the NV. The order quantity problem was addressed for normal and uniform demand distributions. Furthermore, for cases where the initial price is also under the control of the NV, the problem was solved for uniform demand distribution.

For the discounting problem, as the numerical examples show, the use of multiple discounts may increase the revenue for any Q and realization of demand. The effect of the new discounting scheme is more significant for higher demand elasticity and smaller fixed discounting cost. The more elastic the demand, the larger the number of prices the NV should use and the smaller the difference between consecutive prices. The higher the fixed discounting cost, the smaller the number of prices the NV should use and the larger the difference between consecutive prices. Actually, the proposed model sheds some light on the behavior of retailers where multiple items are usually discounted together. By discounting multiple items together (such as all men's and/or women's apparel), the fixed discounting cost per item is reduced, the number of prices used to sell the items are increased, and revenue is increased.

The effect of the incorporation of multiple discounts on the optimal order quantity depends both on the elasticity of demand and the fixed discounting cost. From Eq. (17) one obtains $Q^*_{h+1} - Q^*_h = P_0(h^3 - (2h+1)\sum_{i=0}^{h-1} i)/bh^2(h+1)^2 - F/P_0$. Simple analysis shows that the first term to the right of the equality is positive and decreasing in h. Thus, the smaller the value of b and the smaller the fixed discounting cost, the more likely that $Q^*_{h+1} \geqslant Q^*_h$ and that Q^*_{h+1} has higher expected profit than Q^*_h, especially at smaller values of h. In this case, because demand elasticity is high and the fixed discounting cost is low, the NV orders larger quantities knowing that any excess inventory can be sold by using small discounts without incurring a large fixed discounting cost.

Future research can address several extensions of the above model. An extension dealing with prices that are not equally spaced provides a useful generalization of the model. In this case, the NV only restricts the discounts to be at least a certain percentage off the original price so that customers see them as meaningful. The decision variables become the optimal prices $(P_0, P_1, \ldots, P_n)$ for the discounting problem. Other extensions can deal with other types of demand–price relationships and other probability distributions of demand.

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Appendix A

Proof of Lemma 1. Suppose for $h = k$, $P_0(k-1)/k \leqslant P_a < P_0$ and for $h = N = k-1$, $R_B(N) > R_B(k)$. From (8),

$$R_B(k) = \frac{1}{k}P_0 x_0 + \frac{k-1}{k}P_0 Q - F. \tag{A.1}$$

Since $P_0(k-1)/k - P_0(k-2)/(k-1) = P_0/k(k-1) > 0$, $P_0(k-1)/k \leqslant P_a < P_0$ implies that $P_0(k-2)/(k-1) \leqslant P_a < P_0$. Thus from (8),

$$R_B(N) = \frac{1}{k-1}P_0 x_0 + \frac{k-2}{k-1}P_0 Q - F. \tag{A.2}$$

Subtracting (A.1) from (A.2) and simplifying gives

$$R_B(N) - R_B(k) = \frac{P_0}{k(k-1)}(x_0 - Q) < 0, \tag{A.3}$$

which leads to a contradiction.

Proof of Lemma 2. From (8),

$$R_B(v+1) = \frac{j}{v+1}P_0 x_0 + \frac{v+1-j}{v+1}P_0 Q + \frac{P_0^2}{b(v+1)^2}\sum_{i=0}^{j-1} i - jF \tag{A.4}$$

and

$$R_B(v) = \frac{j}{v}P_0 x_0 + \frac{v-j}{v}P_0 Q + \frac{P_0^2}{bv^2}\sum_{i=0}^{j-1} i - jF. \tag{A.5}$$

Subtracting (A.5) from (A.4) and simplifying gives:

$$R_B(v+1) - R_B(v) = \frac{P_0(Q - x_0)}{v^2(v+1)^2(P_0 - P_a)} \times \left[jv(v+1)(P_0 - P_a) - (2v+1)P_0 \sum_{i=0}^{j-1} i \right]. \tag{A.6}$$

For any discounting to be needed, $Q > x_0$ (or equivalently $P_0 > P_a$). Thus, the sign of $R_B(v+1)$

$-R_B(v)$ is determined by the term inside the brackets. Define

$$g = jv(v+1)(P_0 - P_a) - (2v+1)P_0 \sum_{i=0}^{j-1} i. \qquad (A.7)$$

The largest value P_a can take is computed by taking the smaller limit obtained using $P_a < [(v-j+1)/v]P_0$ and $P_a < [(v-j+2)/(v+1)]P_0$ in (8). Since

$$\frac{v-j+1}{v} - \frac{v-j+2}{v+1} = \frac{-j}{v(v+1)} < 0$$

the upper limit on P_a is $P_a = [(v-j+1)/v]P_0$ which when used in (A.7) gives the following expression for g:

$$g = j(v+1)P_0(j-1) - (2v+1)P_0 \sum_{i=0}^{j-1} i. \qquad (A.8)$$

The values of g are shown in Table 6. Thus $g = j(j-1)P_0/2 \geqslant 0$ which implies $R_B(v+1) \geqslant R_B(v)$.

Proof of Lemma 3. Suppose for $h = k$, $P_0(k-1)/k \leqslant P_a < P_0$. The proof that for $h = N = k-1$, $R_M(N) \leqslant R_M(k)$ is provided in three parts:

1. Since
$$\frac{k-1}{k}P_0 - \frac{k-2}{k-1}P_0 = \frac{P_0}{k(k-1)} > 0,$$
if $\dfrac{k-2}{k-1}P_0(Q-x_0) > F$
then $[(k-1)/k)]P_0(Q-x_0) > F$ and the revenue is given by (8). Thus, by Lemma 1, $R_M(N) < R_M(k)$.

Table 6

j	g
1	0
2	P_0
3	$3P_0$
4	$6P_0$
5	$10P_0$
6	$15P_0$
…	…
j	$j(j-1)P_0/2$
…	…

2. If $[(k-2)/(k-1)]P_0(Q-x_0) < F$ then $R_M(N) = R_M(k-1) = P_0x_0$ and if $[(k-1)/k)]P_0(Q-x_0) < F$ then $R(k) = P_0x_0$ which gives $R_M(N) = R_M(k)$.
3. If $[(k-2)/(k-1)]P_0(Q-x_0) < F$ then $R_M(N) = R_M(k-1) = P_0x_0$. If $[(k-1)/k]P_0(Q-x_0) > F$, then from (9) $R_M(k) = P_0x_0 + [(k-1)/k]P_0(Q-x_0) - F > P_0x_0$ which implies $R_M(N) < R_M(k)$.

From parts 1, 2, and 3, $R_M(N) \leqslant R_M(k)$.

Proof of Lemma 4. The first derivative of $E(\pi(h))$ in (16) is

$$\frac{dE(\pi(h))}{dQ} = \frac{1}{\beta-\alpha}\left[P_0(\beta - Q) + \frac{P_0^2 \sum_{i=0}^{h-1} i}{h^2 b} - (h-1)F - C(\beta-\alpha)\right] \qquad (A.9)$$

and the second derivative

$$\frac{d^2E(\pi(h))}{dQ^2} = \frac{-P_0}{\beta-\alpha}.$$

Since $\dfrac{d^2E(\pi(h))}{dQ^2} < 0$, $E(\pi(h))$ is concave.

Proof of Lemma 5. Simplifying (18) gives

$$\begin{aligned} E(\pi(h)) = &\int_Q^{+\infty} P_0 Q f(x_0)\,dx_0 \\ &+ \sum_{j=1}^{h-1} \int_{Q-jP_0/hb}^{Q-(j-1)P_0/hb} \left(\frac{jP_0x_0}{h} + \frac{(h-j)P_0Q}{h}\right) f(x_0)\,dx_0 \\ &- \sum_{j=1}^{h-1}\left(\frac{P_0^2}{bh^2}\sum_{i=0}^{j-1} i - jF\right) \\ &\times (F(Q-(j-1)P_0/hb) \\ &- F(Q - jP_0/hb)) \\ &+ \int_{-\infty}^{Q-(h-1)P_0/hb} P_0x_0 f(x_0)\,dx_0 \\ &+ \left(\frac{P_0^2}{bh^2}\sum_{i=0}^{h-1} i - (h-1)F\right) \\ &\times F(Q-(h-1)P_0/hb). \end{aligned} \qquad (A.10)$$

The first and second derivatives of $E(\pi(h))$ are

$$\frac{dE(\pi(h))}{dQ} = \frac{P_0}{h}\left[h - \sum_{i=0}^{h-1} F\left(Q - \frac{iP_0}{hb}\right)\right] - F\sum_{i=0}^{h-2} f\left(Q - \frac{iP_0}{hb}\right) - C, \tag{A.11}$$

and

$$\frac{d^2E(\pi(h))}{dQ^2} = \frac{-P_0}{h}\sum_{i=0}^{h-1} f\left(Q - \frac{iP_0}{hb}\right) - F\sum_{i=0}^{h-2} f'\left(Q - \frac{iP_0}{hb}\right). \tag{A.12}$$

Let y be normally distributed with mean μ and standard deviation σ. The probability distribution of y is

$$f(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(y-\mu)^2/2\sigma^2} \tag{A.13}$$

with first derivative

$$f'(y) = -\frac{y-\mu}{\sigma^2}\frac{1}{\sqrt{2\pi}\sigma} e^{-(y-\mu)^2/2\sigma^2} = -\frac{y-\mu}{\sigma^2} f(y). \tag{A.14}$$

Substituting from (A.14) into (A.12) gives

$$\frac{d^2E(\pi(h))}{dQ^2} = \frac{-P_0}{h} f\left(Q - \frac{(h-1)P_0}{hb}\right) + \sum_{i=0}^{h-2}\left[\frac{Fh(Q - iP_0/hb - \mu_0) - P_0\sigma^2}{h\sigma^2}\right] \times f\left(Q - \frac{iP_0}{hb}\right). \tag{A.15}$$

Since $f(Q - (iP_0/hb)) > 0$ for any Q, $d^2E(\pi(h))/dQ^2 \leqslant 0$ for $Q \leqslant \mu_0$.

For, $Q > \mu_0$, the proof that $d^2E(\pi(h))/dQ^2 \leqslant 0$ is provided in three parts:

1. From (A.15), a sufficient condition for $d^2E(\pi(h))/dQ^2 \leqslant 0$ is

$$Fh(Q - iP_0/hb - \mu_0) < P_0\sigma^2, \quad i = 1, \dots, h-2. \tag{A.16}$$

which is satisfied if

$$Fh(Q - \mu_0) < P_0\sigma^2. \tag{A.17}$$

By assumption 1, the largest F is at $F = P_0^2/bh^2$ which when substituted in (A.17) gives

$$Q < \mu_0 + \frac{bh\sigma^2}{P_0}.$$

Thus, $E(\pi(h))$ is concave for any $Q < Q^C = \mu_0 + bh\sigma^2/P_0$.

2. For any value of Q, the expected gain is

$$G(Q) = (P_0 - C)[1 - F_0(Q)] + (P_1 - C)[F_0(Q) - F_1(Q)] + \dots + (P_k - C) \times \left[\sum_{i=0}^{k-1} F_i(Q) - F_k(Q) - (k-1)\right]. \tag{A.18}$$

The largest per unit profit in (A.18) is $(P_0 - C)$ and the largest probability of obtaining this profit is $[1 - F_k(Q)]$. Thus, an upper bound on $G(Q)$ is

$$G^U(Q) = k(P_0 - C)[1 - F_k(Q)]. \tag{A.19}$$

For any value of Q, the expected loss is

$$L(Q) = (F + C - P_{k+1})F_k(Q). \tag{A.20}$$

An upper bound on Q^*, denoted Q^U is one which satisfies $G^U(Q) \geqslant L(Q)$ which gives

$$k(P_0 - C)[1 - F_k(Q)] \geqslant (F + C - P_{k+1})F_k(Q). \tag{A.21}$$

Simplifying (A.21) gives

$$\frac{k(P_0 - C)}{F + C - P_{k+1}} \geqslant \frac{F_k(Q)}{1 - F_k(Q)}. \tag{A.22}$$

Since $F + C - P_{k+1} \geqslant k(P_0 - C)$, (A.22) becomes

$$1 \geqslant \frac{F_k(Q)}{1 - F_k(Q)} \tag{A.23}$$

which simplifies to $F_k(Q) \leqslant 0.5$. Thus, $Q^U \leqslant \mu_k = \mu_0 + kP_0/hb$.

From part 1, $E(\pi(h))$ is concave for any $Q < Q^C$. Thus, if $Q^U \leqslant Q^C$, then Q^* is in the concave region of $E(\pi(h))$ and $dE(\pi(h))/dQ = 0$ is a sufficient condition for optimality. Substituting for Q^U and Q^C gives $\mu_0 + kP_0/hb \leqslant \mu_0 + bh\sigma^2/P_0$

which simplifies to the condition $kP_0^2 \leqslant b^2h^2\sigma^2$ of the Lemma.

3. The smallest per unit profit in (A.18) is $(P_k - C)$ and the smallest probability of obtaining this profit is $[1 - F_0(Q)]$. Thus, a lower bound on $G(Q)$ is

$$G^{\mathrm{L}}(Q) = k(P_k - C)[1 - F_0(Q)]. \tag{A.24}$$

Again an upper bound on Q^*, denoted Q^{U} is one which satisfies $G^{\mathrm{L}}(Q) \geqslant L(Q)$ which gives

$$k(P_k - C)[1 - F_0(Q)] \geqslant (F + C - P_{k+1})F_k(Q). \tag{A.25}$$

Simplifying (A.25) gives

$$\frac{k(P_k - C)}{F + C - P_{k+1}} \geqslant \frac{F_k(Q)}{1 - F_0(Q)}. \tag{A.26}$$

Since $F + C - P_{k+1} \geqslant k(P_0 - C)$, $F + C - P_{k+1} \geqslant k(P_k - C)$ and (A.26) becomes

$$1 \geqslant \frac{F_k(Q)}{1 - F_0(Q)}. \tag{A.27}$$

Since $F_0(Q) \geqslant F_k(Q)$, $F_0(Q)$ can be written as $F_0(Q) = F_k(Q) + \Delta$ where $\Delta \geqslant 0$. Substituting in (A.27) gives $F_k(Q) \leqslant (0.5 - \Delta/2)$. Thus, $Q^{\mathrm{U}} \leqslant \mu_k = \mu_0 + kP_0/hb$ and the same argument dealing with the largest expected gain can be applied resulting in the condition $kP_0^2 \leqslant b^2h^2\sigma^2$ being sufficient.

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