Demand uncertainty and returns policies for a seasonal product: An alternative model

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Abstract

Marvel and Peck [International Economic Review 36 (1995) 691–714] considered the following seasonal-product problem: A manufacturer sets wholesale price ($p_w$/unit) and return credit ($r$/unit); the retailer then sets retailer price ($p_R$/unit) and order quantity ($Q$). How should the manufacturer set $p_w$ and $r$? Demand uncertainty consists of two components: “valuation” and “customers’ arrivals”. Our more realistic models reveal effects unobservable from Marvel–Peck’s. E.g.: (i) Setting $r > 0$ benefits the manufacturer much more than the retailer. (ii) “Valuation” (but not “customer-arrival”) uncertainty is imperative for the retailer; without it, the manufacturer can set $p_w$ and $r$ such that he reaps most of the profits. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Recently, Marvel and Peck [1] posed the following problem in the economics literature:

The manufacturer wholesales a product to the retailer for $p_w$/unit. Units of the product that the retailer could not sell may be returned to the manufacturer for a $r$/unit reimbursement. The retailer buys $Q$ units from the manufacturer and attempts to retail them to his customers at $p_R$/unit. For any given $p_w$ and $r$ set by the manufacturer, the retailer will set $p_R$ and $Q$ that maximize the retailer’s own expected profit. Recognizing this, how would (or should) the manufacturer set $p_w$ and $r$?

The model used by Marvel and Peck [1] to answer the above questions uses a couple of key assumptions. There are several different plausible variations of these assumptions; unfortunately, using a different variation leads to very different model behavior (i.e., conclusions on how the manufacturer and retailer would/should set their prices and quantities). The purposes of this paper are to: (i) present some realistic variations of the Marvel–Peck (hereafter “MP”) model; (ii) sketch the necessary solution methodologies for these variations; and (iii) use the numerical solutions obtained from the modified models to present alternative answers to the question posed by MP. It will be seen that the answers obtained with our models differ considerably from the answers obtained from MP’s model. This paper assumes familiarity with MP’s [1] paper, and uses as much as possible the same symbols defined there; also, their
background discussion of the problem will not be repeated.

1.1. Basic definitions

The “manufacturer” incurs a manufacturing cost of $c/unit for a product, he wholesales them to the “retailer” at $p_w/unit, and pays the retailer $r/unit for returned (unsold) units. The retailer buys $Q$ units from the manufacturer and retails to his “customers” at $p_R/unit. “$Q$” may be subscripted as “$Q_R$, “$Q_M$” or “$Q_1$” when it is desirable to specify the party (retailer, manufacturer, or the vertically integrated firm) making the $Q$-decision. Similarly, $\pi_M$, $\pi_R$ and $\pi_t$ denote the expected profit of, respectively, the manufacturer, the retailer and the integrated firm.

Random variables will be denoted by bold letters. In MP [1] the random demand is characterized by two independent random components:

(i) $v$: the valuation or “reservation price” retail customers place on the product; with density $f(\cdot)$, distribution function $F(\cdot)$, and support $[\underline{v}, \bar{v}]$. A customer will only purchase the product if the customer’s actual $v$-value equals or exceeds the retail price $p_R$.

(ii) $m$: the number of customers who arrive during a particular market cycle; with density $g(\cdot)$, distribution function $G(\cdot)$, and support $[\underline{m}, \bar{m}]$.

MP further assumed that each of the $m$ customers who arrives will purchase (exactly) one unit if $v \geq p_R$, and zero otherwise. MP considered three cases:

Case 1: $m$ deterministic but $v$ stochastic.
Case 2: $v$ deterministic but $m$ stochastic.
Case 3: both $v$ and $m$ are stochastic.

In addition to the above symbols defined in MP [1], we define

\begin{align*}
&d_i \quad \text{demand from a single customer } i \\
&D \quad \text{the total demand from the } m \text{ customers, with support } (D, \bar{D}); h(\cdot) \text{ and } H(\cdot) \text{ are } D's \text{ density and distribution function, respectively} \\
&C_r \quad \binom{n}{r}/[r!(n - r)!]
\end{align*}

$P_{b}(p_R) [1 - F(p_R)]$, the probability that a customer will buy at retail price $p_R$

\begin{align*}
P_n & = 1 - P_b
\end{align*}

For a standard normally distributed variable $z \sim N(0, 1)$, its density and distribution function are denoted by, respectively, $\varphi(\cdot)$ and $\Phi(\cdot)$.

For any random variable $x$, let $\mu_x = E(x)$, $\mu_k(x) = E(x - \mu_x)^k = k$th central moment of $x$, and $\sigma_x = \sqrt{\mu_2(x)}$.

The optimal value of a decision variable $x$ is denoted $x^*$. The optimum of an objective variable (say) $\pi$ optimized over one decision variable is denoted $\pi^*$; whereas $\pi^{**}$ is the value optimized over two decision variables.

1.2. Brief literature review

The seasonal product MP considered is closely related to the “newsboy problem” in the MS/OR literature. The problem is briefly defined below:

The stochastic demand $D$ of a single-period product (e.g., a newspaper) is represented by a given probability distribution. The “unit cost” to the retailer is fixed. The problem is to determine order quantity $Q$ (and perhaps retail price $p_R$).

Compared to the preceding structure, MP’s problem has (among others) two important additional features:

(i) While the newsboy problem (as in most MS/OR inventory models) characterizes demand uncertainty by a statistical demand distribution, MP’s model perceives two separate components affecting demand uncertainty: $v$ and $m$. Both MP’s and our studies show that these two uncertainty components affect optimal decisions quite differently.

(ii) Practically all papers in the huge MS/OR “newsboy” literature (see references in, e.g., Khouja [2] and Lau [3]) consider only how an integrated “manufacturer–retailer entity” would deal with the retail market; or, equivalently, they assume that $p_w$ and $r$ are fixed and hence the manufacturer has no decision variable. MP’s model considers the interaction between a manufacturer and a retailer — which
are now recognized as two separate entities. This adds two more decision variables \( p_w \) and \( r \) to the problem.

Pasternack [4] is the first to consider manufacturer/retailer interaction in the newsboy context. He assumed that the retail price \( p \) is fixed by the market, but the retailer is entitled to return to the manufacturer all unsold units for a refund of Sr/unit. Defining “channel efficiency” as

\[
E_r = \frac{(\pi^{\#}_M + p_R^{\#})}{\pi^R}.
\]

Pasternack showed that: (i) a manufacturer’s decision of \( r = 0 \) will lead to “\( E_r < 1 \)”; (ii) the manufacturer can enforce “\( E_r = 1 \)” by setting a “channel-coordinating” \((w, r)\) (with \( r > 0 \)) such that the following relationship is satisfied:  

\[
(p_R - c)(p_R - r) = (p_R - w)p_R.
\]

He also pointed out that: (i) there exists infinite sets \((w, r)\)-values that satisfy (2); and (ii) different channel-coordinating \((w, r)\)-values lead to different ways of splitting \( \Theta^\# \) between the manufacturer and the retailer. However, Pasternack [4] did not address the mechanism for regulating this profit split between the manufacturer and the retailer (or, in other words, how does or should the manufacturer pick one implementable pair of \((w, r)\)-values out of the infinite possibilities). Essentially, MP not only attempted to answer the above question, but they also did so for an extended model whose stochastic demand is decomposed into \( v \) and \( m \).

Pricing between manufacturers and retailers has been studied by many (see, e.g., [5,6] and their references), but mostly in the context of non-seasonal products with deterministic (though possibly price-dependent) demands. Recently Padmanabhan and Png [7] also considered manufacturer/retailer interactions in a seasonal product with uncertain demand. Two fundamental differences between Padmanabhan–Png’s and the newsboy–MP models are: (i) Padmanabhan–Png’s model assumes that the demand uncertainty is resolved before the retailer sets \( p \) (and hence before the selling begins), whereas the newsboy–MP models assume that demand uncertainty exists throughout the selling season; (ii) market uncertainty is represented by a (discrete) Bernoulli distribution in Padmanabhan–Png’s model, but by more comprehensive formats in the newsboy–MP models. Similar differences apply between the newsboy–MP models and the other models cited in [7].

1.3. Overview

Section 2 shows that, under the stochastic-\( v \) Case 1, how a different interpretation of \( v \) will lead to very different formulations and solutions. Section 3 shows that, under the stochastic-\( m \) Case 2, how a different assumption on the manufacturer–retailer relationship will also lead to very different formulations and solutions. Section 4 gives a summary and briefly discusses fruitful extensions.

2. The assumption on whether different customers have different \( v \)

There are at least two alternatives in interpreting the statement of “\( P_b(\ p_R) = [1 - F(\ p_R)] \)” defined in Section 1.1:

**Alternative A:** At the end of the market period the single realized value of \( v \) becomes known, and (as stated in [1, p. 699]) “all buyers have the same reservation price \( v \)”. \( F(\cdot) \) governs the realization of this universal \( v \)-value.

**Alternative B:** Different customers have different actual \( v \)-values — whose distribution follows \( F(\cdot) \). The probability that any given customer will buy is \( P_b = [1 - F(\ p_R)] \). Given any \( P_b \) (assume \( 0 < P_b < 1 \), some customers (roughly \( 100P_b \) percent of them) will end up buying, others will not.

To illustrate clearly the difference between the two alternatives, assume that: (i) there are \( m = 1000 \) customers; (ii) \( v \sim \mathcal{U}(0, 1) \) (i.e., \( F(\cdot) = v \), like the one used in MP’s numerical illustration); and (iii) \( p_R = 0.8 \) (pre-set). Under Alternative 1, \( \text{Prob}(v < p_R) = 0.8 \), i.e., at the end of the period there is a 0.8 probability that \( D = 0 \) unit, and a 0.2 probability that \( D = m = 1000 \) — note that since \( m \) has been
assumed deterministic, there is no way for $D$ to be anything but either 0 or 1000. Under Alternative B, $P_b = 0.2$, hence the actual $D$ observed at the end of the period can have any value between 0 and 1000, the probability of observing $\{D = r \text{ units}\}$ is given by the binomial probability $[1000\, C_r](0.2)^r(0.8)^{1000-r}$. For example, in a CD/records store, if Alternative A applies, the observed demand for each of the many titles carried would be either 0 or a very high value (equal to the number of potential buyers of the title that visited the store). If Alternative B applies, the observed demands of different titles would be a more continuous range of values. MP's [1] model uses Alternative A.

It appears that Alternative B is at least as plausible as Alternative A. This section shows that substituting Alternative B into MP's model leads to very different mathematical formulations and results for Case 1 (where only $v$ is stochastic); also, many of these results appear to be at least as plausible as the Alternative A results.

2.1. Using Alternative A in Case 1 (no arrival uncertainty)

A convenient consequence of using Alternative A is that, as MP [1, p. 694] stated: “if $m$ is known, the (deterministic-$m$) problem can be reduced to that of dealing with a representative (customer)”. Since the retailer’s problem is to maximize his own expected profit $\pi_R$, considering only one representative customer gives (as stated in the expression above MP's equation [1, Eq. (2) p. 694])

$$\pi_{R1} = \max \left[ p_R \left( 1 - F(p_R) \right) + F(p_R)r - p_w, 0 \right] \quad (3a)$$

$$= \max \left[ p_R P_b + rP_a - p_w, 0 \right] \quad (3b)$$

(adjusting notation $P_b$ and $P_a$). Thus, the decision is: “If $\pi_{R1} > 0$, order; otherwise, do not order”. To illustrate, consider a simplified situation where $p_R$ and $p_w$ are already fixed. Assume $m = 1000$, $p_R = 4$, $P_b = 0.6$, $r = 1$, $c = 1$. (4)

If $p_w = 2.9$, then substituting this $p_w$-value and the values in (4) into (1) gives

$$\pi_{R1} = \max \left[ (4 \times 0.6 + 1 \times 0.4 - 2.9), 0 \right]$$

$$= \max \left[ -0.1, 0 \right] = 0.$$ 

Hence,

when $p_w = 2.9$, $Q^* = 0$ and Retailer’s Aggregate Profit ($\pi_{R1}$) = 0. $(5a)$

In contrast, if $p_w = 2.7$, Eq. (1) becomes

$$\pi_{R1} = \max \left[ +0.1, 0 \right] = 0.1;$$

hence,

when $p_w = 2.7$, $Q^* = 1000$ and $\pi_R = 1000\pi_{R1} = 100$. $(5b)$

Simply put, the decision is: “$Q^* = 0$ if $p_w \geq 2.8$; but $Q^* = 1000$ units if $p_w < 2.8^*$. $Q^*$ abruptly jumps from 0 to 1000 when $p_w$ decreases from 2.8 to (say) 2.7999 — this behavior comes from the logical and convenient consequence of ignoring the important factor $m = 1000$ when one adopts Alternative A.

The next subsection examines Alternative B: different customers have different realized $v$-values.

2.2. Using Alternative B in Case 1 (no arrival uncertainty)

This study assumes that $m$ is sufficiently large so that $D$ under Alternative B can be approximated by a normal distribution. This large $m$ assumption is consistent with the current scenario of a manufacturer using a retailer. If $m$ is small (as when an industrial manufacturer sells to a few industrial users), it is more likely that a retailer is not used (it should, however, be emphasized that a small $m$ formulation requires only a straightforward extension of the following approach).

Under the Alternative B interpretation of $F(\cdot)$, the “demand per customer” $d_i$ is a “Bernoulli distributed” random variable with $\mu_d = P_b$, and the mean and standard deviation of total demand $D$ are (see, e.g., [8])

$$\mu_D = mP_b = m\mu_d, \quad \sigma_D = \sqrt{mP_b P_a}. \quad (6a)$$

Substituting $m = 1000$ and $P_b = 0.6$ for the current case into (6a) gives

$$\mu_D = 1000 \times 0.6 = 600,$$

$$\sigma_D = \sqrt{1000 \times 0.6 \times 0.4} = 15.492. \quad (6b)$$

Also, when $mP_b$ and $mP_a$ are both greater than 5, $D$ is very closely approximated by a continuous normal distribution (central limit theorem). Hence,
if the retailer orders \( Q \) units, the retailer’s expected profit function is

\[
\pi_R = -p_w Q + \int_0^Q \left[ p_R D + r(Q - D) \right] h(D) \, dD
+ \int_Q^P p_R Q h(D) \, dD,
\]

(7)

where \( Q \) is a decision variable. Eq. (7) is the classical “newsboy problem”, and solving for “\( d\pi_R/dQ = 0 \)” gives

\[
H(Q^*) = \frac{(p_R - p_w)}{(p_R - r)} \quad \text{or} \quad Q^* = H^{-1}\left[\frac{(p_R - p_w)}{(p_R - r)}\right].
\]

(8)

\( H(Q^*) \) is commonly known as the “service level” (hereafter “SL”), i.e., the likelihood that not all the merchandise (i.e., all the \( Q^* \) units) are sold at the end of the season.

Since \( H(\cdot) \) is normal, we also have

\[
H^{-1}(y) = \mu_D + z\sigma_D,
\]

where \( z = \Phi^{-1}(y) \) (defined in Section 1.1).

(9)

Combining Eq. (4), (8) and (9) and using a standard normal table, we have

\[
Q^* = 600 + 15.49 \times [\Phi^{-1}(0.3666)] = 594.7
\]

when \( p_w = 2.9 \),

\[
Q^* = 600 + 15.49 \times [\Phi^{-1}(0.4333)] = 597.4
\]

when \( p_w = 2.7 \).

Combining Eq. (7) and (8) and simplifying, it can also be shown that, for any given \( p_R \) and \( p_w \), when the retailer maximizes his \( \pi_R \) by ordering \( Q^* \) prescribed by Eq. (8), his resultant expected profit under any \( h(\cdot) \) will be

\[
\pi_R^*(p_R) = (p_R - r) \int_0^{Q^*} D h(D) \, dD
\]

(\( Q^* \) given in Eq. (8)).

(10)

If \( h(\cdot) \) is normal, combining Eqs. (6a) and (8), (10) and the results in [9] gives

\[
\pi_R^*(p_R) = \mu_D (p_R - p_w) - \sigma_D (p_R - r) \Phi(z^*),
\]

where \( z^* = \Phi^{-1}\left[\frac{(p_R - p_w)}{(p_R - r)}\right] \).

(11)

In Eq. (11), the first term \( \mu_D (p_R - p_w) \) is the expected profit the retailer can extract from his market if he has perfect market information (or if \( \sigma_D = 0 \)), the second term amounts to the “expected value of perfect market information”.

The following tabulation (obtained with Eqs. (4), (8) and (11)) shows the “smooth” variation of \( \pi_R^a(p_R) \) with \( p_w \) under Alternative B (the values of the other parameters are as given in (4)):

<table>
<thead>
<tr>
<th>( p_w )</th>
<th>( Q^* )</th>
<th>( \pi_R^a )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.50</td>
<td>600.0</td>
<td>881.46</td>
</tr>
<tr>
<td>2.70</td>
<td>597.4</td>
<td>761.72</td>
</tr>
<tr>
<td>2.90</td>
<td>594.7</td>
<td>642.50</td>
</tr>
<tr>
<td>3.30</td>
<td>588.7</td>
<td>405.77</td>
</tr>
<tr>
<td>3.70</td>
<td>580.1</td>
<td>171.84</td>
</tr>
<tr>
<td>3.90</td>
<td>571.6</td>
<td>56.55</td>
</tr>
<tr>
<td>3.95</td>
<td>567.0</td>
<td>28.07</td>
</tr>
</tbody>
</table>

(12)

Incidentally, at \( p_w = 2.7 \), if one orders \( Q = 1000 \) as prescribed by Alternative A, Eq. (5b) gave \( \pi_R = 100 \), which is considerably less than the corresponding \( \pi_R^a = 761.72 \) given in (12) for Alternative B. Note also the drastically reduced but yet positive \( \pi_R^a \) when \( p_w \) moves from 2.50 to 3.95 under Alternative B.

The above example (particularly the numerical approach) clarifies: (i) the difference between the MP’s \( \pi_{R1} \)-criterion (Eq. (3)) and our modified \( \pi_R \)-criterion (Eq. (7)); and (ii) a basic characteristic of the current manufacturer–retailer problem that is unfortunately concealed by Eq. (3). \( \pi_R \) (or \( \pi_{R1} \)) of Eq. (3) suggested that when \( p_w \) exceeds $2.8, the retailer will set \( Q = 0 \) because his \( \pi_{R1} \) is 0 (refer to (5a)); in other words, the model contains an internal mechanism to enforce a relatively low \( p_w \) (and hence a relatively high retailer’s profit share). \( \pi_R \) of Eq. (7) indicates that even when the manufacturer raises \( p_w \) to 3.95 (and beyond, as long as it is lower than \( p_R = 4 \), the retailer’s \( \pi_R \) will still exceed 0 (see (12)). However, the retailer may still set \( Q = 0 \) if he feels that (say) \( \pi_R^a = 28.07 \) (see (12)) is too low for him; but then the reason is not because his \( \pi_R \) is 0, but because he has other better things to do. However, if he unfortunately does not, then he will have to “slave” for even a very meager cut of the total “channel profit” that the product earns from the consumer market. In other words, in contrast to Eq. (1), \( \pi_R \) of Eq. (7) reveals that there is no internal
mechanism to enforce a “reasonable” retailer’s profit share; instead, an additional “alternative opportunity” factor is needed to enforce a relatively high retailer’s share. A well-known analogy is the way profit is split between the landowners (aristocrats) and the users (peasants) in many economies in the past and present. Similarly, urban street vendors (of, e.g., newspapers and imported or “fancy” ice-creams/soft-drinks) in many of today’s developing countries get a much smaller profit cut compared to either (i) the manufacturers/importers; or (ii) similar vendors in the developed countries. These vendors will get a larger profit share only when alternative employment opportunities force the manufacturers/importers to raise the retailers’ share in order to retain them. The next section (on Case 2) will consider the explicit imposition of a minimum required retailer’s profit.

2.3. Solution of the no-arrival-uncertainty problem with alternative B assumption

To clarify the implication of our Alternative B interpretation of the statement “\( P_v( p_R) = [1 - F( p_R)] \)”, the preceding subsections considered situations with given \( p_w \) and \( p_R \). This restriction is now removed for solving MP’s Case 1 problem.

When \( p_R \) as well as \( Q_R \) are the retailer’s decision variables, then by combining Eqs. (6a) and (11), the retailer’s problem can be written as

\[
\max_{p_R} \pi_R^R(p_R) = m[1 - F(p_R)](p_R - p_w)
\]

\[
- \sqrt{m[1 - F(p_R)]}F(p_R)(p_R - r)
\times \varphi\{\Phi^{-1}[(p_R - p_w)/(p_R - r)]\}.
\]

(13)

Note that the \( Q_R \)-optimization step (Eq. (8)) is already imbedded in Eq. (13). After finding \( p_R^* \) from (13), it can be substituted into Eq. (8) to find \( Q_R^* \).

The manufacturer’s expected profit \( \pi_M \) consists of: (i) the initial profit from selling \( Q_R^* \) units to the retailer; (ii) less expected reimbursement for the retailer’s returns. Thus, the manufacturer’s problem can be stated as

\[
\max_{p_w, r} \pi_M(p_w, r) = (p_w - c)Q_R^* - \int_{D} r(Q - D)h(D) \, dD,
\]

where for any given \((p_w, r)\)-values, the associated \( Q_R^* \) is obtained by solving Eq. (13) (note the big difference between (14) and MP’s (7) [1, Eq. (7), p. 695]). Although the above problem cannot be solved analytically, Appendix A explains how it can be easily solved numerically.

MP’s numerical examples assumed \( v \sim U[\frac{1}{2} - z, \frac{1}{2} + z] \) and \( m \sim U[\frac{1}{2} - \beta, \frac{1}{2} + \beta] \), these uniform distributions are generalized here by scaling factors \( k_v \) and \( k_m \):

\[
v \sim U[k_v(\frac{1}{2} - z), k_v(\frac{1}{2} + z)] \quad \text{and} \quad m \sim U[k_m(\frac{1}{2} - \beta), k_m(\frac{1}{2} + \beta)],
\]

i.e.,

\[
F(v) = \frac{1}{2} + \left( \frac{[v/k_v] - 1}{2}\right),
\]

\[
G(m) = \frac{1}{2} + \left( \frac{[m/k_m] - 1}{2}\right).
\]

(15a)

In typical retail situations \( k_m \gg k_v \); e.g., a toy retails around $5, and the number of potential customers that will visit the toy store during the toy’s selling season is around 5000 (this scaling consideration was irrelevant under MP’s Alternative A because only one representative customer needed to be considered). Besides uniformly distributed \( v \) and \( m \), we also obtained numerical solutions for normally and beta distributed \( v \)s and \( m \)s to ensure that our reported observations are not restricted to a narrow distributional assumption for \( v \) and \( m \).

The last four columns in Table 1 pertain to the vertically (i.e., manufacturer–retailer) integrated firm, who has to determine only retail price \( p_I \) (corresponding to \( p_R \) in the non-integrated situation) and production quantity \( Q_I \) to maximize the firm’s expected profit \( \pi_I \). Similar to Eq. (13), \( p^*_I, Q^*_I \) and \( \pi^*_I \) are obtained by solving

\[
\max_{p_I} \pi_I(p_I) = m[1 - F(p_I)](p_I - c)
\]

\[
- \sqrt{m[1 - F(p_I)]}F(p_I)p_I \varphi\{\Phi^{-1}[(p_I - c)/p_I]\}.
\]

(16)

Note that the above formulation is again different from MP’s vertically integrated-firm formulation [1, pp. 702–703].
Table 1
Solutions to the “no arrival uncertainty” problem with $k_v = 1$, $k_m = 1000$ (i.e., $m = 500$), $c = 0.1$ and selected $z$-values necessary to illustrate trends (for each variable or column, the minimum or maximum value is in bold print)

<table>
<thead>
<tr>
<th>$x$</th>
<th>Manufacturer $p^*_v$</th>
<th>Retailer $p^*_r$</th>
<th>Vertically integrated firm $E_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.002</td>
<td>0.494</td>
<td>0.493</td>
<td>145.4</td>
</tr>
<tr>
<td>0.006</td>
<td>0.484</td>
<td>0.481</td>
<td>175.0</td>
</tr>
<tr>
<td>0.010</td>
<td>0.474</td>
<td>0.399</td>
<td>175.8</td>
</tr>
<tr>
<td>0.015</td>
<td>0.463</td>
<td>0.394</td>
<td>172.9</td>
</tr>
<tr>
<td>0.020</td>
<td>0.450</td>
<td>0.016</td>
<td>168.7</td>
</tr>
<tr>
<td>0.040</td>
<td>0.398</td>
<td>0.000</td>
<td>146.3</td>
</tr>
<tr>
<td>0.060</td>
<td>0.344</td>
<td>0.000</td>
<td>120.8</td>
</tr>
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<td>0.080</td>
<td>0.324</td>
<td>0.000</td>
<td>91.3</td>
</tr>
<tr>
<td>0.100</td>
<td>0.343</td>
<td>0.094</td>
<td>78.3</td>
</tr>
<tr>
<td>0.140</td>
<td>0.368</td>
<td>0.233</td>
<td>64.6</td>
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<td>0.180</td>
<td>0.384</td>
<td>0.258</td>
<td>57.8</td>
</tr>
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<td>0.220</td>
<td>0.404</td>
<td>0.270</td>
<td>53.9</td>
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<td>0.425</td>
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<td>0.300</td>
<td>0.444</td>
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<td>50.3</td>
</tr>
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<td>0.494</td>
<td>0.386</td>
<td>49.3</td>
</tr>
<tr>
<td>0.480</td>
<td>0.532</td>
<td>0.415</td>
<td>49.7</td>
</tr>
</tbody>
</table>

2.4. Ranges of parameters of numerical solutions obtained in this study

Numerical solutions were obtained for various combinations of the following parameter values: $c = 0.0-0.4^2$ in steps of 0.1; $k_v = 10^n$ with $n = 1-3$ (see Eq. (15)); $k_m = 10^n$ with $n = 3-5$; $x = 0.005-0.490$ in steps of 0.005. Recall that $\beta = 0$ in this section. Table 1 presents solutions for the case of: $k_v = 1$, $k_m = 1000$ (i.e., deterministic $m = 500$), $c = 0.1$, uniform $F(v)$ and selected $z$-values. Solutions were also obtained (not shown) for normal and beta as well as uniform ($v$).

Table 1 and all other tables in this paper are constructed with the following approach: we ascertain the patterns of the problem’s behavior using numerical solutions from a large number of different combinations of parameter values, a very small subset of these solutions are then tabulated here to illustrate the patterns so ascertained. Thus, the patterns we will be generalizing below based on Table 1 have been confirmed with numerical solutions from many different combinations of parameter values and $F(v)$.

2.5. Patterns of behavior of the optimal solutions: Vertically integrated firm

Start with the case of $x = 0$ (i.e., $v = 1$ is now deterministic) — hence $p_k = p_l = \frac{1}{2}$ and (buying probability) $p_b = \frac{1}{2}$ are fixed. Eq. (6a) gives $\mu_D = 250$ and $\sigma_p = 11.18$. For the vertically integrated firm, the corresponding formula for Eq. (8) is $SL^* = (p_l - c) / p_l$ and $Q^*_l = H^{-1}(SL^*)$.

With $c = 0.1$ as depicted in Table 1, $SL^* = 0.8$, $z = \Phi^{-1}(0.8) = 0.842$; hence, $Q^*_l = H^{-1}(0.8) = \mu_D + z\sigma_p = 250 + 0.842 \times 11.18 = 259.41$. Similarly, (11) gives $\pi^*_l = 98.4$.  

---

2 We investigated the situation of $c = 0.0$ because it is the (only) situation analyzed in detail in [1, Section 4]. This zero-cost situation produces solutions exhibiting special patterns not shared by solutions from situations with $c > 0$. We feel that situations with $c > 0$ are at least as common as a situation with $c = 0$, and for brevity sake only characteristics shared by “$c > 0$” situations will be reported in this paper.
Now consider \( x > 0 \). Substituting \( F(v) \) from (15b) into \( P_b \)'s definition (i.e., \( P_b = 1 - F(\rho_R) \)) and differentiating gives \( dP_b/d\rho_R = -1/(2xk_v) \). That is, when \( x \) is non-zero but very small, a very small reduction in unit retail price (either \( p_t \) or \( p_R \)) leads to a very large increase in \( P_b \) — and hence \( \mu_D \). Therefore, the \( \pi^*_R \)-column in Table 1 shows that as \( x \) first becomes non-zero, the initial ability to “manipulate prices” leads to a rapid rise in \( \pi^*_R \). This justifies the larger \( Q^*_R \), which in turn produces the increasing \( \pi^*_R \). However, as \( x \) becomes larger \(( > 0.015 \) in Table 1), the necessary \( p_t \)-reduction is not fully compensated by the desirable \( P_b \)-increment, and \( \pi^*_R \) “peaks out”. Eventually, when \( x \) is too large \(( > 0.14 \) in Table 1), it is not even worthwhile to strive for a high \( P_b \) by maintaining a low \( p_t \), and \( p_R^* \) increases.

Note that \( \pi^*_R \) at \( x > 0 \) is never less than \( \pi^*_R = 98.4 \) computed above for \( x = 0 \). However, the benefit of a non-zero \( x \) to \( \pi^*_R \) peaks at \( x = 0.015 \). This non-monotonic effect of \( v \)'s \( x \) is largely due to the fact that in MP’s (and hence our) model, for a given \( \rho_R \), a change in \( x \) (causing a change in \( P_b \)) causes both \( \mu_D \) and \( \sigma_D \) to change (see (6a)), and the dependence between \( \mu_D \)'s and \( \sigma_D \)'s changes is complex.\(^3\) A useful extension of the current model is to develop a way to specify valuation uncertainty (or a related attribute) such that \( \mu_D \) and \( \sigma_D \) can be controlled independently when one changes the valuation-uncertainty magnitude.

2.6. Patterns of behavior of the optimal solutions: manufacturer versus retailer

For the non-integrated situation, MP showed that under Alternative A, \( r^* = 0 \) always. In contrast, Table 1 shows that under Alternative B, \( r^* \neq 0 \) typically.

\(^3\) For example, if \( v \sim U[0.5 - x, 0.5 + x] \) as assumed in MP [1], straightforward manipulations give

\[
\mu_D = m[(p_t - 0.5)/2]
\]

and \( \sigma^2 = m[(p_t - 0.5)^2]/4 \).

Thus, if \( p_t > 0.5 \), then as \( x \) increases, both \( \mu_D \) and \( \sigma_D \) increase. However, if \( p_t < 0.5 \), then as \( x \) increases, \( \sigma_D \) increases but \( \mu_D \) decreases.

A more interesting phenomenon is revealed by the \( \pi^*_{R*} \) column. Together with the next section, the combined results of this paper will show that the retailer “earns his way” only when he can: either (i) shoulder the risk of bearing the cost of unsold merchandise; and/or (ii) properly manipulate the retail price (which is beyond the manufacturer’s control in the current context). The manufacturer is interested in maximizing his own expected profit, which in the current context implies that he is not interested in sharing his profit with the retailer unless he is “forced to do so”.

Therefore, when possible, the manufacturer will choose not to share with the retailer the risk associated with unsold merchandise, and this is achieved by granting a large \( r \) to the retailer. Thus, in the first few rows of Table 1, the manufacturer sets \( r \) close to \( p_w \), and there is also little retail-price manipulation to be done (since \( x \) is small), hence the retailer’s profit share (\( \pi^*_{R*} \)) is very small. An important and counter-intuitive point to note is a large \( r \) is often not beneficial to the retailer at all! If \( p_w \) were fixed (probably a common implicit assumption), a larger \( r \) is indeed more beneficial to the retailer. However, if \( p_w \) is variable, then a manufacturer can often reduce the retailer’s profit share by matching a larger \( r \) with a suitably larger \( p_w \).

Section 3 below will illustrate a stronger example of this effect for Case 2 (where \( v \) is deterministic).

When one moves down Table 1, the larger \( x \) (i.e., price manipulability) itself leads to a higher retailer’s profit. At the same time, the desirability of inducing the retailer to order more leads the manufacturer to: (i) reduce \( p_w \), but also (ii) compensate himself somewhat against the reduced \( p_w \) by reducing \( r \). The \( r \) reduction then also leads to a higher retailer’s profit. The combination of these two factors brings \( \pi^*_{R*} \) to a peak (at \( x = 0.08 \) in Table 1). Towards the bottom of Table 1 (when \( x \) is too large), as was true for the integrated firm, it is not worthwhile for the “players” to strive for a high \( P_b \) by maintaining a low retail price, hence the manufacturer increases his \( p_w \), which in turn enables him to increase \( r \) (in an attempt to get a larger profit share). Thus, the retailer is again denied by the manufacturer the opportunity to bear much risk on unsold merchandise; however, a larger \( x \) means that there is now much price manipulation.
to do, therefore $\pi_R^e$ is much higher at $z = 0.48$ than at $z = (\text{say}) 0.015$ (when the ratio $r^*/p_w^*$ was much lower, i.e., when the retailer gets to bear most of the risk of unsold merchandise). In other words, for the lower, i.e., when the retailer gets to bear most of the profit opportunity for the retailer (due to a low $r$) at $z = 0.015$ is less influential than the price-manipulating profit opportunity at $z = 0.48$.

The last column in Table 1 gives $E_t$ defined in (1). $E_t$ deteriorates as $z$ increases; it is very close to 1 when $z$ is small — the reason for this phenomenon will become very clear after Section 3.

3. Case 2: No valuation uncertainty

This section shows how the solution of Case 2 is affected when one (in contrast to MP’s approach) explicitly addresses the mechanism necessary for the manufacturer to manipulate the retailer’s purchase-quantity decision. This factor is unrelated to the different interpretations of $v$ considered in Section 2. In this section, $v = V$ (deterministic).

3.1. Summary of MP’s statements, approach, results and summary of our differences

With $V$ deterministic, MP stated that the retailer’s problem is

$$\max_{q} \pi_R = \Pr \left[ \int_{m}^{Q} x \, dG(x) + \int_{Q}^{\infty} Q \, dG(x) \right]$$

$$- p_w Q + r \int_{m}^{Q} (Q - x) \, dG(x). \quad (17)$$

However, in MP’s assumed scenario the retailer never gets to solve this problem because: (i) Case 2 explicitly assumes that $\Pr = V$ is fixed by the market; but (ii) in addition, MP assume that the retailer will order the $Q^*$ units prescribed by the manufacturer (see below).

MP [1, p. 697] stated that “In equilibrium... the manufacturer sets $r = V$ and $p_w = V$... given the manufacturer’s choice of

$$V = r^* = p_w^*, \quad (18)$$

... the manufacturer’s problem (is):

$$\max_{q} \pi_M = V \left[ \int_{m}^{Q} x \, dG(x) + \int_{Q}^{\infty} Q \, dG(x) \right] - cQ \quad (19)$$

and solving for “$\delta \pi_M/\delta Q = 0$” gives the $Q^*$ that maximizes $\pi_M$ (not $\pi_R$) as

$$Q^* = G^{-1}[(V - c)/V]. \quad (20)$$

MP also stated [1, p. 697] that “we assume that the manufacturer can induce the retailer to carry $(Q^*\text{ given in Eq. (20)})$”. The actual mechanism that could induce the required retailer’s behavior was not specified or conjectured.

The basic differences between MP’s and our results are that, under our approach: (i) $p_w^*$ differs from $V$ (contrasting (18)); instead, $c < p_w^* < V$; (ii) $r^*$ differs from $V$ (contrasting (18)); instead, $0 < r < p_w^*$; (iii) an explicit mechanism is given for the retailer to determine his $Q^*$; (iv) the key to our approach is the explicit consideration of the actual magnitude of the retailer’s profit (assumed to be 0 in MP’s model).

3.2. An explicit mechanism to coordinate the manufacturer’s and retailer’s independent decisions

Consider the following numerical example:

$$m \sim U(0, 100), \quad \text{hence} \quad G^{-1}(p) = 100p; \quad c = 1; \quad \text{deterministic} \ p_R \ (\text{or} \ V) = 4. \quad (21)$$

A useful relationship to remember when $m \sim U(a, b)$ is

$$\int_{m}^{Q} x \, dG(x) = (Q^2 - a^2)/[2(b - a)]. \quad (22)$$

Consider first an integrated firm. Eqs. (8), (10) and (22) give (note that $r = 0$ and $p_w = c$ for an integrated firm):

$$\text{SL}^* = H(Q^*) = (V - c)/V; \quad \text{hence} \quad Q^* = G^{-1}[(4 - 1)/4] = G^{-1}(0.75) = 75, \quad (23a)$$

$$\pi^*_r = V \int_{m}^{Q^*} m \, dG(m) = 4 \times [75^2/200] = 112.5. \quad (23b)$$
Consider now the non-integrated system. If the retailer is allowed to solve his problem (Eq. (17)) after the manufacturer sets $p_w$ and $r$, then “$d\pi R/dQ = 0$” leads to (similar to (8))

$$Q^*_R = G^{-1}[(V - p_w)/(V - r)]$$

(note that $p_R = V$ in Eq. (17)).

(24)

However, since $Q^*_R$ in (23a) brings in the maximum possible system profit, we assume tentatively that the manufacturer will want to induce $Q^*_R$ to equate $Q^*_R$ ($= 75$, from (23a)). This can be achieved by setting $p_w$ and $r$ to satisfy

$$(V - p_w)/(V - r) = (V - c)/V \quad \text{or}$$

$$r^* = V(p_w - c)/(V - c).$$

(25)

Eq. (25) comes from combining (23a) and (24), and is identical to (2) due to Pasternack [4].

Consider now the implications of (25). Among the infinite pairs of feasible $(p_w, r)$-values satisfying condition (25), assume $p_w = 3.99$, hence (25) gives $r = 3.9866$. Substituting this $r$ into (10) and (22) gives $\pi^*_M = (4 - 3.9866) \times [75^2/200] = 0.375$. Similarly, substituting this $p_w$ and $r$ into (14) and (22) gives $\pi^*_M = (3.99 - 1) \times 75 - 3.9866 \times [(75 \times 0.75) - (75^2/200)] = 112.125$. Note that $(\pi^*_M + \pi^*_R) = 112.5 = \pi^*_R$ (see (23b)); i.e., $E_r = 1$. Realizing this, we could have computed $\pi^*_M$ as

$$\pi^*_M = (\pi^*_R - \pi^*_R) = (112.5 - 0.375)$$

(26)

which is much easier than using (14) and (22) (this is an important point for an easy solution to (32), to be considered in the next subsection).4

However, the manufacturer can increase $p_w$ further to 3.999 and still induce the retailer to order $Q^*$ by granting (according to (25)) $r = 3.99866$; then (with the same procedures used in the preceding paragraph) $\pi^*_R = 0.0375$, $\pi^*_M = 112.4125$, and still $E_r = 1$. The above example illustrates that the manufacturer can set $p_w$ as close to $V$ as he wants and yet can induce the “right” behavior from the retailer — a direct result from (2) (due to Pasternack [4]) and recognized by MP in their Footnote 6. However, what we want to emphasize is that when $p_w \approx V$, $\pi^*_R$ becomes so small that the retailers could not possibly stay in business (unless he receives a fixed “handling fee”/unit, which then nullifies the entire manufacturer/retailer pricing problem under consideration). Recall that a less extreme example of this phenomenon appeared earlier in the first few rows of Table 1 when $x$ (i.e., retail-price manipulation opportunity) is very small. In the current section, there is no retail-price manipulation for the retailer to do, and naturally the manufacturer wants to shoulder more risk (which brings higher expected profit); hence the very high $r$ (but always slightly lower than $p_w$ as required by Eq. (25)). The retailer is left with negligible risk, hence negligible profit.

Both Pasternack and MP assume that the above issue is outside the scope of the problem; instead, we propose below that the problem can be made more meaningful by simply adding a constraint guaranteeing a minimum profit that keeps the retailer in business.

3.3. Our formulation and solution of the Case-2 “Stochastic-m” problem

Assume now that the retailer will “stay in business” (or, more realistically, “be willing to carry the product”) only if $\pi^*_R \geq T$ (a minimum target). Thus, the manufacturer’s problem is to maximize $\pi^*_M$ while making sure that “$\pi^*_R \geq T$”. Combining (10), (14) and (25), this problem can be stated as (where (27c) is a combination of (24) and (25)):

$$\max_{p_w, r} \pi_M(p_w, r) = (p_w - c)Q^*_R - \int_{P_c}^{Q^*_R} r(Q - m) \, dG(m)$$

(27a)

subject to

$$(V - r)\int_P^{Q^*_R} m \, dG(m) \geq T,$$

(27b)

$$Q^*_R = G^{-1}[(V - c)/V].$$

(27c)

\[4\] Earlier, before proposing (25), we assumed tentatively that the manufacturer wants to induce $Q^*_R$. It is now clear that this assumption is valid, since the manufacturer could not raise his own $\pi^*_R$ by having $E_r < 1$. His best strategy is to maximize $\pi_1$ and $E_1$, and then to get as big of a cut as possible from $\pi^*_R$. 


For any random variable $x$ with distribution function $H(x)$, define $x$’s “partial expectation” as

$$E_p(x; H, Q) = \int_x^Q x \, dH(x).$$  \hspace{1cm} (28)

Eq. (27b) can then be rewritten as $(V - P)E_p(m; G, Q^*) \geq T$. Given the discussion in the preceding subsection, it is obvious that for any given $T$, the optimum of formulation (27) will occur when $(V - P)E_p(m; G, Q^*) = T$ in (27b), i.e., no more but no less than $T$ is given to the retailer. Noting that $E_p(m; G, Q^*)$ is not a function of $r$ or $p_w$ (see (27c) and (28)), the following solution to (27) can be obtained using (in the order given) (27c), (27b), (25), (23b) and (26):

$$Q^*_r = G^{-1}[(V - C)/V],$$
$$r^*_s = V - [T/E_p(m; G, Q^*)],$$
$$p^*_w = c + r^*_s - (r^*_s c)/V),$$
$$\pi^*_r = V E_p(m; G, Q^*), \quad \pi^*_R = T,$$
$$\pi^*_M = \pi^*_R - \pi^*_R.$$

For example, if $T = 56.25$, substituting the parameters in (21) into (29a) gives

$$Q^*_r = Q^*_T = 75, \quad E_p(m; G, Q^*) = 28.125,$$
$$r^*_s = 2, \quad p^*_w = 2.5,$$
$$\pi^*_r = 4 \times 28.125 = 112.5, \quad \pi^*_R = 56.25 = T,$$
$$\pi^*_M = 56.25$$

(to cross-check, (10) gives $\pi^*_R = (4 - 2) \times 28.125 = 56.25 = T$). Specifying a different $T$ will lead to different numerical answers in (29b).

We showed above that: (i) $p_w < V$; and (ii) $r^*_s < p^*_w$. In contrast, MP’s numerical solutions to the same problem are: $r^*_s = p^*_w = 4$ (by model’s definition), $Q^*_R = 75$ from (20); $\pi^*_R = 112.5$ from (23b) or (19); $\pi^*_R = 0$, and $Q^*_R = Q^*_M$ via an unspecified mechanism.

3.4. Effect of $\beta$’s (i.e., $m$’s uncertainty) magnitude

With $m \sim U(a, b)$ in the illustrative problem stated in (21), define $m$’s mean as $\mu_m = (a + b)/2$. Simple manipulations starting from (10) and (22) lead to

$$\pi^*_T = \pi^*_R - \pi^*_R,$$

$$\pi^*_R = \mu_m - \beta k_m(1 - SL^*).$$

Hence, if $SL^* > 0.5$, then $Q^*_R$ increases as $\beta$ increases. This differs from the classical results (e.g., in Leland [10]) based on multi-period products, but is actually a standard phenomenon (see, e.g., Lau and Lau [11]) for a single-period product considered in this, MP’s and related studies.

4. Concluding remarks

4.1. Brief summary

Recall that Table 1 in Section 2 showed that when $\alpha$ is small, $\pi^*_R$ is small and $E_t \approx 1$. This phenomenon is now clearly explained in Section 3 where $\alpha = 0$. One may therefore note that a failure to achieve “$E_t = 1$” is due to valuation uncertainty only.

Given below is a (perhaps over-simplified) summary of the differences between MP’s [1] and our models:

<table>
<thead>
<tr>
<th>Case</th>
<th>Results</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1: no arrival uncertainty</td>
<td>$r^*_s = 0$</td>
<td>$r^*_s \neq 0$ generally</td>
</tr>
<tr>
<td>Case 2: no valuation uncertainty</td>
<td>$r^<em>_s = p^</em>_w = V$</td>
<td>$r^<em>_s &lt; p^</em>_w &lt; V$ (stated as a self-evident condition/ result)</td>
</tr>
</tbody>
</table>
Solutions in Sections 2 and 3 indicate that, whenever practical, a manufacturer should give return credits to the retailer; however, the return credit should be somewhat less than the original wholesale price ($p_w$) paid by the retailer. The solutions also show that, without an additional constraint on minimal retailer’s profit share, the manufacturer can often devise a price-cum-return-credit scheme such that the manufacturer will reap a larger share of the product-market’s profit than the retailer. Note that Sections 2 and 3 have considered only Cases 1 and 2 — in which only one of $v$ or $m$ is stochastic. Case 3 (in which both $v$ and $m$ are stochastic) is examined in Lau et al. [12], where it is shown that the above conclusions are also valid for Case 3.

4.2. Extensions

Fruitful extensions include: (i) modeling valuation uncertainty such that the mean and standard deviation of market demand do not have to vary together and dependently (stated earlier in Section 2); (ii) an in-depth study of the effects of errors in estimating $f(v)$ and $g(m)$; (iii) considering situations in which the manufacturer and the retailer do not share the same market information.

Appendix A. Solving Eqs. (13) and (14) for the Case-1 problem

For any given $(p_w, r)$-values, the value of $p_R$ that maximizes $\pi_R(p_R)$ in (13) was obtained with the IMSL [13] subroutine UVMGS, which executes the “golden section” search technique. The above step is imbedded in the function-program $\pi_M(p_w, r)$ (Eq. (14)), which was optimized with the IMSL subroutine BCPOL — which executes Nelder and Mead’s [14] “simplex method”. Both UVMGS and BCPOL are designed for non-smooth functions, and we also executed BCPOL with multiple initial points. This is to eliminate any question about the possible effect of function smoothness and unimodality on the validity of our optimal solutions.

References