Robust multi-item newsboy models with a budget constraint

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Abstract

In this paper we present robust newsboy models with uncertain demand. The traditional approach to describing uncertainty is by means of probability density functions. In this paper we present an alternative approach using deterministic optimization models. We describe uncertainty using two types of demand scenarios; namely interval and discrete scenarios. For interval demand scenarios we only require a lower and an upper bound for the uncertain demand of each item, while for discrete demand scenarios we require a set of likely demand outcomes for each item. Using the above scenarios to describe demand uncertainty, we develop several minimax regret formulations for the multi-item newsboy problem with a budget constraint. For the problems involving interval demand scenarios, we develop linear time optimal algorithms. We show that the corresponding models with discrete demand scenarios are NP-hard and that they are solvable by dynamic programming. Finally, we extend the above results to the case of mixed scenarios where the demand of some of the items is described by interval scenarios and the demand of the remaining items is described by discrete scenarios.

Keywords: Inventory; Production; Management; EOQ models; Robust optimization

1. Introduction

In this paper we consider the multi-item newsboy model with a budget constraint and demand uncertainty. We present an alternative approach to stochastic optimization by using integer programming models with minimax objectives. We assume that the items of our newsboy model may be of continuous or discrete nature, or a combination of the two. This prompts the introduction of two types of demand scenarios for the items; namely interval and discrete. Before we proceed to further describe our modelling approach as well as to justify our motivation for the structuring of uncertainty via scenarios, we will position this research within the current state of decision theory.

In decision theory there are three different chance environments; namely certainty, risk and uncertainty. Among the three, the certainty and risk environments have received the most attention. In a certainty environment all problem parameters are considered to be accurately known and the objective is to optimize...
a carefully selected performance measure. For the risk environment, parameters are assumed to take on values from carefully selected probability distributions and the objective is to optimize expected system performance.

Certainty and risk environments however, may be inadequate in some cases. For instance, when introducing new products, approximating their demand may be impossible. Other products exhibit relatively stable demand, but even then the volume of the demand fluctuates within a small range. These sources of uncertainty are sometimes captured using risk models. Risk models capture the behavior of uncertain demand with the use of probability distributions. This approach, too, is subject to a few drawbacks. Often, the selected probability distributions only partially capture the behavior of uncertain parameters. Or, the decision situation is of unique nature and hence no historical data are available for the uncertain parameters thus making almost impossible to determine reasonable distributions that capture the behavior of the parameters. In addition, more often than not the incorporation of probability distributions in the risk model make it very difficult to solve, thus forcing the problem solver to resort to assumptions (such as independence assumptions) that may render the model a mediocre representation of reality. Also, it is customary in risk models to optimize expected system performance. Then, the optimal solution of the risk model provides a guideline for optimizing long-term performance. This may not be in line with the objective of a manager whose survival in the market is based on short-term performance. Several deterministic and stochastic inventory models are presented in [1–3]. In [4,5], distribution free newsboy problems are considered where the demand probability distribution is unknown, except for its first two moments, and the objective is to optimize the worst case expected cost over all possible demand distributions with the given first two moments. Similarly, distribution free problems for assemble-to-order, assemble-in-advance, and composite policies are considered in [6]. Additional objective functions for distribution free inventory problems include minimax regret approaches. For instance, infinite horizon \((Q, R)\) models with backorder costs are considered in [7,8].

A manager who is interested in short-term performance has to exploit the opportunities available to him/her at the time of the decision. This, along with the traditional risk averse attitude of managers, render minimax regret approaches very important in identifying robust solutions, i.e., solutions that perform well for any realization of the uncertain demand parameters. In this paper we present a number of minimax regret formulations for the multi-item newsvendor problem with a single budget constraint, when the demand distribution is completely unknown. The only assumption made about the demands for individual items is that they may be described with the use of scenarios, or the demand for each item falls within a known lower and upper bound. Under this assumption, in [9] a dynamic programming approach is presented to solve the multi-period newsvendor problem, while in [10] the distribution free newsvendor problem is studied with the objective of minimizing maximum regret.

In this article, our modelling approach incorporates minimax regret objective functions together with the use of scenarios and/or intervals for uncertain demands. Similar approaches have been recently applied in many operations research areas including location theory (see [11,12]), scheduling (see [13,14]), and inventory control (see [15]).

We can now proceed to describe the motivation of the scenario approach for capturing demand uncertainty. First consider items of discrete nature (e.g. magazines, newspapers, tires); for these items we capture demand uncertainty by means of discrete scenarios. A discrete scenario is a collection of possible demand realizations of the item in question. As a result, for every item of discrete nature we associate a scenario, i.e., a set of possible demand realizations. Note that we do not associate probabilities with every possible demand realization; a scenario is just a collection of all possible values for the demand of a particular item. Determining the demand scenario of each item is based on understanding the manager on the sources that affect uncertainty (such as market and competitive environment) for each item. In this paper we assume that the demand scenarios associated with each discrete item is known. For items of continuous nature (e.g. oil, petroleum, sugar) we make use of interval scenarios. To specify the interval scenario for
a particular item of continuous nature we only need to carefully select a lower and an upper bound for the demand. Again, all values in the specified interval are considered likely to occur, but there are no probabilities associated with these values. The upper and lower bounds of interval scenarios are subjectively selected by the decision maker. Demand values that are realized with infinitesimal probability should only be used if the decision maker is willing to hedge against extremely unlikely scenarios. Else, the range of values in each interval scenario should only contain likely realizations.

In some cases, robust optimization can be used in risk environments where, even though there is distributional knowledge of uncertain input parameters, the intractability of the model makes it extremely difficult to derive answers based on stochastic optimization. In this case, one can identify upper and lower bounds using the 90% quartile or other appropriate percentage. When the distribution of the uncertain parameters is concentrated around the mean, then the 90% quartile can provide relatively tight and at the same time realistic bounds. Else, the decision maker would have to balance the difference of the lower bound from the upper, with the corresponding quartile of the demand distribution. An example for the feasibility of this approach is provided in [16] who report an application from the apparel industry; for details about this application see [17]. In [16], a scatter diagram is provided of initial demand forecasts (obtained using experts’ opinions) versus season sales for fashion products. In 24 of the 35 points in this diagram, the experts’ opinion is very close to the actual sales, and in the remaining 11 points the ratio of initial-to-actual sales is no greater than 7. Hence, for all practical purposes, in this application, one can develop reasonable bounds by accounting for about two standard deviations on either side of the mean (in this application the demand distribution for every product is assumed to be normal). Another possible use for the robust optimization approach presented in this article, is for building consensus amongst the various experts. For instance, in [17] the mean demand of each product is taken as the mean of the most likely demand values obtained by seven experts. Rather than obtaining a single value from each expert (i.e., the mean), we can obtain a likely range of values and use the robust quantity as the mean. Using this approach we can minimize the regret suffered by all experts as a group.

In this paper we consider the multi-item newsboy model with a budget constraint. The demand for each of the items is uncertain. Our objective is to make a single period stocking decision prior to realizing the demand for any of the items. At the end of the period we can salvage any unused units at a prespecified price. Stocking decisions for subsequent periods are independent of decisions made in previous periods. Demand uncertainty is captured by means of scenarios that are constructed as described above. The budget constraint may represent the load or volume constraint of a truck, or the space available by the retailer, or the total budget available for the purchase of the items. The deterministic version of this model has appeared in many textbooks (see [1,2]) and it is first presented in [18]. The solution of this model (see [19] for a good summary of the known results on deterministic inventory problems with constraints) is based on Lagrangean relaxation. In this paper we consider the robust version of the problem, which is appropriate for the uncertainty environments mentioned earlier. In the next section we define three different robustness objectives and formally present the models considered.

2. Problem formulation and basic results

\( n \)         number of items,
\( c_i \)     cost per unit for item \( i \),
\( r_i \)   revenue per unit \( (r_i \geq c_i) \) for item \( i \),
\( s_i \)   salvage value per unit \( (s_i \leq c_i) \) for item \( i \),
\( l_i \)   cost of lost sales per unit \( (beyond the lost profit \( r_i - c_i \)) \) for item \( i \),
\( Q_i \)   order quantity for item \( i \),
$d_i$ demand for item $i$,
$\pi_i(Q_i, d_i)$ the profit function for item $i$, for the order quantity $Q_i$ and demand $d_i$.

The profit function for the $i$th item is given by

$$
\pi_i(Q_i, d_i) = \begin{cases} (r_i - c_i + l_i)Q_i - l_id_i & \text{if } Q_i \leq d_i, \\ (r_i - s_i)d_i - (c_i - s_i)Q_i & \text{if } Q_i > d_i. \end{cases}
$$

(1)

For a particular item, the classical newsboy approach identifies the quantity $Q^N$ that maximizes the expected profit (see [2]). This approach results in

$$
F(Q^N) = \frac{r - c + l}{r - s + l},
$$

where $F(\cdot)$ is the cumulative distribution function of the probability density function (p.d.f.) $f(\cdot)$ of the demand for that particular item. If the p.d.f. is uniformly distributed in $U[D, \bar{D}]$, a straightforward calculation shows that

$$
Q^N = \frac{(c - s)D + (r - c + l)\bar{D}}{r - s + l}.
$$

(2)

For our robust formulations we need the following definitions. Let $D^i(i)$ be the collection of all possible demand realizations for item $i$, $i = 1, 2, \ldots, n$. Then, the solution to any of the multi-item problems that will be stated below must be a $n$-tuple in $D^1(1) \times D^2(2) \times \cdots \times D^n(n)$. For discrete demand scenarios we consider three different objective functions. Our first objective is called absolute robustness and it is formulated as follows:

$$
\begin{align*}
\text{(AR)} & \quad \max_{(Q_1, \ldots, Q_n) \in D^1(1) \times \cdots \times D^n(n)} \min_{(d_1, \ldots, d_n) \in D^1(1) \times \cdots \times D^n(n)} \sum_{i=1}^{n} \pi_i(Q_i, d_i) \\
\text{s.t.} & \quad \sum_{i=1}^{n} c_iQ_i \leq W,
\end{align*}
$$

(3)

where $W$ is our budget constraint. The absolute robustness approach attempts to find a $n$-tuple of order quantities that maximize the worst case profit over all possible demand realizations.

The second formulation is called robust deviation, and is given by

$$
\begin{align*}
\text{(DR)} & \quad \min_{(Q_1, \ldots, Q_n) \in D^1(1) \times \cdots \times D^n(n)} \max_{(d_1, \ldots, d_n) \in D^1(1) \times \cdots \times D^n(n)} \sum_{i=1}^{n} (\pi_i(d_i, d_i) - \pi_i(Q_i, d_i)) \\
\text{s.t.} & \quad \sum_{i=1}^{n} c_iQ_i \leq W.
\end{align*}
$$

(4)

This formulation provides a solution that minimizes over all choices of order quantities the maximum profit loss due to demand uncertainty. This is a minimax regret approach where the regret is captured by the difference $\pi_i(d_i, d_i) - \pi_i(Q_i, d_i)$; i.e. the profit that could be realized if there was no demand uncertainty (in which case we would order $Q_i = d_i$), minus the profit made for the order quantity $Q_i$. 

The third robust formulation is called relative robustness and the corresponding formulation is given by

\begin{equation}
(RR) \quad \min_{(Q_1, \ldots, Q_n) \in D'(1) \times \cdots \times D'(n)} \max_{(d_1, \ldots, d_n) \in D'(1) \times \cdots \times D'(n)} \frac{\sum_{i=1}^{n} \pi_i(d_i, d_i)}{\pi_i(d_i, d_i)}
\end{equation}

s.t. \quad \sum_{i=1}^{n} c_i Q_i \leq W,

which minimizes the relative profit loss per unit of profit that could be made if there was no demand uncertainty. Note that the relative profit loss measures the lost profit as a percentage of the profit that could be made if we knew the actual demand.

In the rest of our analysis it will become clear that the three objectives result to very different choices of order quantities. Similar formulations can be written for the case of interval scenarios. The only difference in modeling the continuous case, is that the discrete scenarios \(D'(i)\) in models (3)–(5) are replaced by the interval scenarios \([D_{li}, D_{ri}]\) where \(D_{li}, D_{ri}\) are the lower and upper bounds for the possible demand realizations of item \(i: 1 \leq i \leq n\).

As we will show, some of the above problems are related to the continuous knapsack problem (see [21]):

\begin{equation}
\max \sum_{i=1}^{n} w_i x_i \\
\text{s.t.} \quad \sum_{i=1}^{n} c_i x_i \leq C \\
0 \leq x_i \leq 1, \quad i = 1, 2, \ldots, n.
\end{equation}

This problem can be solved using a property established in [20] which can be stated formally as follows. Suppose that the \(n\) items are ordered according to nonincreasing order of the ratio \(w_i/c_i\) (i.e. weight per unit cost). Then, an optimal solution of the continuous knapsack is given by

- \(x_i = 1\) for \(i = 1, 2, \ldots, s - 1\),
- \(x_i = 0\) for \(i = s + 1, \ldots, n\), and
- \(x_s = \bar{c}/c_s\),

where \(s\) is the critical item for which \(s = \min\{i: \sum_{j=1}^{i} c_j > C\}\), and \(\bar{c} = C - \sum_{i=1}^{s} c_i\). To compute the above solution in \(O(n)\) time (and thus avoiding the ordering of the \(n\) items) can be done by identifying the critical item \(s\) using a procedure proposed in [22]. Then, each ratio \(w_i/c_i\) can be compared against \(w_s/c_s\) which prompts the assignment of the appropriate value for \(x_i\). Hence, the continuous knapsack problem is solvable in time linear in the number of variables.

The outline of the rest of the paper is as follows. In Section 3 we consider robust models for a single item. In Section 4 we develop algorithms for the multi-item case with a budget constraint, in the presence of interval demand uncertainty. The case where demand uncertainty is structured by use of discrete scenarios is considered in Section 5. Mixed scenarios where uncertain demand is described by means of discrete scenarios for some items and interval scenarios for others, are considered in Section 6. We close Section 7 with some conclusions.

### 3. Single-item models

To facilitate the development of robust inventory models for the multi-item case, we first need to consider the special case of a single item. For notational simplicity, in this section we will not use indices for the
variables defined in Section 2, e.g. $Q_i$ will be denoted by $Q$, etc. Below we obtain the robust order quantity for each of our three objectives (disregarding the budget constraint).

3.1. Absolute robustness

By observing the profit function $\pi(Q, d)$, we have that

$$\min_{d \in \{D, \bar{D}\}} \pi(Q, d) = \begin{cases} \frac{(r - c + l)Q - l\bar{D}}{(r - s)D - (c - s)Q} & \text{if } Q \leq Q^A \leq \bar{D}, \\ \frac{(r - c + l)D - l\bar{D}}{(r - s)D - (c - s)Q} & \text{if } Q \geq Q^A \geq D, \end{cases}$$

(6)

where $Q^A$ is the absolute robust-order quantity. For every fixed $Q \in [D, \bar{D}]$, the first branch of (6) is linear increasing function of $Q$, while the second is linear decreasing; see Fig. 1a.

Hence, $Q^A$ is in the intersection of the two lines, i.e., the solution of $\pi(Q^A, D) = \pi(Q^A, \bar{D})$, which is

$$Q^A = \frac{(r - s)D + l\bar{D}}{r - s + l}. \tag{7}$$

Evidently, $Q^A$ is a convex combination of $D$ and $\bar{D}$ weighted by $(r - s)/(r - s + l)$ and $l/(r - s + l)$, respectively. Intuitively, $Q^A$ hedges against demand uncertainty by charging lost sales on the maximum possible demand $\bar{D}$ (i.e., the term $l\bar{D}$ in the numerator), and by charging $r - s$ units of lost income for every unit of the minimum possible volume $D$.

3.2. Robust deviation

The deviation robust-order quantity is the solution of

$$\min_{Q \in \{D, \bar{D}\}} \max_{d \in \{D, \bar{D}\}} (\pi(d, d) - \pi(Q, d)).$$

The function $f^D(Q) = \max_{d \in \{D, \bar{D}\}} (\pi(Q, d) - \pi(d, d))$ is given by

$$f^D(Q) = \begin{cases} \frac{(r - c + l)(D - Q)}{(c - s)(Q - D)} & \text{if } Q \leq Q^D, \\ \frac{(r - c + l)(\bar{D} - Q)}{(c - s)(Q - D)} & \text{if } Q \geq Q^D, \end{cases}$$

and it is minimized at the intersection $Q^D$ of the two branches; that point is

$$Q^D = \frac{(c - s)D + (r - c + l)\bar{D}}{r - s + l}. \tag{8}$$
Note that $Q^D$ is also a convex combination of $D$ and $\bar{D}$, weighted by $(c - s)/(r - s + l)$ and $(r - c + l)/(r - s + l)$, respectively. Observe that $Q^D \geq Q^R$, because $r - c + l \geq l$ which indicates that in $Q^D$, in addition to the lost sales $l$, the lost profit $r - c$ is charged against the maximum possible volume $\bar{D}$. As a result, $Q^D$ results to higher service levels and hedges less (as compared with $Q^R$) against overestimating actual demand.

In light of (2) and (8) we have the following corollary.

**Corollary 1.** The single-item deviation robust-order quantity for the interval demand scenario $[D, \bar{D}]$, coincides with the optimal solution of the single-item newsboy model with uniformly distributed demand in $[D, \bar{D}]$.

### 3.3. Relative robustness

We have that

$$\min_{Q \in [D, \bar{D}]} \max_{d \in [D, \bar{D}]} \frac{\pi(d, d) - \pi(Q, d)}{\pi(d, d)} = \min_{Q \in [D, \bar{D}]} \max_{d \in [D, \bar{D}]} \left\{ \frac{(r - c + l)d - (r - c + l)Q}{(r - c)d} \right\} \text{ if } Q \leq d,$$

$$\min_{Q \in [D, \bar{D}]} \max_{d \in [D, \bar{D}]} \frac{\pi(d, d) - \pi(Q, d)}{\pi(d, d)} = \min_{Q \in [D, \bar{D}]} \max_{d \in [D, \bar{D}]} \left\{ \frac{(r - c + l)d - (r - c + l)Q}{(r - c)d} \right\} \text{ if } Q \geq d.$$  

(9)

For fixed $Q = Q_0$ and $d \geq Q_0$, the first branch of (9) is maximized for $d = \bar{D}$, and the second branch is maximized for $d = \bar{D}$. Hence, the function

$$f^R(Q) = \max_{d \in [D, \bar{D}]} \frac{\pi(d, d) - \pi(Q, d)}{\pi(d, d)} = \left\{ \frac{(r - c + l)d - (r - c + l)Q}{(r - c)d} \right\} \text{ if } Q \leq Q^R,$$

$$= \left\{ \frac{(r - c + l)d - (r - c + l)Q}{(r - c)d} \right\} \text{ if } Q \geq Q^R,$$

has the form depicted in Fig. 1b, and $Q^R$ is the solution of $\pi(Q, \bar{D})/\pi(\bar{D}, \bar{D}) = \pi(Q, D)/\pi(D, D)$, i.e.,

$$Q^R = \frac{(r - s + l)\bar{D}}{(r - c + l)\bar{D} + (c - s)\bar{D}}.$$  

(10)

$Q^R$ does not follow a form similar to $Q^A$ or $Q^R$. In fact, $Q^R$ is proportional to $D\bar{D}$. With these basic results, we turn our attention to the multi-item newsboy model. We start in the next section with demand uncertainty described by interval scenarios.

### 4. Multiple items and interval demand scenarios

In case that the demand realizations for item $i$ take values from the interval $[D_i, \bar{D}_i]$, our absolute robust formulation with a budget constraint becomes

$$(\text{AR–IS}) \quad \max_Q \quad \min_{d \in [D, \bar{D}]} \quad \sum_{i=1}^{n} \pi_i(Q_i, d_i)$$

s.t.

$$\sum_{i=1}^{n} c_i Q_i \leq W,$$

$$Q_i \in [D_i, \bar{D}_i].$$

(11)

(12)

The following observations can be made for model AR–IS.
Lemma 1. There exists an optimal solution \( \{Q^*_i\}_{i=1}^n \) for AR-IS such that \( Q^*_i \in [D_i, Q^*_i] \), where \( Q^*_i \) is the absolute robust quantity (7) for item \( i \).

The proofs of all results are included in the appendix.

In light of this lemma, \( \pi_i(Q_i, d_i) \) can be replaced in AR-IS by \( (r_i - c_i + l_i)Q_i - l_iD_i \). Then, (11) can be replaced by

\[
\max_{Q_i \in [D_i, D_i]} \sum_{i=1}^n [(r_i - c_i + l_i)Q_i - l_iD_i]. \tag{14}
\]

The optimal solution of (14) maximizes the quantity \( \sum (r_i - c_i + l_i)Q_i \) and therefore AR-IS reduces to a continuous knapsack problem. We assume that \( W < \sum c_iQ^*_i \) otherwise the trivial solution \( Q^*_i = Q^*_i \), \( 1 \leq i \leq n \), is optimal. Below we adapt the continuous knapsack procedure of Section 2, for the formulation AR-IS; we refer to the resulting algorithm for interval demand scenarios, as AID.

4.2. Algorithm AID

1. Index the items so that \( w_1/c_1 \geq w_2/c_2 \geq \cdots \geq w_n/c_n \), where \( w_i = r_i - c_i + l_i \).
2. If \( W > \sum c_iQ^*_i \), then \( Q^*_i = Q^*_i \), \( 1 \leq i \leq n \), is an optimal solution; Stop. Else, identify the critical item \( s \) such that

\[
s := \min \left\{ i : \sum_{j=1}^i c_i(Q^*_i - D_i) + \sum_{i=1}^n c_iD_i \geq W \right\}.
\]

3. Set \( Q^*_i := Q^*_i \) for \( i < s \), \( Q^*_i := D_i \) for \( i > s \) and \( Q^*_i := \left( W - \sum_{1 \leq i \leq n, i \neq s} c_iQ^*_i \right)/c_i \).

The complexity of AID is \( \mathcal{O}(n \log n) \), due to the sorting at step 1. However, if the critical element \( s \) is identified first (as discussed in Section 2), the sorting can be avoided and the complexity can drop to \( \mathcal{O}(n) \).

Consider now the deviation robust formulation for interval demand scenarios.

\[
(DR-IS) \quad \min_{Q_i \in [D_i, D_i]} \max_{d_i \in [D_i, D_i]} \sum_{i=1}^n (\pi_i(d_i, d_i) - \pi_i(Q_i, d_i)) \tag{15}
\]

s.t. \( (12), (13) \).

Similar arguments as for AR-IS indicate that there exists an optimal solution for DR-IS where \( Q^*_i \leq Q^*_i \), where \( Q^*_i \) is the deviation robust quantity given in (8) for item \( i \). Also, (15) is equivalent to

\[
\min_{Q_i \in [D_i, D_i]} \sum_{i=1}^n (r_i - c_i + l_i)(D_i - Q_i),
\]

which accepts the same solution as (14). Hence,

Theorem 1. An absolute robust solution \( \{Q^*_i\}_{i=1}^n \) (that solves AR-IS) is also deviation robust (i.e., solves DR-IS).
Finally, consider the relative robustness model with interval demand scenarios

\[
\text{(RR-IS)} \quad \min_{Q, \Delta} \max_{d_i \in [D_i, D'_i]} \sum_{i=1}^{n} \frac{\pi_i(d_i, d_i) - \pi_i(Q_i, d_i)}{\pi_i(d_i, d_i)} \\
\text{s.t. (12), (13).} \tag{16}
\]

Similar arguments show that AID optimally solves RR-IS if \( w_i \) is replaced by \( w_i := (r_i - c_i + l_i)/((r_i - c_i)D_i) \). Hence, the item ordering is in nonincreasing order of

- \( (r_i - c_i + l_i)/c_i \) for the absolute and deviation robust objectives, and
- \( 1/c_iD_i + l_i/(c_i(r_i - c_i)D_i) \) for the relative robustness objective,

as can be seen by calculating the ratio \( w_i/c_i \) for each of the objectives. The absolute and deviation robust objectives favor items with larger ratio of (profit + lost sales)/cost. On the other hand, RR-IS favors low-cost items, whose maximum possible demand is low (due to the term \( c_iD_i \) in the denominator of \( w_i/c_i \)), and large lost sales to profit ratio \( l_i/(r_i - c_i) \). A typical example of such environment is a supermarket, where profits are of the order of 2%, the maximum daily volume for most items is low, and the lost sales costs are significant, since stocking out in a particular item may force the consumer to buy all groceries from a competitor.

Example. Consider the cost and demand parameters provided below for five different items. By identifying the nonincreasing order of the quantities \( (r_i - c_i + l_i)/c_i \) and \( 1/c_iD_i + l_i/(c_i(r_i - c_i)D_i) \) we indicate in Table 1 which items appear to be given higher priority by each of the three objectives AR, DR and RR for absolute, deviation and relative robustness, respectively.

This example indicates that item 2 receives top priority by all three objectives because it happens to result to a high (profit + lost sales)/cost ratio, and at the same time low \( D_i \) and large lost sales to profit ratio \( l_i/(r_i - c_i) \). Item 1 is favored by the RR objective while its priority is deemed low for the AR and DR objectives. Similar conclusions can be made for the remaining items. As a result, for any budget level, and for every one of the three objectives, our formulations result to ordering the maximum possible number of units starting with high priority items and continuing on with items of lower priority.

5. Discrete demand scenarios

In this section we assume that for each item \( 1 \leq i \leq n \) we are given a scenario \( D^i(i) \) of demand quantities that may be realized. The number of likely demand outcomes is \( |D^i(i)| \). The formulation for the absolute

<table>
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<th>Item</th>
<th>( r_i, c_i, l_i )</th>
<th>( D_i, D'_i )</th>
<th>( (r_i - c_i + l_i)/c_i )</th>
<th>( 1/c_iD_i + l_i/(c_i(r_i - c_i)D_i) )</th>
<th>Priority</th>
<th>AR, DR</th>
<th>RR</th>
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<td>1, 6</td>
<td>0.75</td>
<td>0.0625</td>
<td>5</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>
robust objective is

\[
(\text{AR–DS}) \quad \max_{Q_i \in D^*(i)} \min_{d_i \in D^*(i)} \sum_{i=1}^{n} \pi_i(Q_i, d_i)
\]

\[\text{s.t. (12).} \quad (17)\]

In the next section we show that AR–DS is \(NP\)-hard.

5.1. \(NP\)-hardness result

We will show that AR–DS is \(NP\)-hard using a reduction from a well known \(NP\)-hard problem; the even–odd partition (see [23]).

\textbf{Input}: A collection of positive integers \(a_i, 1 \leq i \leq 2n\), such that \(\sum_i a_i = A\).

\textbf{Output}: “Yes” iff \(\{a_i\}_{i=1}^{2n}\) can be partitioned into 2 disjoint sets \(A_1, A_2\) such that \(\sum_{a_i \in A_k} a_i = A/2\) for \(k = 1, 2\) and precisely one of \(a_{2i}, a_{2i-1}\) belongs to \(A_1\), for \(1 \leq i \leq n\).

\textbf{Theorem 2}. Problem AR–DS is \(NP\)-complete.

Similar arguments can be used to show the \(NP\)-hardness of the deviation and relative robustness formulations for discrete demand scenarios. In what follows we present optimal solution procedures for these formulations based on dynamic programming.

5.2. Solution procedures

In this section we present dynamic programming optimal algorithms for the case of discrete scenarios and each of the three robust objectives. We first consider the absolute robust objective. We assume that the lowest and highest demand realizations in \(D^*(i)\) are \(D_i\) and \(\overline{D_i}\), respectively. Then, the function \(f^A(Q_i) = \min_{d_i \in D^*(i)} \pi_i(Q_i, d_i)\) is depicted in Fig. 2a.

Observe that Fig. 2a is a discrete version of Fig. 1a precisely because we deal with discrete demand values \(d_i \in D^*(i)\) rather than interval scenarios \(d_i \in [D_i, \overline{D_i}]\) as in Fig. 1a. Note that some of the order quantities in

![Fig. 2. The undominated order quantities \(U^h\) and \(U^p\).](image)
\[ D^i(i) \] are redundant. In Fig. 2a, the redundant quantities are depicted by unfilled circles; and they are the quantities that are greater than the quantity \( Q^i \in D^i(i) \) that maximizes \( f^i_n(Q_i) \). All these redundant order quantities have a smaller contribution to the objective function value even though they require a greater expenditure. As a result, they should never be considered in an optimal solution. Hence,
\[ f^i_n(\bar{Q}^i) \geq f^i_n(q) \quad \forall q \in D^i(i). \]

Then, the set \( U^A_i \) of undominated-order quantities consists of those order quantities that are less than or equal to \( \bar{Q}^i \), namely
\[ U^A_i := \{ q \in D^i(i): q \leq \bar{Q}^i \}. \]

Intuitively, the absolute robust performance of the order quantities \( q \in D^i(i) - U^A_i \) are dominated by the performance of \( \bar{Q}^i \) because \( f^i_n(\bar{Q}^i) \geq f^i_n(q) \) and \( c_i \bar{Q}^i \leq c_i q \). Hence, in any optimal solution for AR–DS, only the order quantities in \( U^A_i \) need to be considered for item \( i \). With these observations, AR–DS reduces to

\[
\begin{align*}
\max_{q \in U^A_i} & \quad \sum_{i=1}^{n} f^i_n(Q_i) \\
\text{s.t.} & \quad (12),
\end{align*}
\]

which is an integer knapsack problem and can be solved via the following dynamic program:

Define \( F_i(w) \) to be the maximum profit that can be incurred for items 1, 2, \ldots, \( i \), if \( \sum_{j=1}^{i} c_j Q_j \leq w \).

Boundary Condition: \( F_0(\sum_{j} c_j D_j) := f^0_n(D_i) \). This condition captures the requirement that at least \( D_j \) units must be ordered for item \( i \), \( 1 \leq i \leq n \).

Recurrence Relation: \( F_i(w) = \max_{q \in U^A_i} F_{i-1}(w - c_i Q_i) \).

Solution: \( F^* = \max_{\sum_{j} c_j D_j \leq w} F_n(w) \).

The state space of the above DP is \( \mathcal{C}(nW) \) due to the \( n \) items and the \( W \) units of resource available. In each iteration, the DP requires \( \mathcal{C}(\max_i D^i(i)) \) effort. Let \( k := \max_i D^i(i) \); the total effort required by DP is no more than \( \mathcal{C}(nW) \).

The dynamic programs that solve the deviation and relative robustness formulations for discrete demand scenarios are identical to the one presented above, except for the definition of the \( U^A_i \)’s. Fig. 2b depicts by filled circles the undominated order quantities for item \( i \) and the deviation robust objective. In general, for the deviation robust objective the quantity \( \bar{Q}^P_i \) and the set \( U^P_i \) are defined by
\[ f^P_n(\bar{Q}^P_i) \leq f^P_n(q) \quad \forall q \in D^i(i) \quad \text{and} \quad U^P_i := \{ q \in D^i(i): q \leq \bar{Q}^P_i \}. \]

The definitions of \( \bar{Q}^R_i \) and \( U^R_i \) are analogous.

6. Mixed demand scenarios

In this section we consider the most general case where some items are of continuous nature while others are of discrete nature. We assume that the demand and order quantities for the former items can be described by an interval scenario as in Section 4 while the latter items can be described by discrete scenarios; in both cases we use the generic notation \( D^i_i \) to denote that scenario. Without loss of generality we can assume that the items are indexed so that all the items of continuous nature come first. Assuming that there are \( m \) continuous items, they are followed by \( n - m \) items of discrete nature.

Let us first consider the absolute robust formulation AR where the demand scenarios are replaced by \( D^i_i \). Let \( \{ Q^*_i \}_{i=1}^{n} \) be an optimal choice of order quantities for the AR formulation with mixed scenarios. Then, the portion \( W_1 := \sum_{i=1}^{m} c_i Q^*_i \) of the budget is spent on items of continuous nature, and the portion
W_2 \leq W - W_1 \text{ is spent on items of discrete nature. As seen in Sections 4 and 5, the following properties should hold for an optimal solution for the case of AR with mixed scenarios:}

(a) For every item i of discrete nature, Q_i^* \in \mathbb{U}_i^D \text{ (or } U_i^D, U_i^R \text{ depending on the objective}).

(b) For every item i of continuous nature, Q_i^* \in [D_i, Q_i^A] \text{ (or } [D_i, Q_i^P], [D_i, Q_i^R] \text{ depending on the objective}).

(c) AID optimally assigns the W_1 units of resource to the continuous items.

(d) DP optimally assigns the W_2 units of resource to the discrete items.

The above observations motivate the development of an optimal algorithm that utilizes AID and DP along with a search on W_1 \in [\sum_{i=1}^m c_i D_i, \sum_{i=1}^m c_i D_i]. The following algorithm provides the main steps of the proposed procedure.

6.1. Algorithm MIXED

1. Let z_c(W_1) be the optimal objective value for the continuous items with a resource of W_1 units;
   
   W_1 \in [\sum_{i=1}^m c_i D_i, \sum_{i=1}^m c_i D_i].

2. Apply DP on the discrete items with a budget limit of W units.

3. Let Z(W_1) := z_c(W_1) + f_a(W - W_1) \text{; for all } W_1 \in [\sum_{i=1}^m c_i D_i, \sum_{i=1}^m c_i D_i].

4. Compute Z^* = \max_{W_1 \in [\sum_{i=1}^m c_i D_i, \sum_{i=1}^m c_i D_i]} Z(W_1).

Step 1 of MIXED can be performed in O(n \log n) time by first sorting the items in nonincreasing order of w_i/c_i and then computing z_c(W_1): W_1 \in [\sum_{i=1}^m c_i D_i, \sum_{i=1}^m c_i D_i] as a piecewise linear increasing function (see [20]). Step 2 requires O(nW) time, while steps 3 and 4 require no more than O(W) time. Hence, the overall complexity of MIXED is O(nW).

Minor adaptation of the above procedure produces an optimal solution to the mixed scenario version of the formulations DR and RR. The only change that needs to be made is that step 4 should be replaced by Z^* = \min_{W_1 \in [\sum_{i=1}^m c_i D_i, \sum_{i=1}^m c_i D_i]} Z(W_1) due to the fact that f_a(Q_i) and f_r(Q_i) are convex functions, unlike f_a(Q_i) which is concave.

7. Conclusion

In this article we considered robust newsboy models with a budget constraint. The models are applicable when demand uncertainty can be captured using discrete or interval scenarios. We formulated three minimax regret objectives and developed efficient algorithms for nine combinations of objectives and demand scenarios. Below we summarize the findings in Table 2, where we indicate the complexity of the proposed algorithm for each problem and the section in which it was considered.

Our future research directions include the application of robustness on other inventory models.

Table 2
Summary of results for robustness criteria

<table>
<thead>
<tr>
<th>Robustness objective</th>
<th>Scenario type</th>
<th>Interval</th>
<th>Discrete</th>
<th>Mixed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Absolute Deviation</td>
<td>( O(n) ), &amp; 4</td>
<td>( O(n W \max_i</td>
<td>D_i</td>
<td>) ), &amp; 5</td>
</tr>
<tr>
<td>Absolute Relative</td>
<td>( O(n) ), &amp; 4</td>
<td>( O(n W \max_i</td>
<td>D_i</td>
<td>) ), &amp; 5</td>
</tr>
<tr>
<td>Relative Deviation</td>
<td>( O(n) ), &amp; 4</td>
<td>( O(n W \max_i</td>
<td>D_i</td>
<td>) ), &amp; 5</td>
</tr>
</tbody>
</table>
Acknowledgements

The author would like to thank Professor Panos Kouvelis for bringing this problem to his attention. Also, thanks are due to two anonymous referees whose suggestions greatly improved the presentation and interpretation of our results.

Appendix A

Proof of Lemma 1. Let $f_i^*(Q_i) = \min_{d_i \in [D_i, D_i]} \pi_i(Q_i, d_i)$. For every item $i$, $f_i^*(Q_i)$ is a piecewise linear concave function maximized at $Q_i = Q_i^*$ (see (7)). For every $1 \leq i \leq n$ and $Q_i \in [Q_i^*, D_i]$ we have that $f_i^*(Q_i) \leq f_i^*(Q_i^*)$ (due to optimality of $Q_i^*$) and $c_i Q_i \geq c_i Q_i^*$. This means that by selecting a quantity $Q_i$ greater than the robust quantity $Q_i^*$ for item $i$, the objective of AR–IS does not improve and at the same time a greater portion of the budget $W$ is consumed. Hence, the optimal quantity $Q_i^*$ belongs in $[D_i, Q_i^*]$. □

Proof of Theorem 2. Given an instance \{\(a_i\)\}_{i=1}^{2n} of even–odd partition, consider the $n$-item instance of AR–DS with $c_i = 1, s_i = 0, l_i = 2, r_i = 2a_{2i}/(a_{2i-1})$, $D^i(i) = \{a_{2i-1}, a_{2i}\}$ for $1 \leq i \leq n$, and $W = \frac{A}{2}$. To see that the above values result in a well defined instance of AR–DS, note that $-(c_i - s_i) = -1$ is negative, and $r_i - c_i + l_i = 2a_{2i}/(a_{2i-1}) + 1 > 0$ which agree with the desired slopes of the lines in Fig. 1a for every item $i$. For the above data, and according to (7) we have

$$Q_i^* = \frac{2a_{2i-1} a_{2i}}{a_{2i-1} + a_{2i}},$$

however, $Q_i^* \notin D^i(i)$ in general. To see that $Q_i^* \in [a_{2i-1}, a_{2i}]$ note that if $k := a_{2i-1}$ and $\delta := a_{2i} - a_{2i-1}$, then

$$a_{2i-1} = k \leq k \frac{2(k + \delta)}{2k + \delta} = \frac{2k(k + \delta)}{2k + \delta} = Q_i^*$$

$$= \frac{2k^2}{2k + \delta} + \frac{2k\delta}{2k + \delta} \leq k + \delta = a_{2i}.$$ 

Let us denote $f_i^*(Q_i) = \min_{d_i \in D^i(i)} \pi_i(Q_i, d_i)$. For the above instance we have $f_i^*(a_{2i-1}) = (r_i - c_i + l_i)a_{2i-1} - l_i a_{2i} = a_{2i-1}$. Similarly, $f_i^*(a_{2i}) = a_{2i}$. Hence, AR–DS becomes

$$\max_{Q_i \in [a_{2i-1}, a_{2i}]} \sum_{i=1}^{n} Q_i,$$

s.t. $\sum_{i=1}^{n} Q_i \leq \frac{A}{2}$. (A.1)

Clearly, (A.1) has a solution with objective value $\frac{A}{2}$ if and only if there exists a solution to the even-odd partition problem. This completes the proof of the theorem. □

References