The \((r, Q)\) control of a periodic-review inventory system with continuous demand and lost sales

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Abstract

In this paper we consider a periodic review inventory model with lost sales during a stockout and with the constraint that at most one replenishment order may be outstanding at any time. Demands in successive review periods are independent, identically distributed variables from a continuous distribution. The fixed lead time is an integral number of review periods. We explore control policies of the \((r, Q)\) type – that is a replenishment order of size \(Q\) is placed when the inventory position (stock in hand plus stock on order) falls to or below the re-order level \(r\). We use asymptotic renewal theory results to estimate the ‘undershoot’ of the re-order level \(r\) and also to estimate the cycle stockholding cost (which turns out to take a relatively simple form). Based on these approximations we set out a policy improvement solution methodology and illustrate this with some numerical examples for which demand is normally distributed. These numerical examples suggest that a relatively simple approach, based on the economic order quantity, can provide results which are very close to optimal.

Keywords: Inventory; Periodic review; Lost sales; Renewal theory; Undershoot

1. Introduction

In this paper we consider a periodic review inventory model with lost sales during a stockout. Demands in successive review periods are independent, identically distributed variables from a continuous distribution. We explore control policies of the \((r, Q)\) type – that is a replenishment order of size \(Q\) is placed when the inventory position (stock in hand plus stock on order) falls to or below the re-order level \(r\). It is assumed that the optimal policy will be such that there will never be more than one order outstanding at any time, which means that \(Q > r\). The lead time is fixed and is an integral multiple of the review interval – this last assumption simplifies the computation of the stockholding cost but it can be relaxed by allowing the review interval and the lead time to be integral multiples of some (small) basic period, such as a working day, as suggested in Janssen et al. [1]. The analysis extends to variable lead time but its practicality then depends on whether or not a closed form exists for the distribution of demand in lead time. Note that for a lost sales formulation the optimal control policy will, in general, be
neither of the \((r, Q)\) type nor of the \((s, S)\) type but will depend in a much more complex way on the physical stock at the time of placing an order. The complexity of optimal lost sales control policies contrasts with optimal backorder control policies, which, for this formulation, can be shown to be of the \((s, S)\) type. Note also that, for variable lead time, the computation of the optimal policy is more complicated if the lead time is not unimodal.

In many inventory formulations, including this one, the treatment of the lost sales case is more difficult than that of the corresponding backorder case – the main reason being that the inventory position is constant during a stockout. The result is that there is less published work on lost sales models. Morton [2] and Nahmias [3] are two of the few authors to consider the continuous-demand periodic-review lost-sales model – they both used a myopic approach which, although approximate, does permit more than one order to be outstanding at any time. The continuous-review lost-sales model with at most one order outstanding has received rather more attention and most of the standard texts [4–7] contain some reference to it. The most common approach is to ignore stockouts in estimating the mean order cycle time, in which case it is possible to set up an iterative scheme to determine the optimal policy.

A rigorous treatment of the continuous review lost sales model with continuous demand requires a demand process which is infinitely divisible. The only two processes which would appear to meet this criterion are deterministic demand and Wiener demand. A Wiener demand process is one where the demand in any finite interval follows a normal distribution with mean and variance proportional to the duration of the interval, the constants of proportionality being the same for all intervals. The problem with using a normal (or Wiener) demand process is that the coefficient of variation of demand tends to infinity as the time interval tends to zero and therefore the probability of negative demands becomes large as the time interval decreases and a particular stock threshold can be crossed many times. A rigorous treatment requires some assumptions about how the system actually behaves and one assumption, which makes use of absorbing barriers, is that an order is placed the first time \(r\) is crossed and all demand transactions cease when the system first runs out of stock. Beyer [8] performed such a rigorous analysis of an \((r, Q)\) model with Wiener demand, a fixed lead time and lost sales but the analysis is complex and the solution procedure is computationally heavy. With a periodic review and a normal distribution of demand per review interval we only require that the probability of negative demand is so small that it can be ignored.

The approach taken here is to use an asymptotic approximation for the ‘undershoot’, the amount by which stock has fallen below the re-order level by the time an order is placed, and another asymptotic approximation for the mean time-weighted stockholding in an order cycle. The model then has the same structure as the equivalent continuous review model and can therefore be tackled by solution procedures designed for a continuous review model. We employ a solution procedure which differs from the iterative schemes generally used and from the procedure, based on a square-root formula, recently developed by Rosling [9]. Our procedure uses a policy improvement approach which is closer in spirit to the policy iteration algorithms discussed in Tijms [10].

2. Analysis

2.1. Definitions

We make use of the following terminology:

- \(Q\) replenishment order quantity
- \(r\) re-order level
- \(I\) interval of time between successive reviews (defined to be the time unit)
- \(L\) (fixed) lead time of a replenishment order
- \(X_I\) demand in a review interval, with realisation \(x_I\), mean \(\mu_I\) and variance \(\sigma^2_I\)
- \(X_u\) undershoot of the re-order level, with realisation \(x_u\), mean \(\mu_u\) and variance \(\sigma^2_u\)
- \(X_L\) demand during lead time, with realisation \(x_L\), mean \(\mu_L\) and variance \(\sigma^2_L\)
- \(X\) undershoot plus the lead time demand (which we shall call the demand during the ‘period of risk’), with realisation \(x\), mean \(\mu\), variance \(\sigma^2\) and density function \(f\)
stockholding cost per unit per review interval

- time-weighted stockholding until stock is depleted given an initial stock of \( y \)

- renewal function for the demand process – the expected number of review cycles, including the first one, for which cumulative demand does not exceed \( z \)

- fixed cost of placing a replenishment order

- expected lost sales incurred during an order cycle when the re-order level is \( r \)

- probability that the system runs out of stock during an order cycle given \( r \)

- expected cost incurred during an order cycle given policy \( (r, Q) \)

- expected duration of an order cycle given policy \( (r, Q) \)

- long-run average total cost per unit time given policy \( (r, Q) \)

2.2. Assumptions

In addition to those already stated or implied, the following assumptions are made:

(i) It is required that \( (Q - r) \gg \mu_t \) or, equivalently, that an order cycle should be quite long in comparison with the review interval. This assumption is needed for the asymptotic distribution of the undershoot of \( r \) to offer an acceptable approximation.

(ii) Demand \( X_t \) in a review interval cannot be negative. This means that if demand is normally distributed then the parameters are such that the probability of negative demand is negligible.

(iii) The distribution of demand \( X \) during the ‘period of risk’ is known and is assumed to be normal in our numerical examples.

2.3. The undershoot of \( r \)

The ‘undershoot’ is the amount by which stock has fallen below \( r \) at the review point when an order is placed. Derivations of the asymptotic (as \( (Q - r)/\mu_t \to \infty \)) distribution of the undershoot may be found in, for example [10] or [11]. This asymptotic distribution has mean and variance

\[
\mu_u = \frac{E(X^2_t)}{2\mu_t}
\]

and

\[
\sigma_u^2 = \frac{E(X^3_t)}{3\mu_t} - \mu_u^2.
\]

Discussion of the numerical accuracy of these approximate measures in a non-asymptotic setting are contained in Sahin [12] and Baganya et al [13].

In what follows we shall use expressions (1) and (2) for the mean and variance of the undershoot.

2.4. The demand during the period of risk

The period of risk is defined as the time between the inventory position reaching \( r \) and the arrival of the resulting replenishment order. The demand during this period has two components – the undershoot of the re-order level and the demand during the lead time. The mean and variance of the demand in lead time are, assuming independence,

\[
\mu_L = L\mu_t
\]

and

\[
\sigma_L^2 = L\sigma_t^2.
\]

The total demand during the period of risk therefore has mean and variance, again assuming independence,

\[
\mu = \mu_L + \mu_u
\]

and

\[
\sigma^2 = \sigma_L^2 + \sigma_u^2.
\]

It is the density function \( f \) of demand \( X \) during the period of risk which is used to determine the expected lost sales and the probability of a stockout during an order cycle:

\[
\text{ELS}(r) = \int_r^\infty (x - r)f(x) \, dx
\]
and
\[
\text{PSO}(r) = \int_r^\infty f(x)\,dx. \quad (7b)
\]

An alternative approach to estimating the expected lost sales during the period of risk is suggested in Tijms [10, pp. 61–68] in which it is approximated by the difference of two integrals – one relating to demand in lead time and the other relating to demand in lead time plus a review interval. However, in the context of the problem considered here, the formulation which results is more complex than the one we derive.

2.5. The stockholding cost

The unit stockholding cost may be defined in three ways. We could attach a cost to each unit held at the beginning of a review interval (after any receipt of stock), attach a cost to each unit held at the end of a review interval (before any receipt of stock) or attach a cost to the average of these two. We shall use the last measure because it offers the closest correspondence to the equivalent continuous review measure. The stock at the end of a review interval will be the stock at the beginning less the demand met during the interval. Since the average demand met during an order cycle is \( Q \), the mean time-weighted stockholding for an order cycle based on the starting stock will be \( IQ \) more than the mean time-weighted stockholding based on the ending stock. Therefore, it is easy to switch from a formulation based on one interpretation of unit cost to one based on another interpretation.

We shall first estimate the time-weighted stockholding for an order cycle based on the starting stock. Arguing intuitively and starting at a review point, the \( z \)th unit of stock sold will have been held in stock for an average of \( M(z) \) review intervals. Therefore, the mean time-weighted stockholding given an initial stock of \( y \) is given by
\[
H(y) = \int_0^y M(z)\,dz. \quad (8)
\]

This is shown in Theorem 1.1.9(b) of Tijms [10, p. 9], slightly adjusting the result to account for the inclusion of the first period, to take the asymptotic (as \( y \to \infty \)) value
\[
H(y) \approx \frac{y^2}{2\mu_t} + \frac{E(X_t^2)\mu_t}{2\mu_t^2} + k, \quad (9)
\]

where \( k \) depends on the parameters of demand in a review cycle but is independent of \( y \).

We introduce here the positive part operator \( \cdot^+ \) defined by \( y^+ = y \) if \( y > 0 \) and \( y^+ = 0 \) otherwise. The mean time-weighted stockholding for a typical order cycle is the mean stockholding until stock is depleted based on a stock level of \( [r - X]^+ + Q \) just after the receipt of an order, \( H([r - X]^+ + Q) \), less the corresponding measure based on a stock level of \( [r - Z]^+ \) just before the arrival of the next order, \( H([r - Z]^+) \). The overall mean time-weighted stockholding in an order cycle, using Section 2.2(i) to justify the assumption that \( X \) and \( Z \) are (asymptotically as \( Q \to \infty \)) independent, is therefore
\[
E(H([r - X]^+ + Q)) - E(H([r - Z]^+))
\]

\[
= E(H([r - X]^+ + Q)) - E(H([r - X]^+))
\]

\[
= E(H([r - X]^+ + Q) - H([r - X]^+))
\]

\[
= E\left(\frac{2Q[r - X]^+ + Q^2}{2\mu_t} + \frac{E(X_t^2)Q}{2\mu_t^2}\right). \quad (13)
\]

The asymptotic expression for \( H \), which is given by (9) and which is used to derive (13) from (12), should be fairly accurate as an approximation for \( H([r - X]^+ + Q) \) but may not be a good approximation for \( H([r - X]^+) \) unless the stock service level is high. Our belief is that the use of expression (13) does not introduce a significant error and has the benefit of producing a tractable formulation. This expression can be written as
\[
\frac{Q}{\mu_t} \left\{ E([r - X]^+ + Q) + \frac{Q}{2} + \frac{E(X_t^2)}{2\mu_t} \right\}
\]

\[
= \frac{Q}{\mu_t} \left\{ Q - \mu + ELS(r) + \frac{Q}{2} + \mu_t \right\}
\]

\[
= \frac{Q}{\mu_t} \left\{ Q - \mu_L + ELS(r) + \frac{Q}{2} \right\}. \quad (16)
\]

This gives the mean stockholding based on starting stock. We subtract \( IQ/2 = Q/2 \) since \( I \) is the unit of time) to obtain the corresponding measure
based on the average of starting and finishing stock:

\[
\frac{Q}{\mu_t} \left\{ r - \mu_L - \frac{\mu_t}{2} + \text{ELS}(r) + \frac{Q}{2} \right\}. \tag{17}
\]

The mean demand in an order cycle is \((Q + \text{ELS}(r))\) and therefore the mean duration of an order cycle (using the review interval as the unit of time) is

\[
T(r, Q) = \frac{(Q + \text{ELS}(r))}{\mu_t}. \tag{18}
\]

The appropriate measure for average stock is therefore

\[
\frac{Q}{(Q + \text{ELS}(r))} \left\{ r - (L + \frac{1}{2})\mu_t + \text{ELS}(r) + \frac{Q}{2} \right\}. \tag{19}
\]

We have therefore recovered the fairly standard expression for average stock for a stochastic lost sales inventory model. It is the product of two terms, the first term giving the fraction of time the system is in stock and the second giving the average stock level while the system is in stock.

2.6. The problem formulation

The approximate mean total cost associated with an order cycle is

\[
C(r, Q) = \frac{hQ}{\mu_t} \left( r - (L + \frac{1}{2})\mu_t + \text{ELS}(r) + \frac{Q}{2} \right) + A + p\text{ELS}(r) \tag{20}
\]

and the long-run average total cost per unit time (which for the remainder of this paper we shall refer to simply as the ‘cost’) for the system (using the renewal result often attributed to Ross [11]) is therefore

\[
g(r, Q) = \frac{C(r, Q)}{T(r, Q)} \tag{21}
\]

\[
= \frac{1}{(Q + \text{ELS}(r))} \left\{ hQ \left( r - (L + \frac{1}{2})\mu_t + \text{ELS}(r) + \frac{Q}{2} \right) + A\mu_t + p\mu_t\text{ELS}(r) \right\}. \tag{22}
\]

The objective is to minimise this as a function of \(r\) and \(Q\) (or to conclude that it is not economically worthwhile stocking the product).

2.7. The policy-iteration algorithm

Suppose the cost of the best solution found so far is \(g\). Define the ‘\(g\)-adjusted cycle cost function’ as

\[
Y_g(r, Q) = C(r, Q) - gT(r, Q). \tag{23}
\]

In order to find a policy \((r, Q)\) which gives a lower cost than \(g\) we need to find a policy for which \(Y_g\) is negative. The general approach is therefore to minimise \(Y_g\) with respect to \(r\) and \(Q\) and use the result if it gives a negative minimum to obtain an improved policy. If no negative value for \(Y_g\) exists (within the constraints imposed on \(r\) and \(Q\)) then the policy cannot be improved upon and we already have the optimal solution. The spirit of this approach is closer to policy iteration as discussed in [10] than it is to the use of an iterative scheme to minimise (22) directly as suggested by Rosling [9].

Differentiating \(Y_g\) with respect to \(r\) and \(Q\) respectively and setting the results to 0 gives

\[
\text{PSO}(r) = \frac{hQ}{(hQ + p\mu_t - g)} \tag{24}
\]

and

\[
Q = \frac{g}{h} - (r - (L + \frac{1}{2})\mu_t + \text{ELS}(r)). \tag{25}
\]

We shall now describe our policy-iteration algorithm. The algorithm is initialised with some reorder level \(r_N\) and order quantity \(Q_N\) for which the cost is \(g_N = g(r_N, Q_N)\). Each iteration consists of the following three steps.

Step 1: Solve (25) with \(r = r_N\) and \(g = g_N\) for \(Q\), and call the result \(Q_M\).

Step 2: Solve (24) with \(Q = Q_M\) and \(g = g_N\) for \(r\), and call the result \(r_M\).

Step 3: Compute \(g_M = g(r_M, Q_M)\). If the change in the cost or the solution parameters is negligible then stop. Otherwise set \(r_N = r_M\), \(Q_N = Q_M\) and \(g_N = g_M\) and go to Step 1.

The algorithm is stopped at Step 3 either when the improvement in cost is negligible or when the
changes in the policy parameters are negligible, although the latter requires more iterations to achieve a given level of accuracy.

2.8. Is it optimal to stock the item?

We first determine whether it is optimal not to stock the item at all. Set \( g = g_{\text{NS}} = p \mu t \), the cost of not stocking the item, and \( r = 0 \) in (25) and solve for \( Q \) to obtain \( Q_0 \) (with the constraint \( Q \geq 0 \)). If \( Q_0 = 0 \) then, since the optimal value of \( Q \) given \( r \) is a decreasing function of \( r \), it is optimal not to stock the item. If \( g(0, Q_0) < g_{\text{NS}} \) then it is obviously optimal to stock the item, otherwise

\[
\frac{dY_{g_{\text{NS}}}}{dr} = \frac{hQ}{\mu t} \left( 1 - \text{PSO}(r) \right) + \text{PSO}(r) \left( \frac{g_{\text{NS}}}{\mu t} - p \right)
\]

which is \( \geq 0 \) for all \( r \geq 0 \) and so \( Y_{g_{\text{NS}}}(r, Q) \geq Y_{g_{\text{NS}}}(0, Q) \) for all \( r \geq 0 \). But, by the definition of \( Q_0 \), \( Y_{g_{\text{NS}}}(0, Q) \geq Y_{g_{\text{NS}}}(0, Q_0) \) and hence \( Y_{g_{\text{NS}}}(r, Q) \geq Y_{g_{\text{NS}}}(0, Q_0) \). Finally, if \( g(0, Q_0) \geq g_{\text{NS}} \) then \( Y_{g_{\text{NS}}}(0, Q_0) \geq 0 \). It follows that \( Y_{g_{\text{NS}}}(r, Q) \geq 0 \) and, therefore, that the policy of not stocking cannot be improved upon.

2.9. The determination of a good initial policy

Assuming that it is optimal to stock the item, we next determine a (good) starting policy. This is done by ignoring ELS(r) throughout (22) and computing the optimal value of \( Q \). This is the economic order quantity (EOQ) and takes the value \( \sqrt{2A/\mu t/h} \). We then ignore ELS(r) in the denominator of (22), differentiate what remains with respect to \( r \), set the result to 0 and solve for \( r \) to obtain \( r_{\text{EOQ}} \), using

\[
\text{PSO}(r) = \frac{h\text{EOQ}}{p\mu t + h\text{EOQ}}.
\]  

An alternative starting value for \( r \) is obtained by using, in place of (27), the equivalent result for a backorder formulation (which comes from (22)) by ignoring ELS(r) in the denominator and, in the numerator, in the bracketed expression to be multiplied by \( hQ \):

\[
\text{PSO}(r) = \frac{h\text{EOQ}}{p\mu t}.
\]  

In numerical work the use of (28) rather than (27) has frequently produced lower cost initial solutions. The reason for this is apparent from (24) where \( hQ \) will provide a much better approximation for \( g \) than 0 will. When the distinction is needed we use \( r_{\text{LS}} \) as the (lost sales) solution to (27) and \( r_{\text{EOQ}} \) as the (backorder) solution to (28).

Compute \( g_{\text{EOQ}} = g(r_{\text{EOQ}}, \text{EOQ}) \). If \( g_{\text{EOQ}} < g_{\text{NS}} \) then set \( g_N = g_{\text{EOQ}}, r_N = r_{\text{EOQ}} \) and \( Q_N = \text{EOQ} \). Otherwise set \( g_N = g(0, Q_0), r_N = 0 \) and \( Q_N = Q_0 \). Note that we could have started with policy \((0, Q_0)\) anyway but this would generally result in more iterations.

3. Numerical illustrations

A testbed of eight problems and the results obtained are presented in Table 1. For each of these it was assumed that demand in a review interval (and hence in lead time) was normally distributed. It was also assumed that demand \( X \) during the period of risk (the lead time demand plus the under-shoot) was normally distributed, with mean and variance given, from Eqs. (1)–(6), by

\[
\mu = L\mu + \frac{(\sigma^2 + \mu^2)}{2\mu t}\]

and

\[
\sigma^2 = L\sigma_t^2 + \frac{(\mu_t^2 + 3\sigma_t^2)}{3} - \left( \frac{(\sigma_t^2 + \mu_t^2)}{2\mu t} \right)^2,
\]

noting that, for a normal distribution, \( E(X_t^3) = \mu_t^3 + 3\mu_t\sigma_t^2 \). Numerical work on other problems reinforce the general findings discussed below.

For each problem we evaluated the policy arising from three different approaches (App) – BO represents an initial solution based on \( r = r_{\text{BO}} \) and \( Q = \text{EOQ} \), LS represents an initial solution based on \( r = r_{\text{LS}} \) and \( Q = \text{EOQ} \) and Opt represents the
optimal solution (subject to the assumptions and approximations in the model) obtained by our policy-iteration algorithm from an initial solution – this typically required five iterations to obtain results for which the order quantity is correct to five places of decimal. The last column (SL) gives the resulting stock service level (the percentage of demand met).

All the above policies incorporated the asymptotic approximation for the undershoot. One point to note is that BO gives solutions which are very close to the ones given by Opt and consistently better than those given by LS. Policies which did not make an adjustment for the undershoot produced costs which were some 2–3% higher than those for equivalent policies adjusted for undershoot – not surprisingly suggesting lower re-order levels than were required. Policies which employed an iterative scheme along the lines discussed in [4, pp. 168–172], with the adjustment for undershoot, generally produced results which were poorer than BO but slightly better than LS. A solution method based on optimising \( r \) (iteratively), subject to \( Q = EOQ \) and retaining \( ELS(r) \) in the denominator of (22), produced no measurable improvement on BO.

We attempted, by decreasing the review interval, to obtain results which could be compared with those from the corresponding continuous review model, but encountered problems in doing so.
Firstly, decreasing the review interval required us to contravene assumption (ii) of Section 2.2, since the probability of negative demand increases markedly as the time interval tends to zero. Secondly (but related to the first point), the asymptotic expression for the mean undershoot does not tend to zero, as the time interval tends to zero, but tends to $\sigma^2/2\mu$, where $\mu$ and $\sigma^2$ are the mean and variance of demand per unit time. We do not here offer a formulation for a continuous review normal/Wiener demand model but just draw attention to the need for care in setting out the assumptions and analysis required for such a model to be logically consistent.

We used readily available routines for calculating and inverting ELS($r$) and PSO($r$) for the normal distribution (see, for example, [10] pp. 69–71). Although we have concentrated on the normal distribution equivalent routines are available for other distributions, such as the gamma family of distributions.

4. Conclusions

We have considered a periodic-review, continuous-demand, lost-sales model with a fixed lead time and at most one order outstanding and derived a formulation which is structurally the same as that for the equivalent continuous-review model. We have proposed a novel solution methodology, based on policy iteration. In numerical work we have established that a solution based on the economic order quantity and a simple calculation for the re-order level (our BO approach) gives solutions which are very close to optimal under a wide range of parameter settings and which should suffice for most practical purposes. One key area which requires further investigation is how the analysis for this model might be extended to allow for the possibility of more than one replenishment order being outstanding at any time.

References