Looking for arbitrage

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Abstract

We consider financial contracts that are tradable in any quantities at fixed prices. A bundle of such contracts constitutes an arbitrage if it offers non-negative payoff in any future state, but commands negative present cost. This article brings together fairly recent results on how to find an arbitrage provided some exists. Otherwise, a state-contingent, non-profit price vector will be identified. As vehicle we use a simply-constrained least squares problem, minimizing the distance to arbitrage-free pricing. © 2000 Elsevier Science Inc. All rights reserved.

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1. Introduction

Suppose the financial contract (security for short) \( j \in J \) promises to pay \( a_{sj} \) units of account (or money) if and when state \( s \in S \) becomes manifest. Before the true \( s \) is revealed that security \( j \) can be bought in any quantity \( x_j \in \mathbb{R} \) at fixed unit cost \( c_j \). By convention, \( x_j < 0 \) means a short sale. There are no sign restrictions on \( c_j \). Both sets \( S \) and \( J \) are finite, the first providing an exhaustive list of relevant and mutually exclusive contingencies. For convenience, let \( A = [a_{sj}] \) denote the \( S \times J \) payoff matrix. The decision vector \( x \in \mathbb{R}^J \) will be referred to as a portfolio. Any such \( x \), and \( c \in \mathbb{R}^J \) as well, is henceforth regarded as a column vector. Thus, letting \( T \) mean transposition of matrices, the overall cost of portfolio \( x \) equals \( c^T x \).

Such a situation, offering virtually unrestricted purchase of portfolios, makes it most natural, and sometimes quite rewarding, to ask: Does there exist an opportunity...
for arbitrage? Can something be obtained for nothing? Is there a money machine or, as Koopmans (1951) said, a land of Cockaigne? Of course, at a purely semantic level no deep problems emerge: either an arbitrage can be found or not. However, in formal or practical terms the above questions are more challenging. Existence of arbitrage is here taken to mean that the linear inequalities

$$Ax \geq 0, \quad c^T x < 0$$  \hspace{1cm} (1)

admit a solution $x \in \mathbb{R}^t$. Letting $a_s$ henceforth denote the $s^{th}$ row of $A$, Eq. (1) says that non-negative dividend $a_s x$ becomes available in every possible state $s$, provided one buys $x$ at negative overall cost $c^T x$. So, by choosing an arbitrage $x$ you first receive $c^T x$, and subsequently you obtain at least zero in any state. In other words, an arbitrage is a financial investment strategy that generates positive profits at no risk. Evidently, such prospects are rather attractive. Most likely, they will be exploited by competent traders.

So, absence of arbitrage amounts to the insolubility (inconsistency) of Eq. (1). A famous, classical lemma of Farkas (1902) tells that such absence prevails if and only if the dual system

$$A^T y = c, \quad y \geq 0$$  \hspace{1cm} (2)

can be solved for some vector $y \in \mathbb{R}^s$ of state-contingent prices. More precisely, the said lemma asserts that Eq. (1) or Eq. (2) is solvable, but never both. Note, however, that it does not tell which alternative holds. (Clearly, adding more states, or reducing the set of securities, makes the existence of an arbitrage less likely.)

A simple argument helps to understand why Eq. (1) excludes Eq. (2). An arbitrage $x$, if any, furnishes a non-negative valuation $a_s x$ of every row $a_s$ in $A$, and hence of every non-negative combination $A^T y$ of those rows ($y \geq 0$). Since, at the same time, $c^T x < 0$, the vector $c$ itself cannot possibly be such a combination. More formally, the incompatibility of Eqs. (1) and (2) follows from the contradiction

$$0 > c^T x = (A^T y)^T x = y^T A x \geq 0 .$$

Admittedly, existence is a more difficult issue than exclusion. It says that if no price regime $y \geq 0$ on state-dependent dividends yields the non-profit matching $A^T y = c$, then some arbitrage $x$ must be available. Most modern proofs of this last statement (Mangasarian, 1969; Stoer & Witzgall, 1970)—that is, of Farkas’ lemma—invoke separation of two appropriately defined convex sets. That argument is non-constructive. It neither decides which alternative Eq. (1) or Eq. (2) holds nor does it provide explicit solutions. Clearly, to obtain more precise information on existence, one may solve Eq. (1) or Eq. (2) as linear programs, appending then dummy objectives. Doing so is, however, fraught with some inconveniences. First, if one program proves inconsistent, one must start afresh on the other. Second, when a program indeed has a solution, the latter will often be non-unique or unbounded. Thus there is ample motivation for determining, in one shot, which system is solvable while, at the same time, exhibiting a solution.

Such a constructive approach has recently been developed by Dax (1993, 1997).
His procedure appears central, elegant, and of practical importance. It also has pedagogical value. Therefore, a chief purpose of this article is to spell out his construction in finance jargon, a subsidiary objective being to interpret and motivate the procedure in simple economic terms. As minor novelties we offer yet another method and relate arbitrage to a saddle point problem.

2. Arbitrage and least norm problems

Let $\|\cdot\|$ denote the customary Euclidean norm. As a stepping stone to our inquiry we use the following

Proposition 1 (Pricing and least square problems).

1. Eq. (2) is solvable if and only if the constrained least squares problem

   $$\min \frac{1}{2} \| A^T y - c \|^2 \quad \text{subject to} \quad y \succeq 0,$$

   has optimal value 0.

2. Eq. (3) admits, in any case, at least one optimal solution $y^*$, possibly non-unique. However, the image $z^* = A^T y^*$ of any solution $y^*$ will be unique and minimize the distance from $c$ to the closed convex set $Z := \{ A^T y : y \succeq 0 \}$.

3. $y^*$ solves Eq. (3) optimally if the following linear complementarity conditions hold:

   $$y^* \succeq 0, \quad Ar^* \succeq 0, \quad \text{and} \quad y^T Ar^* = 0$$

   where $r^* := A^T y^* - c$ is named the residual vector.

Proof. Statement (1) is immediate. For Statement 2 we use the nontrivial fact that $Z$ is closed. Since $Z$ contains 0, the two problems

$$\min \{ \| z - c \|^2 : z \in Z \} \quad \text{and} \quad \min \{ \| z - c \|^2 : z \in Z, \| z - c \| \leq \| c \| \}$$

have the same solutions. The strictly convex function $z \mapsto \| z - c \|^2$ must have a unique minimum $z^*$ on the non-empty compact convex set $\{ z \in Z : \| z - c \| \leq \| c \| \}$.

Statement (3) results from

$$\frac{\partial}{\partial y} \frac{1}{2} \| A^T y - c \|^2 = A(A^T y - c) = Ar \quad \text{with} \quad r := A^T y - c,$$

and the fact that the differentiable, convex function $g(y) := 1/2 \| A^T y - c \|^2$ attains its minimum over the convex set $Y := \mathbb{R}^n_+$ at some $y^* \in Y$ if the directional derivative $g'(y^*; y - y^*) = (y - y^*)^T \nabla g(y^*)$ is non-negative along all feasible directions $y - y^*, y \in Y$. This happens if

$$y^* \succeq 0 \quad \text{and} \quad (y - y^*)^T Ar^* \succeq 0 \quad \text{for all} \quad y \succeq 0.$$

It remains only to argue that Eq. (4) and the variational Eq. (5) are equivalent.
That argument is standard, but repeated all the same: Eq. (5) implies Eq. (4) because first, the two choices \( y = 2y^* \) and \( y = 0 \) in Eq. (5) yield \( y^*Ax^* = 0 \), and second, now knowing from Eq. (5) that \( y^*Ax^* \geq 0 \) for all \( y \geq 0 \), we get \( Ax^* \geq 0 \). The converse (i.e., Eq. (4) ⇒ Eq. (5)) is straightforward.

We remark that Statement 3 also derives from the Kuhn-Tucker conditions associated to the Lagrangian \( g(y) = Ty \) of Eq. (3). These say that \( Ty^* = 0, \quad y^* = g(y^*) = Ar^* = \lambda \geq 0, \quad y^*T \lambda = y^*Ax^* = 0 \).

For any contingent price vector \( y \geq 0, y_s \) is construed as the valuation of one unit of account in state \( s \). Thus, \( r_j := \sum_{s \in S} a_{js}y_s - c_j = \sum_{s \in S} a_0y_s - c_j \) is the corresponding net return on security \( j \). In price-taking (competitive) equilibrium, if any, that return \( r_j \) should be nil for all securities. Therefore, to solve Eq. (3) will henceforth be named an attempt at competitive pricing. The usefulness of that program has been brought out by Dax (1993):

**Theorem 2 (Explicit decidability of the arbitrage problem).** Let \( y^* \) be any optimal solution to the constrained least squares Eq. (3). Then the associated residual

\[
r^* := A^T y^* - c
\]

is unique, and \( r^* = 0 \) if \( y^* \) solves Eq. (2). Otherwise, when \( r^* \neq 0 \), this vector provides an arbitrage. More precisely, \( Ar^* \geq 0 \) and \( c^T r^* = -||r^*||^2 \).

Proof. The first assertion derives from Proposition 1, statement (2). For the last assertion, given \( c = A^T y^* - r^* \), we need only record that

\[
c^T r^* = (A^T y^* - r^*)^T r^* = y^* A r^* - r^* T r^* = -r^* T r^* = -||r^*||^2 \,.
\]

Thus, the solution of Eq. (3) explicitly decides which alternative holds. More important, it either quotes a state-contingent price vector \( y^* \) solving Eq. (2), or it displays a specific arbitrage \( r^* \). In any case, the economic nature of these two objects \( y^* \) and \( r^* \) is easy to grasp: given a price regime \( y^* \geq 0 \), then security \( j \) provides net revenue \( r_j^* := \sum_{s \in S} a_{js}y^*_s - c_j \) per unit. Recall that price-taking behavior and absence of quantity restrictions are essential features of our setting. To render those features realistic we should reasonably have each \( r_j^* = 0 \). Otherwise, when offering unrestricted choice \( x_j \) at fixed unit cost \( c_j \), some profit or portfolio will be unbounded. So, we naturally want \( r_j^* = 0, \forall j \), that is, \( \sum_{j \in J} r_j^* = 0 \). This observation tells that \( y = y^* \) ought better be chosen so as to

\[
\text{minimize} \quad \sum_{j \in J} r_j^* = \sum_{j \in J} \left( \sum_{s \in S} a_{js}y^*_s - c_j \right)^2 = \sum_{j \in J} \left( \sum_{s \in S} a^T_{js}y^*_s - c_j \right)^2 = ||A^T y - c||^2 .
\]

We already noted that when the optimal value of the last problem equals zero, any optimal solution \( y^* \) solves Eq. (2). Otherwise, when \( r^* \neq 0 \), each nonzero component

\[
r_j^* = \sum_{s \in S} a_{js}y^*_s - c_j
\]

points to a security \( j \) which, at best, remains “wrongly” priced. Specifically, when \( r_j^* > 0 \), that security \( j \) is worth buying; it exhibits positive return (net of its purchase
Depending upon the current $y$, the complementarity condition $y^TA^*y - c 
eq 0$. Then the unit vector $r^* = ||r^*||$ solves the following problem concerning steepest cost descent:

$$\text{minimize } c^T x \text{ subject to } Ax \geq 0 \text{ and } ||x|| = 1. \quad (6)$$

Proof. Any admissible $x$ in Eq. (6) yields $y^TAx \geq 0$ and thus

$$c^T x = (A^Ty^* - r^*)^T x = y^TAx - r^T x \geq -r^T x \geq -||r^*|| \geq -||r^*||.$$

In particular, since $x = r^*/||r^*||$ is feasible, and $c^Tr^*/||r^*|| = -||r^*||$, the conclusion follows.

We note that given any portfolio $x$ and state-contingent price vector $y$, the associated net return $\pi(x,y) := y^TAx - c^T x = (A^Ty - c)^T x$ has gradient $[(\partial/\partial x)]\pi(x,y) = A^Ty - c = r$. So, $r$ is also the direction of steepest payoff ascent.

In light of Theorem 1, to make Eq. (3) not merely illuminating but practical as well, we need an efficient and economically motivated solution procedure. In fact, again following Dax (1997), a constructive, direct method is already available. Before listing details we give a broad overview of its structure. The restriction $y \geq 0$ and the identity $r = A^Ty - c$ will be enforced throughout, this causing little extra effort. In view of Eq. (4) the remaining task is to ensure $Ar \geq 0$ and $y^TAr = 0$. Let us briefly discuss the last condition first. At any stage of the computation there is a set $S^+ \subseteq S$, depending upon the current $y$, that contains exactly those states $s$ for which $y_s > 0$, the other contingent prices being nil. Therefore, since $y \geq 0$, and in the end $Ar \geq 0$, the complementarity condition $y^TAr = 0$ amounts to

$$y_s = 0 \quad \text{or} \quad a_r = 0 \text{ (or both) for every } s \in S;$$

that is, to the requirement that $a_r = 0$ for all $s \in S^+$. Therefore the algorithm has a part (I), concerned with complementarity, from which it exits if indeed $a_r = 0$ for all $s \in S^+$. While still in part (I) we attempt to reduce $S^+$ or the gap $||A^Ty - c||^2 = \sum_{s \in S^+} a_s y_s - c ||^2$ to competitive pricing. Upon exiting part (I) (always with $y \geq 0$, $r = A^Ty - c$, and $y^TAr = 0$) the algorithm enters part (II), whose prime concern is to check the only remaining condition: whether $Ar \geq 0$. If that concern has no bite (i.e., if $a_r \geq 0$, $\forall s \not\in S^+$) then, invoking Eq. (4), we are done: the actual $y$ solves Eq. (3). Otherwise, there is a possibility to reduce the gap $||A^Ty - c||^2$ to competitive pricing by increasing merely one of those prices $y_s = 0, s \not\in S^+$, for which actually $a_r < 0$.

After these preparations we are now ready to state a direct algorithm for the constrained least square Eq. (3): Begin with any price vector $y \in \mathbb{R}^S$.

(I) Identify the positive prices $S^+ \leftarrow \{s \in S : y_s > 0\}$. Go on to compute the gap to competitive pricing; that is, find the residual $r \leftarrow A^Ty - c$. If $S^+$ is empty, or $r = 0$,
or \( a_r = 0 \) for all \( s \in S^+ \), then test optimality as described below. Otherwise, reduce the gap to competitive pricing; that is, find a direction \( d \) along which strictly positive prices (and only those) are to be changed. Specifically, solve the unconstrained least squares problem

\[
\minimize \| \sum_{s \in S^+} a_l(y_s + d_s) - c \|^2
\]

(7)

with respect to the direction \( d \in \mathbb{R}^{S^+} \). This done, change positive prices along \( d \) as follows:

\[
y_s \leftarrow \begin{cases} 
  y_s + \theta d_s & \text{if } y_s > 0 \\
  y_s & \text{otherwise},
\end{cases}
\]

where, to keep prices \( \geq 0 \) and the positive step size \( \theta \) bounded, we use

\[
\theta := \min\{1, -y_s/d_s : s \in S^+, d_s < 0\},
\]

If \( \theta < 1 \), then some price \( y_s \), which most recently was positive, has just become zero, and \( s \) drops out of \( S^+ \). In that case, return to identify positive prices. Otherwise, when \( \theta = 1 \), (II) Test optimality. If \( a_r \geq 0 \) for all states \( s \) with \( y_s = 0 \), then \( y \) solves Eq. (3) and the algorithm terminates. Otherwise, increase a zero price as follows. Among those states \( s \) having \( y_s = 0 \) select one for which the payoff \( a_r \) is minimal, whence negative. For that state quote a larger price

\[
y_s \leftarrow -a_r/\|a_s\|^2,
\]

(8)

and return to identify positive prices.

Proposition 4. The algorithm described above converges after finitely many steps.

Proof. The objective value, namely the competitive gap \( \|A^Ty - c\|^2 \) decreases strictly each time Eq. (7) or Eq. (8) is executed. In fact, Eq. (8) is the solution \( \theta \in \mathbb{R} \) to the auxiliary, unconstrained problem: minimize the function

\[
h(\theta) := \|A^T(y + \theta e_s) - c\|^2 = \|\theta a_s^T + r\|^2 = \theta^2\|a_s\|^2 + 2\theta a_r + \|r\|^2
\]

with respect to \( \theta \). (Here \( e_s \in \mathbb{R}^S \) is the standard unit vector having 1 in component \( s \) and 0 elsewhere.) Since \( h'(0) = 2a_r < 0 \), upon solving the latter problem, a strict reduction in the objective value obtains. Note that part (I) requires finitely many steps each time it comes into operation. Indeed, whenever \( \theta = 1 \), the algorithm leaves this part. Otherwise, if \( \theta < 1 \), the set \( S^+ \) will be reduced. Finally, observe that we enter part (II) only a finite number of times. In fact, since \( a_r = 0 \) for all \( s \in S^+ \) each time we enter, the price vector \( y \), being brought into (II), solves the problem

\[
\minimize \|A^Ty - c\|^2 \text{ subject to } y_s \geq 0 \text{ for } s \in S^+, \text{ and } y_s = 0 \text{ otherwise.}
\]

There are finitely many such problems, and on each encounter with one of them the objective value has been strictly reduced.

The algorithm above makes frequent calls on unconstrained squared norm minimi-
zation Eq. (7) for which there are many good algorithms Demmel (1997). The following, indirect method avoids such calls and is remarkably simple. It is easier to implement than the direct method but does not, in general, produce the very desirable finite convergence. For any \( y \in \mathbb{R}^S \) let \( y_+ := \max(0, y_+) \) denote its projection onto the non-negative orthant \( \mathbb{R}^S_+ \).

Proposition 5 (convergence of an indirect method). Define \( M := AA^T \), \( q := Ac \), and let \( \sigma \) be a positive number such that

\[
y^TMy \geq \sigma \|My\|^2 \quad \text{for all} \quad y \in \mathbb{R}^S.
\]  

(One possible choice \( \sigma \) equals the inverse of the spectral radius of \( M \).) Pick any numerical sequence \( \{\rho_k\} \) contained in a compact subinterval of \((0,2\sigma)\). Then, the following alternative, indirect algorithm converges, for any initial point \( y^0 \in \mathbb{R}^S_+ \), to an optimal solution of Eq. (3): Update the current price vector \( y^k \) iteratively at stages \( k = 0, 1, \ldots \) by the rule

\[
y^{k+1} := [y^k - \rho_k(My^k - q)]_+.
\]  

Proof (by reference to the literature). The matrix \( M \) is symmetric and positive semidefinite. Hence, it satisfies what is named the co-coercity condition Eq. (9) with \( \sigma = (\text{spectral radius of } M)^{-1} \), see Goeleven et al. (1997) Theorem 2.1. The result now follows from a splitting method of Tseng (1991) and Gabay (1983) as explained in Goeleven et al. (1997) Lemma 2.1.

One convenient specification is to let \( \rho_k \) in Eq. (10) be a constant \( \rho \in (0,2\sigma) \), typically found by trial and error. Then, taking the convergence \( \lim_{k \to \infty} y^k =: y^* \) of Eq. (10) for granted, we get asymptotically the fixed point condition

\[
y^* = [y^* - \rho(My^* - q)]_+,
\]

whence Eq. (4), namely, \( y^* \succeq 0, My^* - q = A(A^Ty^* - c) = Ar^* \succeq 0, \) and \( y^*^T(My^* - q) = y^*^TA^Ty* - Ar^* = 0. \) We do not take any stand on how Eq. (10) fares compared with the direct method.

3. Conclusions

A fundamental property regarding pricing of financial securities is that any agent with monotone preferences can choose a best portfolio if there are no arbitrage opportunities. Thus, a precondition for a vector of security prices to yield equilibrium is that it renders arbitrage impossible. Otherwise, there could not be a balance between supply and demand in the financial markets. Absence of arbitrage is, however, a more primitive concept than that of financial market equilibrium. This article is exclusively concerned with the first, rather simple concept, about which we end with five observations:

First, we observe that other so-called theorems of the alternative (Dax, 1993), notably that of Gordan, might serve just as well in characterizing absence of arbitrage
(see Magill & Quinzii, 1996). Whatever suitable theorem is chosen there is an associated least norm problem tying the two alternatives constructively together.

Second, we note that the above analysis, albeit not the algorithm, can accommodate infinitely many states or securities. It would then be natural to invoke appropriate Hilbert spaces.

Third, we find it instructive to relate arbitrage to the (min-max) saddle value of the bivariate function

$$f(x,y) := (c - A^Ty)^T x = c^T x - y^T A x = -\pi(x,y)$$

(11)
defined on $X \times Y$ where $X := \{ x \in \mathbb{R}^l : Ax \geq 0, \|x\| \leq 1 \}$ and $Y := \mathbb{R}^m$. Of course, for this consideration to have good meaning, the set $X$ must be non-empty. That property can, at no loss of generality, be guaranteed by adding another security which always pays dividend 0 and presumably is sold at unit cost $0$.

Proposition 6. Suppose $X := \{ x \in \mathbb{R}^l : Ax \geq 0, \|x\| \leq 1 \}$ is non-empty. Then Eq. (11) has a finite saddle value

$$v := \inf_{x \in X} \sup_{y \in Y} f(x,y) = \sup_{y \in Y} \inf_{x \in X} f(x,y).$$

Moreover, there is a portfolio $x^* \in X$ such that

$$v = \sup_{y \in Y} f(x^*,y).$$

It always holds that $v = c^T x^* \leq 0$, and $x^*$ is an arbitrage if $v < 0$.

Proof. The first two statements follow from the min-max theorem in Aubin (1992), Theorem 8.1. Since $0 \in X$ and $f(0,y) = 0$ for all $y$, we get $v \leq 0$. Given $x^*$, any supremum choice $y^* \geq 0$ must satisfy $y^T A x^* = 0$, whence $v = c^T x^*$. If $v < 0$, then $x^*$ is an arbitrage by definition.

Fourth, we note that the passage from stationarity in Eq. (3) to optimality conditions Eq. (4), that is, to

$$y^* \succeq 0, \quad My^* - q \succeq 0, \quad y^T (My^* - q) = 0,$$

with $M = AA^T$ and $q = Ac$ (as in Proposition 5), is well known and goes both ways. This link helps to draw insights about arbitrage from the rich theory on linear complementarity (Cottle et al., 1992).

Fifth and finally, we mention that there are well known probabilistic characterizations of no arbitrage, starting maybe with Arrow (1953), continued in Cox and Ross (1976), Harrison and Kreps (1979). It appears though that many computational aspects remains unexplored, particularly in multi-period settings.

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Notes

1. For a notable exception see Kormornik (1998).
2. This minimal consistency condition is independent of individual preferences and endowments. An elegant study of Ellerman (1984) deals with arbitrage in other, rather general settings.

References


