Probability of call and likelihood of the call feature in a corporate bond

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Abstract

This paper suggests a new way of predicting the likelihood of a corporate bond being callable. We compute the probability that a bond, if callable, would actually be called within a certain period. We also hypothesize a positive relationship between this probability and the likelihood of the bond being issued with a call feature. Comparative static results yield the following empirical implications: the likelihood of a call feature should be an increasing function of coupon rate, corporate tax rate and leverage ratio, and a decreasing function of interest rate and firm risk (volatility). Tests with recently issued corporate bonds provide fairly strong support for the model’s predictions. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

An important component of corporate bond design is the presence (or absence) of a call feature. Traditionally, a significant percentage of long-term corporate debt has been callable, although this percentage changes over time. The determinants of the call feature are therefore of significant interest to finance researchers and practitioners.

Theoretically, a number of reasons have been suggested for the inclusion of a call feature (see Kish and Livingston, 1992, for a brief summary) such as tax advantage, managerial flexibility, interest rate risk, and maturity considerations. The tax advantage hypothesis relies on differential tax rates between the corporation and the marginal bondholder; but this argument was undercut by Miller’s Hypothesis (1977), which proved that the two tax rates must be equal in equilibrium (Kish and Livingston, 1992, p. 690). Additional tax effects have also been examined by Mauer and Lewellen (1987) and Mauer et al. (1991). The managerial flexibility hypothesis states that the call option embedded in the bond (i.e., option to replace high-yield with low-yield debt) is valuable, hence corporate bonds tend to be callable; more recent papers also proposed asymmetric information/signalling and agency problems (Robbins and Schatzberg, 1986; Barnea et al., 1980, etc.) as possible reasons for issuing callable bonds. According to the interest rate risk hypothesis, the value of the option to call is higher when interest rates are high (because of the greater potential for future drops in interest rates), hence bonds are more likely to be callable during periods of high interest rates. The maturity hypothesis states that the option value is higher for longer-maturity bonds, hence longer-maturity bonds are more likely to be callable.

There is a small empirical literature on callable bonds, e.g., Vu (1986), King and Mauer (1999) and Kish and Livingston (1992). Vu found a number of reasons for bond calls, e.g., relaxing restrictive covenants, leverage hypothesis, information effects, and wealth transfer hypothesis, but no single effect dominated. King and Mauer also concluded that no single underlying motive fit the average call, but they did find some support for the bond refunding hypothesis, restrictive covenant elimination hypothesis, and leverage adjustment (or capital structure) hypothesis. Kish and Livingston (1992) identified factors that influenced the likelihood of a call feature in a corporate bond, and reported no support for the managerial flexibility hypothesis, the tax advantage hypothesis or the interest rate risk hypothesis, but some support for the maturity hypothesis.

This paper suggests a different approach to explain and predict the inclusion of the call feature in a long-term corporate bond. We first compute the probability that a bond will be called within a given time horizon, based on the

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1 Kish and Livingston (1992) report that about 83% of public offerings of corporate bonds during 1977–1986 were callable.
optimal call policy and the first passage time distribution. We then test empirically the relationship between the ex-ante probability of call and the likelihood of a call feature in the bond. The test hypothesis is that, holding everything else the same, a bond with a larger probability of call is more likely to be issued with a call feature. Thus, any factor which increases the probability of call (e.g., high coupon rate) will also increase the probability of the bond being callable, ceteris paribus. The empirical tests provide fairly strong support for this hypothesis.

In order to compute the probability of call within a certain time period, we need to know the optimal call policy under taxes and bankruptcy costs; this is derived in our model using an extension of Leland (1994). We also need to compute the optimal call premium, for which we use the agency-cost-minimization approach of Fischer et al. (1989). Our model then computes the probability (at the time of issue) that the bond, if callable, will be called within a certain number of years, assuming that both the call policy and the call premium are optimal. This probability is found to be a function of the bond characteristics (coupon rate), the issuing firm characteristics (firm volatility, tax rate, and leverage ratio) and the risk-free interest rate. From the comparative static results, we derive empirical implications about the effect of each relevant factor on the probability of call, and thereby on the likelihood of the bond being callable. These implications are tested using a sample of recent offerings of long-term corporate bonds. The results offer fairly strong support for the model’s predictions, in spite of the small sample size and the noisy estimates of some sample parameters.

The rest of the paper is organized as follows: Section 2 shows how to determine the optimal call policy for a bond with a given call premium, using an extension of Leland’s (1994) methodology. Section 3 computes the optimal call premium, i.e., the one which minimizes agency costs, as in Fischer et al. (1989). In Section 4, we derive the probability of a bond being called within time \( T \), based on the first passage time distribution, assuming that the bond is called optimally. Section 5 describes comparative static results for the probability of call, and enumerates the empirical implications arising from these results. These implications form the basis for the empirical tests carried out in Section 6, whose results are quite supportive of the model’s predictions. Section 7 summarizes the main results and concludes. The mathematical derivations are presented in Appendices A and B.

2. Theory: Optimal call policy

2.1. The model

The model is similar to Leland (1994), except that the bond is callable. There exists a firm whose unlevered value (i.e., value of its assets) \( V \) follows a continuous diffusion process with constant proportional volatility:
\[ \frac{dV}{V} = (\mu - \delta) \, dt + \sigma \, dz, \]  

(1)

where \( \mu \) is the total expected return on asset value, \( \delta \) the constant fraction of value paid out to all security holders, \( \sigma \) the volatility, and \( dz \) is the increment of a standard Brownian motion process.

There is an outstanding issue of callable debt of infinite maturity \(^2\) with a continuous coupon rate \(^3\) of \( c \) (as a fraction of face value \( SF \); i.e., the coupon amount \(^4\) is \( cF \) per unit time) and a call premium of \( p \) (also as a fraction of face value). The call price is therefore \( (1 + p)F \), which is the amount bondholders receive when the firm calls the bond. There exists a risk-free asset which pays a continuous constant \(^5\) interest rate \( r \), and the corporate tax rate \( \tau \) is constant. Since the tax law allows firms to deduct the call premium paid from its taxable income, there is a tax benefit to the firm associated with the call premium (see Mauer and Lewellen, 1987). Assuming that the bond was originally issued at par (which is the norm), the firm’s loss when it calls the bond is given by the call premium times the face value, or \( pF \). Hence the tax benefit associated with the call decision is the loss times the corporate tax rate, or \( \tau pF \).

When \( V \) falls to a default-triggering level \( \Theta \), equity holders declare bankruptcy and bondholders take over the firm’s assets but incur bankruptcy cost \( \alpha \) (as a fraction of the asset value), where \( 0 \leq \alpha \leq 1 \). Thus the bankruptcy cost is \( \alpha \Theta \), bondholders’ payoff is \( (1 - \alpha)\Theta \), and equity holders are left with nothing.

\(^2\) This is a common assumption in the corporate bond and capital structure literature, e.g., Mauer (1993), Fischer et al. (1989), Leland (1994), Merton (1974), Black and Cox (1976), etc. With long maturity bonds, the return of principal has negligible value and can be ignored (see Leland, 1994); moreover, hundred-year bonds are making a comeback on Wall Street (Schwimmer, 1996). Analytically, infinite maturity permits time-independent valuation formulas and optimal policies, which keeps the analysis tractable.

\(^3\) It is true that firms in real life usually determine the coupon rate and call feature simultaneously. Thus, strictly speaking, the coupon rate is not an exogenous determinant of the call feature. However, we are trying to determine the probability of call for a given coupon rate, hence the coupon rate in our model is exogenous.

\(^4\) Note that the total payout from operations is \( \delta V \). If this is insufficient to make coupon payments, then the shortfall will be financed by issuing additional equity. Of course, shareholders will do this only if it is optimal to keep the firm going; otherwise they will default on the debt, as discussed below. See Leland (1994, footnote 12) for a discussion on this point.

\(^5\) There are two factors driving the call decision – default risk and interest rate risk. Interest rate risk has been examined by Brennan and Schwartz (1977), Kraus (1983), Chiang and Narayanan (1991) and Mauer (1993), etc., and default risk by Fischer et al. (1989) and Longstaff and Tuckman (1994). While a complete treatment would include both sources of risk, it would complicate the model significantly, and has not been attempted yet. We have, therefore, decided to follow the example of Fischer et al. (1989) and focus on default risk. The effect of default risk is important by itself because of the cross-sectional implications from such a model. Moreover, it has been shown empirically (Kish and Livingston, 1992) that that interest rate risk is not significant in determining whether a bond is callable or non-callable.
In our model, the default-triggering level $\Theta$ is determined endogenously as an optimal decision by shareholders. That is, in Leland’s (1994) terminology, the callable bond is not protected debt (as it would have been with an exogenously specified default-triggering level). This is more realistic than protected debt, since long-term corporate bonds in practice are usually unprotected (see Leland, 1994).

The callable bond also ceases to exist when it is called, at which time the bondholder is paid the call price. Suppose the firm’s call policy is to call the bond when the state variable $V$ rises to a certain critical level $V^*$. Then, at the first instant that $V = V^*$, the bond is called and replaced by a perpetual non-callable bond with the same dollar coupon payment $c_F$ per unit time. The refunding cost is given by $\beta$ times the value of the replacement (non-callable) debt, where $0 \leq \beta \leq 1$. $\beta$ is therefore just a measure of the fractional refunding (or flotation) cost; we assume it is constant, but can be made variable (see Mauer, 1993). Like the default trigger, the call trigger $V^*$ is also determined optimally by the firm.

2.2. Optimal call policy

As described above, there are two levels of $V$ which trigger some action by the firm; when $V$ falls to the lower level the firm defaults, and at the upper trigger level the firm calls the bond. Both the triggers should be chosen optimally, i.e., so as to maximize the value of the equity of the firm. This can be done by imposing the smooth-pasting condition for both the triggers (see Leland, 1994). The existence of an optimal lower (default) boundary $H$ has been discussed by Leland (1994) and will not be repeated here.

The upper trigger $V^*$ has been used in the context of call decisions by Fischer et al. (1989) and Leland (1998). Fischer et al. state “...to characterize a firm’s recapitalization policy, we employ a simple strategy in the spirit of the (s,S)-inventory control problem: a firm adjusts its capital structure whenever the state variable reaches a critical upper bound or lower bound” (p. 341). Leland states “... firm calls its debt and restructures with newly issued debt if asset value rises to the call level” (p. 1217). However, although it has been used in the literature, the existence of a finite upper boundary has not yet been demonstrated. It is not clear that the (American) option to call the bond should be exercised prior to maturity. Therefore we examine the issue briefly below, and provide further details in Appendix A.9.

$^6$ The assumption of replacement by a non-callable bond is a standard one in the refunding literature; the arguments justifying this assumption are given in Kraus (1983) and Mauer (1993).

$^7$ This assumption is made to ensure maintenance of debt-service parity, as in Mauer (1993) and Yawitz and Anderson (1977).
First of all, note that when the callable bond is replaced with a non-callable bond, the debt service (or dollar coupon amount, $cF$) remains unchanged. The firm pays the old bondholders the call price, avails of the tax benefit associated with the call premium, and receives the market value of the non-callable bond from the new bondholders. This non-callable bond value is an increasing function of $V$ because of the consequent reduction in default risk when $V$ rises. Hence the firm’s payoff at call is an increasing function of $V$, and the firm should optimally call and replace the existing bond when $V$ is large enough, i.e., $V^*$ will be large. We show below that $V^*$ will also be finite in general.

The payoff at call (as shown in Appendix A.9) is

$$\left\{ \frac{cF}{r} - \left[ \frac{cF}{r} - (1 - \tau)V_B \right] \left( \frac{V}{V_B} \right)^{\gamma_2} \right\} + \tau pF - (1 + p)F,$$

where the term in braces $\{ \cdots \}$ is the market value of the replacement non-callable bond, the second term is the tax benefit from the call premium, and the third term the amount paid to old bondholders (the call price). This expression can be rewritten as

$$\left\{ \frac{cF}{r} - [1 + (1 - \tau)p]F \right\} - \left[ \frac{cF/r - (1 - \tau)V_B}{V_B^{\gamma_2}} \right] V^{\gamma_2}$$

or $\lambda_1 - \lambda_2 X$, where $\lambda_1$ and $\lambda_2$ are constants and $X$ is the rescaled state variable, given by

$$V^{\gamma_2} = X.$$ 

The payoff at call, $\lambda_1 - \lambda_2 X$, can be recognized as the exercise payoff from a put option. Thus the option to call and replace the bond is actually equivalent to a put option.

It is well known in option theory that a put option, unlike a call option, can optimally be exercised prior to maturity even in the absence of dividends. Since the option to call the bond is equivalent to a put option, as shown above, this implies that it will generally be optimal to call the bond when $X$ is small enough but non-zero. Equivalently, since $\gamma_2$ is negative, the bond should optimally be called when $V$ is large enough but finite. For complete details of the argument, see Appendix A.9.

In Appendix A, we show that the optimal default trigger $\Theta^*$ and the optimal call trigger $V^*$ can be determined jointly as follows:

**Result 1.** For a given call premium, the optimal default policy $\Theta^*$ and the optimal call policy $V^*$ can be obtained by solving the following Eqs. (2) and (3) simultaneously.
where the constants $K_3$ through $K_{10}$ are defined in Appendix A, and $V_B$ is the default trigger for a non-callable bond, i.e., firm will default when $V$ falls to $V_B$ (as in Leland, 1994). Appendix A.2 provides an analytical expression for $V_B$.

2.3. Numerical illustration

There being no analytical solution for the optimal policies, we present a numerical example to illustrate the model, using the following parameter values: $F = \$100, c = 8\%, p = 6\%, \sigma = 0.2, r = 6\%, \delta = 3\%, \alpha = 0.5, \beta = 0.01, and \tau = 0.35$. With these values, solving Eqs. (2) and (3) gives: (i) the optimal default policy is to declare bankruptcy as soon as $V$ falls to $\Theta^* = 54.2153$; and (ii) the optimal call policy is to call and replace the bond as soon as $V$ rises to $V^* = 165.7546$.

3. The agency problem of early recapitalization and the optimal call premium

As discussed in Section 2.2 above and Appendix A, the optimal call policy maximizes the ex-post equity value for all values of $V$. However, it does not necessarily maximize the total firm value. To use the terminology of Fischer et al. (1989), the call policy represents the second-best policy, where the first-best policy would result in maximizing total firm value.8

A first-best policy (one that maximizes total firm value) would minimize the losses to outside entities or third parties such as under-utilized tax benefits, bankruptcy and/or financial distress costs paid to outsiders, etc. Equity holders, however, have an additional incentive; when $V$ increases, the bond becomes less risky while the coupon reflects the previous (higher) level of risk. Therefore, it is worthwhile for equity holders to transfer wealth from bondholders to themselves, which they can do by calling the bond at the agreed-upon call price, even if it does not maximize total firm value. This is what

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8 Recall that the ex-ante (i.e., prior to issuing the debt) objective of the firm is to maximize the total firm value, whereas after issuing debt, the objective is to maximize the equity value. See Leland (1994) for details.
Fischer et al. (1989) call “the agency problem of early capitalization” (p. 428). This problem can be mitigated by making recapitalization costly enough, in the form of an appropriate call premium. The optimal call premium is computed in our model such that this agency problem is eliminated, and the second-best (equity maximizing) policy coincides with the first-best (total firm value maximizing) policy.

By not following the first-best policy, the shareholders incur (ex-ante) the agency costs that arise from not maximizing firm value (i.e., the difference between firm value when it is maximized and firm value when equity value is maximized). One way to avoid that is by pre-committing to the first-best or firm-value maximizing policy. Such a commitment will, however, not be credible unless the firm-value maximizing policy also happens to be the (ex-post) equity-value maximizing policy, i.e., there is no incentive incompatibility. This can be ensured by choosing an appropriate call premium. Thus, the optimal call premium is such that maximizing equity value and maximizing total firm value both lead to the same call trigger $V^*$. As shown in Appendix A, the optimal call premium $p^*$ (along with the optimal default and call policies) can be determined as follows:

**Result 2.** The optimal default policy $\Theta^*$, the optimal call policy $V^*$, and the optimal call premium $p^*$ can be jointly obtained by solving Eqs. (2)–(4) simultaneously; Eq. (4) is given below:

$$
\gamma_1 (V^*)^{\gamma_1} (K_5 - K_7 - K_{10}) + \gamma_2 (V^*)^{\gamma_2} (K_6 - K_8 - K_{10})
+ \gamma_2 \left( \frac{V^*}{V_B} \right)^{\gamma_2} \left[ \frac{\tau cF}{r} + \alpha V_B + (1 - \tau) \beta \left( V_B (1 - \alpha) \frac{cF}{r} \right) \right] = 0.
$$

This result is very similar to the Fischer et al. (1989) derivation of the optimal call premium, except that we have included the following, which were ignored by them: tax effects of the replacement bond and the refunding costs, and effect of potential bankruptcy costs associated with the replacement bond. Thus, our boundary conditions are different from Fischer et al.’s model.

For the same parameter values as in Section 2.2 (except that the call premium is now chosen optimally), solving Eqs. (2)–(4) simultaneously gives the following:

$$
\Theta^* = 54.5992, \quad V^* = 171.9174, \quad \text{and} \quad p^* = 8.965\%.
$$

### 3.1. Optimal call policy in terms of bond price

In the traditional literature (see Longstaff and Tuckman, 1994, for some references), the optimal call policy has often been expressed in terms of bond price, i.e., call when bond price first equals call price. However, it can be shown
that this is not true when the call premium is different from the optimal value. In general, it is optimal to call not as soon as the bond price reaches the call price, but only after it has exceeded call price by a certain margin. The bond price first increases and then falls as firm value $V$ increases, and it is possible for the bond price to exceed the call price when the call premium is not optimal. Incidentally, Mauer (1993) has shown, in a model with interest rate risk, that bond price can exceed call price, even without agency problems.

When the call premium is optimal, however, it is always optimal to call as soon as bond price reaches call price, as shown by Fischer et al. (1989). In such cases, the bond price will be an increasing function of $V$ until the bond is called, at which point the bond price will equal the call price. Thus, when the call premium is set optimally, bond price will never exceed call price.

4. The probability of call by time $T$

Here we show how to compute the probability that the bond will be called by time $T$, for a given call premium $p$. The bond should optimally be called the first time $V$ reaches the call trigger $V^*$; thus the time to call can be defined in terms of a hitting time or first passage time distribution (see Harrison, 1985). It is defined formally as follows: the hitting time of the process $\{X(t)\}$ to the level $y$ is $T(y) = \inf \{t \geq 0, X(t) = y\}$. Hitting times of points play a fundamental role in the study of one-dimensional diffusion processes, and the concept is used here to describe a bond call, which is triggered by firm value $V$ rising to the trigger level $V^*$. This idea was also used in the context of default by Black and Cox (1976).

We are interested in the probability of call by time $T$, i.e., the probability that $V$ has reached the call trigger $V^*$ by time $T$. Also suppose that, for the stochastic process $V$, the hitting time to $V^*$ is $t(V^*)$, i.e., $t(V^*) = \inf \{t \geq 0, V(t) = V^*\}$. If the bond is called by time $T$, then it must be the case that the hitting time $t(V^*)$ is shorter than or equal to $T$. Thus, $\text{Prob} \left( \text{Call by } T \right) = \text{Prob} \left( t(V^*) \leq T \right)$. Let this probability be $P(V, t)$, since it is a function of the current value of $V$ as well as the calendar time $t$. (It will also depend on the horizon $T$, the trigger $V^*$, and the parameters of the stochastic process for $V$, all of which we suppress in the notation.) $P(V, t)$ is the probability that the first passage time to $V^*$ is less than $T$. $P(V, t)$ can be written as

$$P(V, t) = \left[ \int_0^T f(V_0, V^*, t) \, dt \mid V_0 = V \right],$$

where $f(V_0, V^*, t)$ is the density function of the hitting time to $V^*$ when starting at $V_0$. Unfortunately, because the process $\{V_t\}$ is absorbed at two boundaries, the density function has no analytical expression; thus the probability $P(V, t)$
cannot be derived analytically. We therefore computed $P(V,t)$ numerically. As shown in Appendix B, $P(V,t)$ must satisfy the partial differential equation (PDE) which is presented in Result 3.

**Result 3.** Let $P(V,t)$ be the probability (at time $t$) of the bond being called by time $T$. Then $P(V,t)$ should satisfy the following partial differential equation:

$$P_t + (\mu - \delta)V_P + 0.5\sigma^2V^2P_{VV} = 0, \quad (5)$$

subject to the boundary conditions:

$$P(\Theta, t) = 0 \quad \forall t < T,$$

$$P(V^*, t) = 1 \quad \forall t < T,$$

$$P(V, T) = 0 \quad \forall V < V^*.$$

There is no closed-form solution for this system of equations, because of the complications brought about by the additional absorption at the lower boundary. However, the PDE (5) along with its boundary conditions can easily be solved numerically for the probability $P(V_0,0)$, by starting at the terminal date $T$ and moving back one period at a time. We used the implicit finite-difference method of discretization to solve the equation, in order to ensure stability and convergence. Using the approach described in Hull (1999, Ch. 16), we computed the current value of the function, $P(V_0,0)$. This is the probability (at time $t = 0$) of call by time $T$, where $V_0 = \text{initial (starting) value of } V$. The probability of call depends directly on the parameters $\mu - \delta, \sigma, T, V_0, V^*, \text{and } \Theta^*$, and indirectly on the other parameters, $F, r, c, \tau$, etc. Thus, all of these factors could potentially affect the probability of call.

Note that we use the actual growth rate $\mu$ and not the risk-neutralized growth rate $r$ (which is used for valuation purposes). This is because the probability of call is dependent on the actual growth rate (see Leland and Toft, 1996). Also note that the probability of call depends crucially on the initial value $V_0$. Because of the convention of issuing corporate debt at (or very close to) par value, we will, in our numerical results section, choose $V_0$ such that bond value equals par value. Since $t = 0$ is the time of issuance of debt, $P(V_0,0)$ gives the probability, at the time of issue, of a call by time $T$.

5. Comparative static results and empirical implications

We computed the probability of call within 10 years for bonds with various parameter values, with the following base case: $F = $100, $c = 7.4\%$, $\sigma = 0.23$, $r = 6.8\%$, $\delta = 3\%$, $\alpha = 0.5$, $\beta = 0.01$, $\tau = 0.33$, and $\mu = 10\%$. These values are chosen to be generally close to the mean values in our empirical sample.
The comparative static results reported below were obtained by repeating the probability computations for a wide range of parameter values around this base case. The call trigger $V^*$, default trigger $\Theta^*$, and the call premium $p^*$ are all determined optimally, and $V_0$ is chosen so that the bond is issued at par, i.e., $D(V_0) = F$. Figs. 1 and 2 illustrate the main comparative static results.

The exact value of $\text{Prob}(\text{Call by } T)$ does depend on $T$, but the comparative static results (effect of parameters on the probability of call) are not affected by the choice of $T$. We report results only for $T = 10$ years, but the same comparative static results were obtained with all other values of $T$.

**Effect of interest rate $r$.** As shown in Fig. 1, $\text{Prob}(\text{Call})$ is a decreasing function of $r$. Thus, for higher interest rates, a bond is less likely to be called if everything remains the same, including the coupon rate. The intuition behind this result is that, if $r$ increases without a corresponding increase in coupon rate, it makes the bond relatively more attractive. Hence the firm is less likely to call the bond, reducing $\text{Prob}(\text{Call by } T)$, for higher values of $r$.

This result is consistent with observed behavior in the corporate bond market. For instance, a large number of bonds were called and refunded when interest rates fell in recent years (Kalotay, 1993), as predicted by our model.

**Effect of coupon rate $c$.** Fig. 2 shows that $\text{Prob}(\text{Call})$ is an increasing function of $c$, i.e., a higher-coupon bond is more likely to be called. This is as expected since a high-coupon bond is less attractive, hence more likely to be called and

![Fig. 1. Shows Prob(Call by 10 years) as a function of the interest rate $r$, for three different levels of volatility: $\sigma = 0.17$, 0.23, and 0.29 (series 1, 2, and 3, respectively). Parameter values: $F =$ S100, $c = 7.4\%$, $\delta = 3\%$, $\alpha = 0.5$, $\beta = 0.01$, $\tau = 0.33$, and $\mu = 10\%$. Call premium and call trigger are always chosen optimally, and $V_0$ is always such that the bond is issued at par, i.e., $D(V_0) = F = 100.$](image-url)
replaced. This result is consistent with the fact that a greater percentage of recently issued (low-coupon) bonds are non-callable relative to bonds issued earlier (with higher coupon rates).

**Effect of firm risk.** As Fig. 1 shows, Prob(Call) is a decreasing function of firm risk $\sigma$. While this result might not be intuitively obvious, it can be explained as follows: we know from basic option theory that the option exercise trigger (here $V^*$) will be farther for higher volatility. This makes it less likely that the bond will be called, everything else remaining unchanged. In our simulations, when $\sigma$ was increased from 0.17 to 0.23 to 0.29, $V^*$ rose from 179.0962 to 299.272 to 564.4028, and the corresponding Prob(Call by 10 years) fell from 76.46% to 65.80% to 55.46%. This result is analogous to the well-known result in option theory that greater volatility causes postponement of American option exercise; for the option to call the bond, the relevant state variable is $V$; thus a higher volatility of $V$ results in delayed exercise, which reduces Prob(Call by $T$). Another possible effect of higher $\sigma$ would be to increase the probability of default, and this might reduce the probability of call. This will reinforce the effect mentioned above.

**Effect of corporate tax rate.** As shown in Fig. 2, Prob(Call) is an increasing function of the corporate tax rate $\tau$. The major tax implication of the call decision is the tax benefit enjoyed by the firm because of the call premium it pays (the benefit is $\tau p F$, as discussed in Section 2.1). Since the benefit is an increasing function of $\tau$, a higher tax rate will increase the incentive to call, increasing Prob(Call).
Other comparative static results (not illustrated) are as follows: Prob(Call) is an increasing function of the leverage ratio, and a decreasing function of bankruptcy costs $z$, payout ratio $\delta$ and refunding cost $\beta$. For the leverage ratio, the intuition is straightforward: the higher the leverage ratio, the faster will debt value $D(V)$ rise with $V$. Since the bond should be called as soon as its price reaches the call price (see Section 3.1), it is clear that the bond will be called earlier when leverage ratio is higher, everything else remaining the same. Therefore, Prob(Call) is an increasing function of leverage ratio. Next, a larger $z$ increases the riskiness of the bond; since a riskier bond would be more costly to replace, the firm will postpone the call, reducing Prob(Call); thus a negative relationship between $z$ and probability of call. Also, a larger $\delta$ reduces the growth rate of $V$, and thus lowers the probability of reaching the trigger $V^*$, hence a negative relationship between $\delta$ and Prob(Call). Finally, a higher $\beta$ will reduce the attractiveness of calling and replacing the bond, hence Prob(Call) is a decreasing function of $\beta$. The comparative static results are summarized in Result 4.

**Result 4.** The probability of call is an increasing function of (i) coupon rate, $^9$ (ii) corporate tax rate and (iii) leverage ratio, and a decreasing function of (i) interest rate, (ii) firm value volatility $\sigma$, (iii) bankruptcy costs, (iv) payout rate and (v) bond refunding cost.

### 5.1. Empirical implications

Result 4 summarizes the effects of various parameters on the probability of call. The next step is to investigate what effect the probability of call (and its determinants) have on the likelihood the bond having a call feature.

As briefly discussed in Section 1, we hypothesize a positive relationship between probability of call and likelihood of a call feature. That is, everything else remaining the same, a bond that is more likely to be called will also be more likely to be callable. This is equivalent to saying that a fairly priced option that cannot be sold (i.e., that can be either exercised or allowed to expire) is more likely to be acquired if the probability of exercise is higher. This assertion can be justified by the existence of intangible factors which are not incorporated in the price, e.g., time and effort required to exercise the option (here, to call and replace the bond). Thus, we have hypothesized that the higher the probability of call, the higher the likelihood of a call feature in the bond.

$^9$ We have used the coupon rate in both the model and the empirical tests. The correct parameter should be the bond yield at issue, which is the true measure of the cost of the bond to the firm. However, since corporate bonds are virtually always issued at (or very close to) par, the coupon rate is not significantly different from the yield at issue.
Thus, any factor which increases (decreases) the probability of call should increase the likelihood of the bond being issued as a callable (non-callable) bond. It is important to note that we are not making a statement about the exact probability of a call feature. Rather, the hypothesis is only about the direction and significance of the effect, e.g., a higher coupon rate will increase the likelihood of a call feature.

Based on the above argument and Result 4, the empirical implications of our model can be summarized by the following.

**Hypothesis.** The probability that a particular bond will be callable is an increasing function of (i) coupon rate, (ii) corporate tax rate and (iii) leverage ratio, and a decreasing function of (i) interest rate, (ii) volatility of firm value, (iii) bankruptcy costs, (iv) payout rate and (v) bond refunding cost.

Kish and Livingston’s (1992) is the only empirical study on the determinants of the likelihood of a call feature. However, their sample period was 1977–1986, and the corporate bond market changed significantly since then (see Emery and Finnerty, 1997). For example, the majority of bonds in Kish and Livingston’s sample were callable (83%) whereas only 24% of our sample of bonds (see Section 6) were callable. We therefore decided to conduct a small-scale preliminary empirical investigation to test the implications of our model. While our empirical methodology is similar to Kish and Livingston (1992), we feel our results might add some incremental empirical information to the literature mainly because of the different sample period and the fact that our study uses a different set of explanatory variables than Kish and Livingston.

6. Empirical investigation

In order to test the empirical implications of our model (Section 5.1), we gathered details of long-term corporate bonds issued in 1996 and the first two months of 1997. Since our theoretical model assumes perpetual debt, the empirical implications are appropriate for long-term debt. We therefore decided to exclude all short- and medium-term corporate debt from our sample. Our empirical investigation is similar to that of Kish and Livingston (1992), but our set of variables is different because our tests are in the context of the specific model of this paper.

6.1. Data sources

The main data sources were the *Standard & Poor’s Corporation Bond Guide* and the *Compustat* files. The former provides basic issue information (issue date, maturity date, coupon rate, face value, whether callable, etc.) and the
latter provides firm-specific information (tax rate, debt level, etc.). Weekly interest rate information was also available from the *S&P Guide*.

From the original list of 186 long-term corporate bonds (maturing in over ten years), the following were deleted: (i) convertible or zero-coupon bonds (14), (ii) bonds issued by foreign corporations or non-corporate entities (23), and (iii) bonds for which sufficient firm-specific information was not available from *Compustat* (45). For example, to estimate the firm value volatility $\sigma$, we require a time series of firm value data; therefore, any firm for which at least 20 continuous quarterly observations were not available was deleted from the list.

The final sample consisted of 104 bonds, of which 25 were callable and 79 were non-callable.

6.2. The variables

We use the dependent variable CALL and the following independent variables (which are suggested by Section 5.1) as potential significant determinants of the call feature: INTRATE, COUPON, VOLA, TAX, LEV, ALPHA, BETA and MATURITY. These variables are described in details below.

The decision to include a call feature is represented by the dichotomous dependent variable CALL, equal to 1 if the bond is callable and 0 if non-callable. INTRATE is the risk-free interest rate, measured by the yield on long-term Treasury bonds (long-term because our model has an infinite-horizon setting) during the week of the issue. COUPON is the coupon rate of the bond. The variable VOLA is the volatility of the firm value or asset value ($\sigma$ in our model). It is estimated as follows: we first generate the quarterly time series $V$ for total firm value (book value of long-term debt plus market value of equity) for each firm for the period ending quarter 4, 1996, from the quarterly *Compustat* database. The database that was used had 12 years of quarterly data, so that a maximum of 48 points were available. From this, we create the discretized version of the time series $\text{d}V/V = V(i).dV/V$, the annualized standard deviation of which gives the volatility $\sigma$ or VOLA. Firms with less than 20 continuous data points were excluded from the sample.

TAX is the corporate tax rate, measured by the average tax rate for the year 1996 as reported in *Compustat* (i.e., firm’s total tax expense divided by the pretax income for the year). This is probably a very noisy measure of $\tau$ in our model; to improve the estimate, Fischer et al. (1989) suggest using the average over a five-year interval following the year of issue. Unfortunately, we could not follow the same procedure because we would need corporate tax data up to the year 2002 (since our sample includes bonds issued in February 1997). Because of data limitations, therefore, we used the tax rate of the year of issue as a (possibly noisy) proxy for the effective tax rate $\tau$.

LEV is the firm’s leverage ratio for the year 1996, measured by the ratio of long-term debt to equity plus long-term debt, where debt is at book value and
equity at market value (both reported in *Compustat*). The book value of debt was used instead of (the more appropriate) market value because of data limitations. However, since the cross-sectional correlation between market and book values of debt is very large (see Bowman, 1980), the misspecification from using book values is probably quite small. It is, in fact, quite common to use book values of debt in studies of leverage ratios.

The parameter $\beta$ (refunding cost) in our model is represented by the variable $BETA$, set equal to the reciprocal of the face value of the bond issue. This is because fractional flotation costs are generally inversely related to the size of the issue (see Fischer et al., 1989, p. 443). Bankruptcy cost $\alpha$ is represented by the variable $ALPHA$, given by the ratio of intangible assets to total assets of the firm (reported in *Compustat*). A firm with a larger percentage of intangible assets would presumably find it more difficult and expensive to realize the fair values of its assets in the event of bankruptcy; hence this measure is used as a proxy for $\alpha$. We acknowledge that both $ALPHA$ and $BETA$ are very crude proxies for the underlying parameters, but better estimates for these parameters were unfortunately not available. The parameter $\delta$ (payout) has not been included in the empirical tests because the appropriate information was available on *Compustat* for very few firms.

Finally, Kish and Livingston (1992) found the time to maturity to be a significant determinant of the call feature. We therefore also included the variable $MATURITY$ (number of years to maturity) to minimize possible misspecification errors owing to omitted variables, although it might be less important in our study which is limited to long-term bonds only.

6.3. T-tests for comparison of means

The first step was to compare the values of the above explanatory variables for callable and non-callable issues. As in Kish and Livingston (1992), we use $t$-tests to compare means. Table 1 summarizes the results of the means tests.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Bond type</th>
<th>$t$-Value</th>
<th>Prob &gt; $t$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Callable (25)</td>
<td>Non-callable (79)</td>
<td></td>
</tr>
<tr>
<td>INTRATE</td>
<td>6.8044%</td>
<td>6.8623%</td>
<td>-0.8762</td>
</tr>
<tr>
<td>COUPON</td>
<td>7.6406%</td>
<td>7.2320%</td>
<td>3.2218</td>
</tr>
<tr>
<td>VOLA</td>
<td>20.6524%</td>
<td>24.0439%</td>
<td>1.5385</td>
</tr>
<tr>
<td>TAX</td>
<td>35.6918%</td>
<td>34.4047%</td>
<td>0.8715</td>
</tr>
<tr>
<td>LEV</td>
<td>0.2351</td>
<td>0.2000</td>
<td>1.2086</td>
</tr>
<tr>
<td>MATURITY</td>
<td>31.92 yrs</td>
<td>31.6076 yrs</td>
<td>0.0556</td>
</tr>
<tr>
<td>ALPHA</td>
<td>0.0496</td>
<td>0.0621</td>
<td>-0.5864</td>
</tr>
<tr>
<td>BETA</td>
<td>0.005711</td>
<td>0.005318</td>
<td>0.6281</td>
</tr>
</tbody>
</table>
From Section 5.1, we expect the callable bond group to have higher values for the variables COUPON, TAX & LEV, and lower values for the variables INTRATE, VOLA, ALPHA & BETA. The results are as expected for all the variables except BETA, which is slightly higher for callable than for non-callable bonds. However, the only variable for which the difference between the two groups is significant is COUPON, with a $t$-statistic of 3.2218 and a $p$-value of 0.0017. As Table 1 shows, the $t$-statistics for differences for the other variables are not significant.

6.4. Logistic regression results

Since a $t$-test with one variable does not control simultaneously for the effects of the other explanatory variables, we next conducted a logistic regression procedure to predict the likelihood of a bond being callable. The general functional form and the rationale for using this model are discussed in Kish and Livingston (1992). The dependent variable is the dichotomous variable CALL, which is a function of the explanatory variables listed above, i.e.,

$$CALL = f(INTRATE, COUPON, VOLA, TAX, LEV, ALPHA, BETA, MATURITY),$$

where the variables have been described above. The logit procedure allows us to test the predictive ability of these variables regarding the inclusion of the call feature. The regression results are summarized in Table 2. A positive coefficient

<table>
<thead>
<tr>
<th>Independent variable</th>
<th>Predicted sign</th>
<th>Logistic regression estimation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Dependent variable: CALL</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Coefficient</td>
</tr>
<tr>
<td>Intercept</td>
<td>–</td>
<td>1.7713</td>
</tr>
<tr>
<td>INTRATE</td>
<td>–</td>
<td>-2.6195</td>
</tr>
<tr>
<td>COUPON</td>
<td>+</td>
<td>1.9221</td>
</tr>
<tr>
<td>TAX</td>
<td>+</td>
<td>0.0415</td>
</tr>
<tr>
<td>VOLA</td>
<td>–</td>
<td>-0.0777</td>
</tr>
<tr>
<td>LEV</td>
<td>+</td>
<td>1.9948</td>
</tr>
<tr>
<td>MATURITY</td>
<td>+</td>
<td>0.0028</td>
</tr>
<tr>
<td>ALPHA</td>
<td>–</td>
<td>1.7124</td>
</tr>
<tr>
<td>BETA</td>
<td>–</td>
<td>31.6475</td>
</tr>
</tbody>
</table>

$\text{aThe dependent variable CALL = 1 if bond is callable, and 0 if non-callable. There are 25 callable and 79 non-callable bonds in the sample; the logit procedure models the probability that bond is callable, hence a plus (+) sign means that callable debt is more likely and a minus sign (−) means that non-callable debt is more likely; model Chi-square = 22.239, with probability = 0.0045; concordant = 76.3%, discordant = 23.4%, tied = 0.3%; Somers’ $D = 0.529$, Gamma = 0.530, Tau - $\alpha = 0.195$, and $c = 0.764.$}
indicates a higher probability of a call feature, i.e., a positive (negative) coefficient implies that an increase in that variable makes the bond more (less) likely to be callable. The predicted signs for the various coefficients are (from Section 5.1):

**COUPON, TAX, LEV and MATURITY** (from Kish and Livingston):

- **positive**

**INTRATE, VOLA, ALPHA and BETA:** **negative.**

As Table 2 illustrates, the logistic regression results support most of the implications of our model. First, we note that all the variables have the expected sign, except ALPHA and BETA, both of which are highly insignificant ($p$-values of 62.70% and 75.73%, respectively). This is not surprising since these are not reliable proxies, as discussed in Section 6.2. The other details are summarized below:

(i) the variable INTRATE has a significant negative effect on the probability of a call feature with a $p$-value or significance level of 1.17%;
(ii) COUPON has a positive and highly significant coefficient ($p$-value = 0.28%);
(iii) VOLA has a negative and significant coefficient ($p$-value = 4.83%);
(iv) LEV has a positive and somewhat significant coefficient ($p$-value = 8.94%).

Also, while the variable TAX has the expected sign (positive) it is insignificant ($p$-value = 36.49%); this could be because we are using a noisy estimate of $\tau$, as discussed in Section 6.2. Finally, the variable MATURITY, while having the expected sign, was insignificant ($p$-value = 78.01%); this is also not surprising, since our study was limited to long-term bonds.

The most significant determinants of the call feature seem to be coupon rate, interest rate, firm value volatility, and leverage ratio. For the overall regression, the Chi-square statistic was 22.239, with a probability of 0.0045; thus the combined effect of the set of explanatory variables is quite significant. Other diagnostics are also reported in Table 2, e.g., Somer’s $D = 0.529$, Gamma = 0.530, Tau-$a = 0.195$, and $c = 0.764$. Finally, in a comparison of observed responses and predicted probabilities, the concordant and discordant percentages were 76.3% and 23.4% respectively, with 0.3% tied (see SAS Manual for details of the concordant and discordant computations).

Repeating the logistic regression without the variables MATURITY, ALPHA and BETA made virtually no difference to the results. For example, the significance of INTRATE was 1.11%, COUPON 0.24%, VOLA 5.34% and LEV 9.71%. TAX was not significant, with probability 40.03%, and the overall model Chi-square statistic was 21.933, with probability 0.05%. We also repeated the logit regression with Bond Rating (as reported by *S&P Guide*) as an additional explanatory variable. Bond Rating was measured by the cardinal-
ized variable RATING, which started at zero for a non-rated bond and increased to 22 for a AAA-rated bond. The variable RATING was found to have a negative coefficient (i.e., a higher rated bond was less likely to be callable) as expected, but was not significant ($p$-value $= 34.14\%$). The results for the other variables were not significantly different, and INTRATE, COUPON, VOLA and LEV remained the only significant determinants of the call feature.

Overall, the logistic regression model seems to do a reasonably good job in explaining the decision to include a call feature. The results of the empirical investigation, even with limited data and noisy parameter estimates, provide fairly strong support for the major implications of our model.

7. Conclusion

In this paper, we have suggested a new way to explain and predict the inclusion of a call feature in a corporate bond. We first show how to compute the probability that a bond (if callable) will be called within a given time period, when the call premium is set optimally and the firm follows the optimal call policy. We then hypothesize that any factor which increases this probability (such as an increase in the coupon rate) will also increase the likelihood of the bond being issued with a call feature. Based on the comparative static results, this gives us a set of empirical implications; these implications are tested with recently issued long-term corporate bonds. The results provide fairly strong support for the implications of our model; coupon rate, interest rate, firm value volatility and leverage ratio are found to be significant predictors of the inclusion of the call feature, as suggested by the model.

Acknowledgements

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Appendix A. Optimal call policy and call premium

A.1. Contingent security valuation

Consider any general claim on the firm (any security contingent on firm’s asset value $V$) that has infinite maturity and pays a continuous coupon amount
$C$ per unit time before bankruptcy. If the value of such a security is $G(V)$, then it can be shown that $G(V)$ should satisfy the differential equation \(^{10}\)
\[
\frac{1}{2}\sigma^2 V^2 G''(V) + (r - \delta)V'G(V) - rG(V) + C = 0
\]
subject to the appropriate boundary conditions. The general solution to such an equation is
\[
G(V) = \frac{C}{r} + K_1 V^{\gamma_1} + K_2 V^{\gamma_2},
\]
where $K_1$ and $K_2$ are constants determined by the boundary conditions, and
\[
\gamma_1 = \frac{1}{2} - \frac{r - \delta}{\sigma^2} + \sqrt{\frac{2r}{\sigma^2} + \left(\frac{r - \delta}{\sigma^2} - \frac{1}{2}\right)^2},
\]
\[
\gamma_2 = \frac{1}{2} - \frac{r - \delta}{\sigma^2} - \sqrt{\frac{2r}{\sigma^2} + \left(\frac{r - \delta}{\sigma^2} - \frac{1}{2}\right)^2}.
\]
Note that $\gamma_1 > 1$ and $\gamma_2 < 0$.

\textit{A.2. Valuation of non-callable bond}

We give below the value of a perpetual non-callable bond $\text{NCD}(V)$ along with the associated tax benefits $\text{NCT}(V)$ and bankruptcy costs $\text{NCB}(V)$. (These are required because the callable bond is replaced by a non-callable bond, see Section 2.1.) The bond has a coupon payment of $\$cF$ per unit time. Since these can be derived as a straightforward extension of Leland (1994), the details are not presented here, but are available from the author on request.

\[
\text{NCD}(V) = \frac{cF}{r} - \left[\frac{cF}{r} - (1 - \alpha)V_B\right]\left(\frac{V}{V_B}\right)^{\gamma_2}, \quad (A.5)
\]
\[
\text{NCT}(V) = \frac{\tau cF}{r} - \frac{\tau cF}{r}\left(\frac{V}{V_B}\right)^{\gamma_2}, \quad (A.6)
\]
\[
\text{NCB}(V) = \alpha V_B\left(\frac{V}{V_B}\right)^{\gamma_2}, \quad (A.7)
\]

\(^{10}\) Since there is no explicit maturity, there will be no time dependency. Hence the security value will not be a function of calendar time $t$. 
where $c_2$ is given by Eq. (A.4) and

$$V_B = \frac{cF}{r} (1 - \tau) \left( -\frac{\gamma_2}{1 - \gamma_2} \right).$$  \hspace{1cm} (A.8)

A.3. Valuation of callable bond

Now take the special case of a security which is a perpetual callable bond with coupon flow $cF$ per unit time. If the callable bond value is denoted by $D(V)$, then we have

$$D(V) = \frac{cF}{r} + K_3V^{\gamma_1} + K_4V^{\gamma_2},$$  \hspace{1cm} (A.9)

where $K_3$ and $K_4$ are determined by the boundary conditions. The upper boundary condition depends on the firm’s call policy. Suppose the firm’s policy is to call the callable bond as soon as $V$ reaches some critical level $V^*$, whereupon the bondholder receives the call price $(1 + p)F$. Then the upper boundary condition will be

$$D(V^*) = \frac{cF}{r} + K_3(V^*)^{\gamma_1} + K_4(V^*)^{\gamma_2} = (1 + p)F.$$  \hspace{1cm} (A.10)

The lower boundary condition will be determined by the bankruptcy-triggering condition. When $V$ falls to the bankruptcy-triggering level $\Theta$, shareholders will declare bankruptcy and bondholders will receive a payoff of $(1 - \alpha)\Theta$. This gives

$$D(\Theta) = \frac{cF}{r} + K_3\Theta^{\gamma_1} + K_4\Theta^{\gamma_2} = (1 - \alpha)\Theta.$$  \hspace{1cm} (A.11)

As we shall show below, both $V^*$ and $\Theta$ will be determined endogenously as optimizing decisions by shareholders.

Eqs. (A.10) and (A.11) can be solved for $K_3$ and $K_4$ in terms of $V^*$ and $\Theta$, giving

$$K_3 = \frac{F \left( 1 + p - \frac{\alpha}{r} \right) \Theta^{\gamma_2} - \left[ (1 - \alpha)\Theta - \frac{\alpha F}{r} \right] (V^*)^{\gamma_2}}{(V^*)^{\gamma_1} (\Theta^{\gamma_2} - (V^*)^{\gamma_2} \Theta^{\gamma_1})},$$  \hspace{1cm} (A.12)

$$K_4 = \frac{\left[ (1 - \alpha)\Theta - \frac{\alpha F}{r} \right] (V^*)^{\gamma_1} - F \left( 1 + p - \frac{\alpha}{r} \right) \Theta^{\gamma_1}}{(V^*)^{\gamma_1} (\Theta^{\gamma_2} - (V^*)^{\gamma_2} \Theta^{\gamma_1})}.$$  \hspace{1cm} (A.13)

\footnote{The optimal policy is commonly expressed in terms of the state variable(s) (here $V$) reaching some critical value. This value is not time-dependent in our model because of the infinite-horizon setting.}
A.4. Valuation of tax benefit

We explicitly value the tax benefit associated with the callable bond, including (a) the effect of the replacement bond when the existing bond is called, and (b) the effect of the call premium, both of which were ignored in Fischer et al. (1989). In this section, we show how the tax benefit is valued, using Leland’s (1994) approach.

Consider a security which pays a constant stream $cF$ per unit time as long as the firm is solvent and the bond is not called, but pays nothing when $V$ reaches the lower limit $\Theta$. Also suppose that when $V$ reaches the upper limit $V^*$, the value of this security becomes

$$
\frac{\tau cF}{r} [1 - (V^*/V_B)^{\gamma_2}] + \tau pF.
$$

(A.14)

The cash flows from this security are identical to the cash flows generated by the tax shield. This is because, when $V = V^*$, the callable bond is replaced by a non-callable bond with the same coupon payments. Thus the tax benefit at the upper limit $V^*$ will be equal to the tax benefit from a non-callable bond, given by $(\tau cF/r)[1 - (V^*/V_B)^{\gamma_2}]$ (see Eq. (A.6)). In addition, the firm will benefit from the tax shield associated with calling the bond; this amount is $\tau pF$, as explained in Section 2.1. Therefore the value of the tax benefit, $T(V)$, should be equal to the value of the above security, which is given by

$$
T(V) = \frac{\tau cF}{r} + K_5 V^{\gamma_1} + K_6 V^{\gamma_2}
$$

(A.15)

with the following boundary conditions: 12

$$
T(V^*) = \frac{\tau cF}{r} + K_5 (V^*)^{\gamma_1} + K_6 (V^*)^{\gamma_2} = \frac{\tau cF}{r} - \frac{\tau cF}{r} \left( \frac{V^*}{V_B} \right)^{\gamma_2} + \tau pF,
$$

(A.16)

$$
T(\Theta) = \frac{\tau cF}{r} + K_5 \Theta^{\gamma_1} + K_6 \Theta^{\gamma_2} = 0.
$$

(A.17)

Solving Eqs. (A.16) and (A.17), we get $K_5$ and $K_6$:

$$
K_5 = \tau F \left[ p - \frac{\xi \left( \frac{V^*}{V_B} \right)^{\gamma_2}}{(V^*)^{\gamma_1} \Theta^{\gamma_2} - (V^*)^{\gamma_2} \Theta^{\gamma_1}} \right],
$$

(A.18)

12 Note that Fischer et al. (1989) ignores the potential tax benefits and potential bankruptcy costs from the replacement bond at the upper limit boundary condition (p. 433, Eq. (8)).
A.5. Valuation of bankruptcy cost

Here we explicitly value the effect of bankruptcy costs. As in the last section, consider a security which pays nothing as long as the firm is solvent and the bond is not called, but pays a fraction $\alpha$ of asset value when $V = \Theta$, and pays an amount $\alpha V_B (V^*/V_B)^{z_2}$ when $V = V^*$. The cash flows from this security are identical to the cash flows associated with bankruptcy costs, using an argument similar to that for tax benefits above. Therefore the above security value will be identical to bankruptcy cost, $B(V)$, given by

$$B(V) = K_7 V^{z_1} + K_8 V^{z_2}$$

(A.20)

with the following boundary conditions (also see footnote 12):

$$B(V^*) = K_7 (V^*)^{z_1} + K_8 (V^*)^{z_2} = \alpha V_B \left( \frac{V^*}{V_B} \right)^{z_2},$$

(A.21)

$$B(\Theta) = K_7 \Theta^{z_1} + K_8 \Theta^{z_2} = \alpha \Theta.$$  

(A.22)

Solving Eqs. (A.21) and (A.22), we get $K_7$ and $K_8$:

$$K_7 = \alpha \frac{V_B (V^* \Theta/V_B)^{z_2} - \Theta (V^*)^{z_2}}{(V^*)^{z_1} \Theta^{z_2} - (V^*)^{z_2} \Theta^{z_1}},$$

(A.23)

$$K_8 = \alpha \frac{\Theta (V^*)^{z_1} - V_B (V^*/V_B)^{z_2} \Theta^{z_1}}{(V^*)^{z_1} \Theta^{z_2} - (V^*)^{z_2} \Theta^{z_1}}.$$  

(A.24)

A.6. Valuation of refunding cost

Consider a security which pays nothing as long as the firm is solvent and the bond is not called, but pays nothing on bankruptcy and pays a fraction $(1 - \tau)\beta$ of the value of the refunding issue (i.e., fraction $(1 - \tau)\beta$ of the value of the replacement non-callable debt at call, given by Eq. (A.5)). This is identical to the cash flows associated with refunding the callable debt. Therefore refunding costs can be evaluated using this security, whose value is given by

$$R(V) = K_9 V^{z_1} + K_{10} V^{z_2}$$

(A.25)
with the following boundary conditions:

\[ R(V^*) = K_9(V^*)^2 + K_{10}(V^*)^2 = (1 - \tau)\beta \text{NCD}(V^*), \]  
(A.26)

where \( \text{NCD}(V) \) is given by Eq. (A.5), and

\[ R(\Theta) = K_9\Theta^2 + K_{10}\Theta^2 = 0. \]  
(A.27)

Solving Eqs. (A.26) and (A.27), we get \( K_9 \) and \( K_{10} \):

\[ K_9 = \frac{\Theta^2(1 - \tau)\beta \text{NCD}(V^*)}{(V^*)^2 \Theta^2 - (V^*)^2 \Theta^2}, \]  
(A.28)

\[ K_{10} = -\frac{\Theta^2(1 - \tau)\beta \text{NCD}(V^*)}{(V^*)^2 \Theta^2 - (V^*)^2 \Theta^2}. \]  
(A.29)

Note that \( \beta \) here is the issue cost (or flotation cost) as a fraction of the replacement bond’s value; since refunding costs are tax-deductible, the after-tax cost at \( V = V^* \) will be \( (1 - \tau)\beta \) of the value of the replacement bond. We assume that \( \beta \) is constant as in Fischer et al. (1989). Mauer (1993) has used a different specification for refunding cost, which could also be incorporated easily in our model.

A.7. Total firm value and equity value

The total firm value is then given by

\[ FV(V) = V + T(V) - B(V) - R(V) \]  
(A.30)

with the following limiting values:

\[ FV(\Theta) = (1 - x)\Theta, \]  
(A.31)

\[ FV(V^*) = V^* - \left( \frac{V^*}{V_B} \right)^{\gamma_2}(\alpha V_B + \tau cF/r) + \tau F(p + c/r) \]

\[ - (1 - \tau)\beta \text{NCD}(V^*). \]  
(A.32)

The value of the firm’s equity is

\[ E(V) = V + T(V) - B(V) - R(V) - D(V) \]  
(A.33)

with the following limiting conditions:

\[ E(\Theta) = 0, \]  
(A.34)

\[ E(V^*) = V^* + \frac{\tau cF}{r} - \frac{\tau cF}{r} \left( \frac{V^*}{V_B} \right)^{\gamma_2} + \tau pF - \alpha V_B \left( \frac{V^*}{V_B} \right)^{\gamma_2} \]

\[ - (1 - \tau)\beta \text{NCD}(V^*) - (1 + p)F. \]  
(A.35)
Note that the limiting value of equity (Eq. (A.35)) is different from that in Fischer et al., because we have included the effects of replacement bond tax shield and potential bankruptcy costs, as well as the tax effect of the call premium, as discussed above.

**A.8. Optimal default trigger $\Theta^*$**

Here we determine the optimal bankruptcy-triggering level $\Theta^*$ for a given call policy $V^*$. The optimal $\Theta$ will maximize shareholders’ equity value for all possible values of the state variable $V$. This can be recognized as an optimal stopping problem; in such cases, the smooth-pasting condition is that the derivatives with respect to the state variable of the maximized value (here, $E(V)$) and the payoff received at stopping (here, 0) are equal at the optimal trigger level $\Theta^*$ (see Leland, 1994, or Merton, 1973, for details). The smooth-pasting condition here is

$$\frac{dE(V)}{dV} \bigg|_{V=\Theta^*} = \frac{dE(\Theta)}{d\Theta} \bigg|_{\Theta=\Theta^*} = 0. \quad (A.36)$$

Differentiating $E(V)$ with respect to $V$, and setting $V = \Theta^*$, Eq. (A.36) simplifies to

$$1 + \gamma_1(\Theta^*)^{\gamma_1-1}[K_5 - K_7 - K_0 - K_3] + \gamma_2(\Theta^*)^{\gamma_2-1}[K_6 - K_8 - K_{10} - K_4] = 0. \quad (A.37)$$

There being no analytical solution, Eq. (A.37) has to be solved numerically, which is a straightforward exercise. The solution gives the optimal $\Theta^*$ in terms of the firm’s call policy $V^*$. Thus, for *any* given call policy $V^*$ (not necessarily optimal), we can determine the appropriate lower boundary $\Theta^*(V^*)$. The next step is to determine the optimal call policy $V^*$.

**A.9. Optimal call policy (optimal call trigger $V^*$)**

Whatever the motivation for attaching the call feature to a corporate bond, once such a bond has been issued it has to be called in an optimal manner, in order to maximize equity value. Therefore, the optimal call policy is given by that value of $V^*$ which maximizes, for any value of $V$, the value of the firm’s equity.

The callable bond, when called, will be replaced by a non-callable bond with the same dollar coupon payment $cF$ per unit time (not the same coupon rate). Since the dollar coupon amounts are identical, the only cash flows at call (to the firm or the equity holders) will be:

1. Pay existing bond holders the call price $(1 + p)F$.
2. Receive tax benefit $\tau p F$ for the call premium.
3. Receive the value of the non-callable bond, given by \( \text{NCD}(V) \) in Eq. (A.5).

Recall that \( \text{NCD}(V) \) is given by

\[
\text{NCD}(V) = \frac{cF}{r} - \left[ \frac{cF}{r} - (1 - \gamma)V_B \right] \left( \frac{V}{V_B} \right)^{\gamma_2},
\]

(A.5)

where \( \gamma_2 \) is negative. As discussed in Leland (1994), the first term represents the default-free value of the bond, and the second term represents the reduction in value because of the possibility of default (and is a decreasing function of \( V \)).

Thus, if the firm calls the bond, the payoff is

\[
\left\{ \frac{cF}{r} - \left[ \frac{cF}{r} - (1 - \gamma)V_B \right] \left( \frac{V}{V_B} \right)^{\gamma_2} \right\} + \tau pF - (1 + p)F,
\]

where the term in braces \( \{ \cdots \} \) is the market value of the replacement non-callable bond, the second term is the tax benefit from the call premium, and the third term the amount paid to old bondholders (the call price). This expression can be rewritten as

\[
\left\{ \frac{cF}{r} - (1 + (1 - \tau)p)F \right\} - \left[ \frac{cF}{r} - (1 - \gamma)V_B \right] \left( \frac{V}{V_B} \right)^{\gamma_2},
\]

where \( \lambda_1 \) and \( \lambda_2 \) are constants and \( V \) is the state variable. If we use the substitution

\( V^{\gamma_2} = X \),

then the payoff at call becomes \( \lambda_1 - \lambda_2 X \). Since the firm has the (American) option to call the bond (without the obligation to call), the payoff is given by

\[
\text{Max} \{0, \lambda_1 - \lambda_2 X_t\} \quad \forall t, \quad \text{or} \quad \lambda_2 \text{Max} \left\{0, \frac{\lambda_1}{\lambda_2} - X_t\right\} \quad \forall t,
\]

This is equivalent to \( \lambda_2 \) put options on asset \( X \) with exercise price \( \lambda_1/\lambda_2 \). We know from option theory that, even without any dividend flow, a put option may be optimally exercised prior to maturity if \( X \) falls far enough. Therefore the optimal exercise policy for this option should be: exercise when \( X \) falls below some critical limit, say \( X^* \) (which is time-independent because the option is perpetual). Since \( \gamma_2 \) is negative, this implies that the optimal policy is to exercise when \( V \) is large enough, or exceeds some critical value \( V^* \), which could be finite because it is a put option. Thus there can be a finite critical value \( V^* \) (of course, in some situations, \( V^* = \infty \)).

Similar to the optimal default trigger, the optimal \( V^* \) is identified by using the first-order smooth-pasting condition at the upper limit

\[
\left. \frac{\text{d}E(V)}{\text{d}V} \right|_{V=V^*} = \frac{\text{d}E(V^*)}{\text{d}V^*},
\]

(A.38)
which simplifies (using Eqs. (A.33) and (A.35)) to

$$\gamma_1 (V^*)^{(1)} [K_5 - K_7 - K_9 - K_3] + \gamma_2 (V^*)^{(2)} [K_6 - K_8 - K_{10} - K_4]$$

$$+ \gamma_2 \left( \frac{V^*}{V_B} \right)^{\gamma_2} \left[ \frac{\tau cF}{r} + x V_B + (1 - \tau) \beta \left( V_B(1 - x) - \frac{cF}{r} \right) \right] = 0.$$  (A.39)

Given a lower limit $\Theta$, we can find the optimal upper limit $V^*$ by solving Eq. (A.39) numerically. But in order to jointly determine the optimal values $\Theta^*$ and $V^*$, we need to solve Eqs. (A.37) and (A.39) simultaneously. This gives Result 1 of our paper.

This is similar to Fischer et al. (1989), but with the following differences: (i) our model incorporates explicit valuation of tax benefits, bankruptcy costs, and refunding (replacement) costs, which makes it easy to see the exact effect of these various frictions; (ii) they ignored the potential tax benefits and bankruptcy costs associated with the replacement bond; and (iii) they ignored the tax effect of the call premium paid at call, as well as the tax effect of the refunding cost incurred when calling (and replacing) the bond. Thus their boundary conditions, including the smooth-pasting or high-contact condition, are different from ours.

A.10. Optimal call premium

As discussed in Section 3 of the paper, the optimal call premium is such that maximizing equity value will be equivalent to maximizing total firm value, i.e., maximizing $E(V)$ or $FV (V)$ will both result in the same call trigger $V^*$. If, instead of maximizing $E(V)$, the firm was to maximize $FV(V)$, then the first-order smooth-pasting condition would be

$$\left. \frac{dFV(V)}{dV} \right|_{V=V^*} = \frac{dFV(V^*)}{dV^*},$$  (A.40)

which simplifies to

$$\gamma_1 (V^*)^{(1)} [K_5 - K_7 - K_9] + \gamma_2 (V^*)^{(2)} [K_6 - K_8 - K_{10}]$$

$$+ \gamma_2 \left( \frac{V^*}{V_B} \right)^{\gamma_2} \left[ \frac{\tau cF}{r} + x V_B + (1 - \tau) \beta \left( V_B(1 - x) - \frac{cF}{r} \right) \right] = 0.$$  (A.41)

If the optimal call premium $p^*$ is used, this should give the same $V^*$ as Eq. (A.39). Therefore, in order to derive jointly the optimal $\Theta^*$, $V^*$ and $p$, we must solve Eqs. (A.37), (A.39) and (A.41) simultaneously. This gives Result 2 of our paper.
Appendix B. Probability of call by time $T$

We are interested in $\text{Prob}[t(V^*) < T]$, where $t(V^*)$ is the first passage time of $V$ to $V^*$. Let this probability be $P(V,t)$. Note that the process $\{V_t\}$ is also absorbed at the lower boundary $V = \Theta$. When $V$ reaches the lower boundary, $\text{Prob}[t(V^*) < T] = 0$. Thus the lower boundary condition

$$P(\Theta, t) = 0 \ \forall t < T.$$  

The upper boundary is similar: when $V$ reaches the call boundary before time $T$, the probability is 1.

$$P(V^*, t) = 1 \ \forall t < T.$$  

The terminal condition $P(V, T) = 0$ when $V < V^*$ is obvious.

The partial differential equation can be heuristically derived as follows: Consider a $V$ such that $V^* > V > \Theta$ at time $t$, and choose a time increment $dt$ sufficiently small so that the probability of $V$ reaching either $V^*$ before time $(t + dt)$ is negligible. At time $(t + dt)$, the probability of $V$ reaching $V^*$ before time $T$ is given by (using Ito’s lemma):

$$P(V_{t+dt}, t + dt) = P(V, t) + P_t dt + P_V dV + 0.5P_{VV}(dV)^2.$$  

From the law of total probabilities, we get

$$P(V, t) = E[P(V_{t+dt}, t + dt)|V_t = V] + 0(dt).$$  

Therefore,

$$P(V, t) = P(V, t) + E[P_t dt + P_V dV + 0.5P_{VV}(dV)^2]$$

$$= P(V, t) + P_t + P_V(\mu - \delta) + 0.5P_{VV}\sigma^2V^2,$$

which can be simplified to Eq. (5) of the paper:

$$P_t + (\mu - \delta)V P_V + 0.5\sigma^2V^2P_{VV} = 0.$$  

References


