Price volatility, welfare, and trading hours in asset markets

R. Todd Smith

Department of Economics, University of Alberta, 8-14 Tory Building, Edmonton, Alberta, Canada T6G 2H4

Received 12 April 1999; accepted 11 November 1999

Abstract

This paper studies the consequences of opening asset markets more often for the properties of asset prices and social welfare. For all reasonable parameter values, increasing trading hours lowers average asset prices, increases unconditional asset price volatility at a given point in time, and decreases unconditional asset price volatility when averaged over the period of time that includes the additional hours that markets are open. Unconditional social welfare is increased by opening markets more often, although the welfare gains are small – well below 1% of lifetime consumption. In contrast, because expanding hours of trading affects agents’ information sets, the welfare effect of more trading hours conditional on information available to agents can be large and the effect can be negative. © 2001 Elsevier Science B.V. All rights reserved.

JEL classification: G10; G28; D60

Keywords: Securities markets; Trading hours; Asset prices; Welfare

1. Introduction

This paper studies the consequences for asset prices and social welfare of increasing the amount of time that asset markets are open for trading.
Currently the New York Stock Exchange is open for trading five days a week for less than 7 hours a day. Many other asset markets are open much less often. For example, the Bulgarian Stock Exchange is open just on Tuesdays and only for 2 hours, and the market for pollution permits in California opens one day every three months. Recently there has been a push to open asset markets more often, as reflected for example by the introduction of after-hours trading on stock exchanges and international agreements between exchanges that allow some assets to be traded virtually round-the-clock. There is a presumption underlying this move toward opening asset markets more often that it is beneficial to investors. This paper seeks to understand whether this is likely to be the case, and in any case, what the mechanism is linking the amount of time asset markets are open to social welfare.

The direct monetary cost of opening asset markets more often has in some cases become small where electronic trading systems have displaced traditional, labor-intensive trading systems. But even in these circumstances a formal analysis of the consequences of opening markets more often is interesting for two reasons. First, Hart’s (1975) point that the introduction of more markets (when markets are incomplete) can lower welfare makes one wonder whether opening existing markets more often could similarly lower welfare. This possibility may have some merit in light of a number of authors’ suggestions (e.g. Economides and Schwartz, 1995; Kregel, 1995) that the indirect, non-monetary costs of opening markets more often may be large because of less informative prices and higher price volatility. Second, even if one believes that most reasonable models ought to have the prediction that opening markets more often raises welfare, it is of interest to understand this link and assess its magnitude.

This paper makes contributions along two lines. First, the paper develops a dynamic general equilibrium model in which investors can choose how often, when, and how much to trade in the asset market. This provides a rich environment to compare the consequences of opening markets more often. Second, because all agents in the model are rational it is possible to obtain explicit welfare measures of the benefits and costs of opening markets more often. The availability of explicit welfare measures means that one does not have to rely just on comparing features of asset prices that may be related to welfare (e.g. volatility, informativeness of prices). When there are countervailing benefits and costs of opening markets more often, only direct welfare measures allow for a concrete determination of the desirability of increasing the number of hours a market is open.

A key finding of the paper is that opening asset markets more often increases unconditional welfare for all reasonable parameter values. One reason for this welfare benefit is that asset price volatility is lower when measured over the...

---

1 I thank Preston McAfee for the pollution permit example.
period of time that includes the additional time that markets are open. In drawing this link between asset price volatility and welfare, it is critical that the measure of price volatility includes the additional time that markets are open. This is so because opening markets more often tend to increase asset price volatility at a given point in time. The paper also considers ‘conditional social welfare’, where welfare functions are evaluated based on the information set actually available to investors at a given point in time. Although the model yields the prediction (noted by Schwartz, 1988, p. 436) that opening markets more often increases the flow of information from markets, this can either increase the welfare benefits (above the unconditional welfare benefits) of opening markets more often, or it can cause a welfare loss. The main determinant of which way the conditional welfare results go is the qualitative information revealed by opening markets more often.

The outline of the paper is as follows. Section 2 describes the model, Section 3 characterizes decision problems, and equilibrium is discussed in Section 4. Sections 5 and 6 study the implications of opening markets more often for the properties of asset prices and welfare, respectively. Section 7 concludes the paper.

2. Model

The model builds on the overlapping generations (OG) framework. Because the focus of the paper is opening markets more often, the prototype OG model in which markets open once each period must be modified. Denote a time period by \( t = 1, 2, \ldots \) Time periods are distinct from trading periods, for there can be several trading periods during each time period. For reference purposes, all notation are collected in Table 1.

Agents born during time period \( t \) die at the end of \( t + 1 \). Generation \( t \) traders have preferences

\[
\ln(C_t) + \beta \ln(C_{t+1}),
\]

where \( C_{t+i} \) denotes consumption of the single good during time period \( t + i \) and \( \beta > 0 \) is the subjective discount factor. An implication of this preference structure is that traders’ asset demands will not depend on asset prices in future time periods. Asset demands will, however, depend on (the distribution of) asset prices in other trading periods within the same time period – preferences are not log-separable across trading periods within each time period. The implications of the above preference structure will therefore be less important if one interprets the length of time between time periods as significant.

Each agent receives \( y \) units of the consumption goods at birth. Goods can be stored costlessly within a time period, but it is prohibitively costly to store goods between time periods. The role of this assumption is twofold: it generates a demand for assets (other than storage) to satisfy life-cycle concerns and it
also allows traders flexibility within a time period as to the time they may trade these assets. Goods held across trading periods in the same time period therefore captures the idea that there is an asset that serves well as a short-term store of value but would not be held for long-term investment.

Long-term investment (i.e. across time periods) is accomplished by purchasing shares of an asset in a competitive market. This asset could be interpreted as stocks, bonds, pollution permits, or any other asset that is traded. The supply of this asset is \( K \) shares, and shares are perfectly divisible. Ownership of a unit of the asset at the start of any time period entitles the investor to a real distribution of \( D \) units of the single good. This dividend plus the future resale value of the asset are the focus of traders because these determine their consumption when old.

It will matter to traders how much asset markets are open during a time period because uncertainty about aggregate demand for the asset is resolved throughout a time period. Formally, the number of traders newly entering the market at a point in time is represented by the random variable \( \bar{x} \). Denoting by \( t_1 \) and \( t_2 \) two distinct moments during time period \( t \), the total number of traders born during time period \( t \) is the sum of the realizations of the process \( \bar{x} \) at each of \( t_1 \) and \( t_2 \). That is, \( Q_t \equiv x_{t_1} + x_{t_2} \) is the total number of traders born in time period \( t \) across the two possible trading periods.

---

2 This underlying source of asset-price uncertainty exists in the models studied by Mendelson (1982, 1987) and Kraus and Smith (1989) for example, and it is also a feature of the model studied by Kyle (1985) and related studies.
period \( t \), where \( x_t \) traders are born at \( t \). The univariate stochastic process \( \tilde{x} \) is assumed to be a strictly positive and bounded, nondegenerate Markov process.\(^3\) It will be convenient to refer to an agent as being of type \( j \in \{1, 2\} \), where a type-1 trader is born at \( t_1 \) and a type-2 trader is born at \( t_2 \).

Another possible source of uncertainty that could be included in this model is the dividend \( D \). As will be discussed below, all of the findings of the paper hold when \( D \) is stochastic or deterministic. With this in mind, to simplify the exposition the focus is on the case in which \( D \) is deterministic.

The final aspect of the model that needs to be discussed is the amount markets are open for trading. With expanded market access (EMA), investors are free to trade at any instant during a time period. Because new traders arrive only at moments \( t_1 \) and \( t_2 \) during time period \( t \), then the EMA regime is equivalent to opening asset markets only at these two moments – there is no distinct motive for trade at any other moment. By contrast, with restricted market access (RMA) investors may trade only at certain moments within a time period. The idea is to restrict agents’ temporal access to the market, and thus the interesting case is when markets open only at \( t_2 \). Any desired trades prior to \( t_2 \) must wait until the market opens at \( t_2 \). In each trading period \( t_i \), the formal trading process involves traders submitting orders – technically, a schedule of asset demands as functions of the price – to the market center where an auctioneer clears the market by single-price auction. The index \( k \in \{EMA, RMA\} \) will subsequently be used to denote the trading regime.

There are two motives for trade in this model. First, finite horizons and lumpy endowments create saving and dissaving motives across generations in each time period. Second – and the key feature of the model – agents may wish to stagger their trading across trading periods in the same time period because of randomness in the price process associated with stochastic aggregate demand for the asset. Because the asset price is stochastic, if this price reveals information about the underlying exogenous variable \( \tilde{x} \), then one must be careful to account for this when studying the consequences of opening markets more often. Schwartz (1988) points out that regulations that affect the amount markets are open for trading will have consequences for the information sets of traders, but this issue has not been studied formally; it is studied formally in this paper.

3. Decision problems

In any time period there are two generations of agents alive, and within each generation there are type-1 and type-2 traders. There are therefore four sets of

\(^3\) To motivate the price-taking assumption one could interpret \( x_t \) as the measure of traders born at \( t \) and \( K \) as the supply of shares per unit measure (say) of the population.
decision problems to consider in each time period, for each of the EMA and RMA regimes. The decision problems of agents in their last period of life is examined first.

3.1. Old traders

Let $V_{t_2-1}^{j,k}$ denote a type-$j$ ($j \in \{1, 2\}$) old trader’s shareholdings at the start of time period $t$ in trading regime $k$ ($k \in \{EMA, RMA\}$). These shareholdings are dated $t_2 - 1$ because they were determined in the last trading period of the previous time period. Let $\mu_{t_1}^{j,k}$ denote the fraction of $V_{t_2-1}^{j,k}$ sold at $t_1$. If $P_t^{k}$ denotes the asset price (in units of the consumption good) in trading period $t$ in trading regime $k$, then the decision problem of old traders at $t_1$ is

$$\max_{\mu_{t_1}^{j,k}} E_{t_1}^{R} \ln(C_t^{j,k}),$$

subject to

$$C_t^{j,k} = V_{t_2-1}^{j,k} \left[ D + \mu_{t_1}^{j,k} P_t^{R} + (1 - \mu_{t_1}^{j,k}) P_t^{k} \right],$$

where $E_{t_1}^{R}$ denotes the mathematical expectation, conditional on information at $t_1$. This expectation is indexed by the trading regime because information sets of traders are in general different in the two regimes.

In the RMA regime the market is closed at $t_1$ and thus the solution to (2) and (3) is trivial:

$$\mu_{t_1}^{R} = 0.$$  \hspace{1cm} (4)

In the EMA regime, the first-order condition is

$$E_{t_1}^{E} \left[ \frac{P_t^{E} - P_t^{R}}{D + \mu_{t_1}^{j,k} P_t^{E} + (1 - \mu_{t_1}^{j,k}) P_t^{R}} \right] = 0.$$  \hspace{1cm} (5)

Note from (4) and (5) that the optimal choice of $\mu_{t_1}^{j,k}$ is the same for all old traders: $\mu_{t_1}^{1,k} = \mu_{t_1}^{2,k}$. One can therefore drop the index $j$ here and write simply $\mu_{t_1}^{j}$ in what follows.

---

4 $\mu$ could lie outside the unit interval: if $\mu < 0$, then an old trader is purchasing shares at $t_1$ and, if $\mu > 1$, then an old trader is short-selling shares at $t_1$. 
3.2. Young traders

Conditional on the optimal \( \mu^k_{t+1} \), young type-2 traders solve

\[
\max_{V^{2,k}_{t}} \ln(C^{2,k}_{t}) + \beta E^k_{t} \ln(C^{2,k}_{t+1}),
\]

subject to:

\[
C^{2,k}_{t} = y - P^{k}_{t} V^{2,k}_{t},
\]

\[
C^{2,k}_{t+1} = V^{2,k}_{t} \left[ D + \mu^k_{t+1} P^{k}_{t+1} + (1 - \mu^k_{t+1}) P^{k}_{t+1} \right].
\]

The solution to (6)–(8) is straightforward:

\[
V^{2,k}_{t} = \frac{\beta y}{P^{k}_{t}(1 + \beta)}, \quad k \in \{\text{EMA, RMA}\}.
\]

In comparison, young type-1 traders’ decision problem at \( t_2 \) reflects the possibility that they may have traded at \( t_1 \) (at least in the EMA regime):

\[
\max_{V^{1,k}_{t}} \ln(C^{1,k}_{t}) + \beta E^k_{t} \ln(C^{1,k}_{t+1}),
\]

subject to:

\[
C^{1,k}_{t} = y - P^{k}_{t} V^{1,k}_{t} + V^{1,k}_{t+1}(P^{k}_{t+1} - P^{k}_{t}),
\]

\[
C^{1,k}_{t+1} = V^{1,k}_{t} \left[ D + \mu^k_{t+1} P^{k}_{t+1} + (1 - \mu^k_{t+1}) P^{k}_{t+1} \right],
\]

conditional on \( \mu^k_{t+1} \), and where \( V^{1,k}_{t} \) represents previous share acquisitions during trading period \( t_1 \).

In the RMA regime it is necessarily the case that

\[
V^{1,R}_{t} = 0,
\]

and thus in this regime

\[
V^{1,R}_{t} = \frac{\beta y}{P^{R}_{t}(1 + \beta)}.
\]

In the EMA regime, the solution to (10)–(12) is

\[
V^{1,E}_{t} = \frac{\beta \left[ y + V^{1,E}_{t+1} (\frac{P^{E}_{t+1} - P^{E}_{t}}{P^{E}_{t}}) \right]}{P^{E}_{t}(1 + \beta)}.
\]

Finally, in the EMA regime young type-1 traders choose \( V^{1,E}_{t} \) at \( t_1 \) to solve
\begin{align*}
\max_{V_{t_1}^E, E_{t_1}} & \quad E_n^E \left[ \ln(C_{t}^{1,E}) + \beta \ln(C_{t+1}^{1,E}) \right], \\
\text{subject to} & \quad (11), (12) \text{ and } (15), \text{ and conditional on the optimal } \mu_{t_1+1}^k. \text{ The optimal choice of } V_{t_1}^{1,E} \text{ solves} \\
& \quad E_n^E \left[ \frac{P_{t_2}^E - P_{t_1}^E}{y + V_{t_1}^{1,E}(P_{t_2}^E - P_{t_1}^E)} \right] = 0. \quad (17)
\end{align*}

4. Equilibrium

Drawing on the above discussion the total number of shares on the market in trading period \( t_1 \) is \( \mu_{t_1}^k K \) (which is zero in the RMA regime) and in trading period \( t_2 \) it is \( K \). Equilibria in the two trading regimes satisfy the following definition:

**Definition 1.** Given the stochastic process \( \tilde{x} \), a competitive equilibrium in the EMA regime is a pair of price processes \( \{P_{t_1}^E, P_{t_2}^E\} \), and a set of asset-trading strategies \( \{V_{t_1}^{1,E}, V_{t_2}^{1,E}, V_{t_2}^{2,E}, \mu_{t_1}^E\} \) that solve (5), (9), (15) and (17) given prices, and for \( k = \text{EMA} \) markets clear

\begin{align*}
\forall_{t_1, V_{t_1}^{1,k}} = \mu_{t_1}^k K, \\
\forall_{t_1, V_{t_2}^{1,k}} + \forall_{t_2, V_{t_2}^{2,k}} = K.
\end{align*}

**Definition 2.** Given the stochastic process \( \tilde{x} \), a competitive equilibrium in the RMA regime is a price process \( \{P_{t_2}^R\} \), and a set of asset-trading strategies \( \{V_{t_1}^{1,R}, V_{t_2}^{1,R}, V_{t_2}^{2,R}, \mu_{t_1}^R\} \) that solve (4), (9), (13) and (14) given prices, and markets clear – (18)–(19) hold for \( k = \text{RMA} \).

Using (9) and (14) in (19) the equilibrium price in the single trading period of the RMA regime is \(^5\)

\begin{equation}
\frac{\beta Q \delta y}{(1 + \beta)K}.
\end{equation}

In the EMA regime, the equilibrium asset price in trading period \( t_2 \) (conditional on \( \{\mu_{t_1}^E, P_{t_1}^E\} \)) is obtained by substituting (9) and (15) into (19):

\(^5\) See Appendix A for additional discussion of equilibrium and for all proofs.
For trading period $t_1$ of the EMA regime, the equilibrium asset price can be shown to be (see Appendix A)

$$P_{t_1}^E = \frac{\beta[Q_t y - P_{t_1}^E \mu_{t_1}^E K]}{[1 + \beta(1 - \mu_{t_1}^E)]K}.$$  \hspace{1cm} (21)

This expression includes as an argument $\mu_{t_1}^E$. To derive an explicit expression for $\mu_{t_1}^E$, first substitute (22) into (21) to give

$$P_{t_2}^E = \frac{\beta[y(Q_t - x_{t_1} (1 - \mu_{t_1}^E)) + \mu_{t_1}^E KD]}{[1 + \beta(1 - \mu_{t_1}^E)]K}.$$  \hspace{1cm} (23)

Finally, substitute (22) and (23) in (5), note that $x_{t_1}$ can be inferred from $P_{t_1}^E$, and simplify to give

$$\mu_{t_1}^E = \frac{(1 + \beta)yx_{t_1} [E_{t_1}^E G(Q_t)]}{1 + \beta yx_{t_1} [E_{t_1}^E G(Q_t)]},$$  \hspace{1cm} (24)

where $G(Q_t) = 1/[y(x_{t_1} + \beta Q_t) + (1 + \beta)KD]$. Substituting (24) in (22) yields and expression for $P_{t_1}^E$ in terms of only exogenous variables. An implication of (24) that is useful in what follows is:

**Lemma 1.** In the EMA regime, a positive fraction of beginning-of-period shareholdings are sold during both the $t_1$ and $t_2$ trading periods: $\mu_{t_1}^E \in (0, 1)$.

5. **Implications for asset prices**

This section studies the effects of opening markets more often on the level of asset prices and the variability of asset prices. The effects on unconditional moments are considered first and then informational consequences of opening markets more often are addressed in the discussion of conditional moments.

5.1. **The level of asset prices**

**Proposition 1.** If $EP_{t_1}^k$ denotes the unconditional mean of $P_{t_1}^k$, then

$$EP_{t_2}^E > EP_{t_2}^R > EP_{t_1}^E,$$

and $EP_{t_2}^R$ is greater than the weighted average selling price in the EMA regime:

$$EP_{t_2}^R > E\left[\mu_{t_1}^E P_{t_1}^E + (1 - \mu_{t_1}^E)P_{t_2}^E\right].$$
The intuition behind these inequalities is that the asset price tends to be low in the first trading period of the EMA regime to entice young agents to purchase assets and therefore bear the risk associated with carrying assets into the second trading period. In effect, a low asset price in the first trading period is the cost old agents incur for being able to share the risk with young agents of holding assets during the current time period. This also means that the asset price tends to increase across trading periods in the EMA regime which, in turn, works (through young agents’ saving functions) to increase demand for assets at \( t_2 \), and thus leads to a higher asset price at \( t_2 \) in the EMA regime than in the RMA regime.

How robust is this link between opening markets more often and mean asset prices? In the EMA regime, lower asset prices at \( t_1 \) derive from risk sharing between young and old agents, and higher asset prices at \( t_2 \) stem from the fact that young type-1 agents carry an inventory of assets into \( t_2 \) which introduces a pure income effect into type-1 agents’ saving functions at \( t_2 \) (on top of the usual income and substitution effects of a price change). With more general preferences, these factors are still the main influences on mean asset prices stemming from opening markets more often. However, the net effects will be complicated by the usual income and substitution effects. For instance, regardless of preferences, type-1 agents’ wealth in the EMA regime at \( t_2 \) is \( y + V^{1,E}_{t_1}(P^{E}_{t_2} - P^{E}_{t_1}) \). Because this is an increasing function of \( P^{E}_{t_2} \), the pure income effect discussed above would still counteract the usual income effect of a price change and therefore tend to reduce the price elasticity of saving relative to the RMA regime. This would therefore tend to raise average asset prices at \( t_2 \) in the EMA regime. Nonetheless, with more general preferences the net effect of opening markets more often on mean asset prices would also be affected by expected asset prices at \( t + 1 \), which could either strengthen or offset the above result depending on risk aversion.

Conditional asset price moments are also of interest because opening markets more often may affect traders’ information sets. In the EMA regime traders can infer \( x_{t_1} \) from \( P^{E}_{t_1} \), but in the RMA regime markets are closed at \( t_1 \), and thus \( x_{t_1} \) is not revealed by market activity. This is relevant to traders at \( t_1 \) because \( x_{t_1} \) is useful information for forecasting the asset price at \( t_2 \). It is also relevant at \( t_2 \), because it may not be possible to disentangle \( x_{t_2} \) from \( Q_{t} \) (see (20)) – which affects the ability to forecast asset prices in period \( t + 1 \) (because \( \tilde{x} \) is Markov), and consequently has a direct bearing on conditional welfare.

This ‘information effect’ was noted by Schwartz (1988) in the context of a comparison of alternative systems for processing orders. Specifically, Schwartz noted that a key advantage of processing orders continuously rather than periodically is the revelation of floor information (including volume and price information) to actual and potential market participants. In the context of the model developed above, in the RMA regime traders’ information sets will in-
clude historical values of \( \{Q_t\} \) (inferred from asset prices) as well as any additional information obtained from other sources. Specifically, in the RMA regime traders may also observe some additional information at each moment \( t_i \) (e.g. electronic order books), which is imperfectly correlated with \( x_{t_i} \). Thus, whereas \( x_{t_i} \) summarizes the information sets of traders in the EMA regime at \( t_i \) (because \( x \) is Markov), let \( S_{t_i} \) denote the (generally coarser) information sets of traders in the RMA regime (which do not in general contain \( x_{t_i} \)).

Consider next whether the above effects of opening markets more often on mean asset prices are valid conditional on information available to traders at \( t_1 \). Drawing on the above discussion, if \( S_{t_1} \) denotes agents’ information set at \( t_1 \) in the RMA regime, then it can be shown (see Appendix A) that

\[
E_{t_1}^{R} P_{t_2}^{R} > E_{t_1}^{E} \left[ \mu_{t_1}^{E} P_{t_1}^{E} + (1 - \mu_{t_1}^{E}) P_{t_2}^{E} \right]
\]

if and only if \(^6\)

\[
E(Q_t | S_{t_1}) > E(Q_t | x_{t_1}) + \left( \frac{x_{t_1}}{\beta} \right) \left( 1 - \frac{E[G(Q_t | x_{t_1})]}{G[E(Q_t | x_{t_1})]} \right),
\]

where

\[
G[E(Q_t | x_{t_1})] = 1/\beta [x_{t_1} + \beta E(Q_t | x_{t_1})] + (1 + \beta)KD].
\]

The final term on the right-hand side of (25) is necessarily negative because \( G(\cdot) \) is a convex function. Thus, the conditional mean of the weighted average asset price in the EMA regime is less than the conditional mean of the asset price in the RMA regime provided the forecast of the aggregate demand shock in the RMA regime is not too much lower than this forecast in the EMA regime. Equivalently, if (25) is not satisfied then the information sets of traders are so different in the two regimes at \( t_1 \) that the effect on unconditional mean asset prices identified in Proposition 1 is dominated by the pure information effect of opening markets more often.

5.2. The variability of asset prices

Consider next the variability of asset prices in the two regimes. To interpret the formal results that follow, it is helpful to think about this issue in terms of the market-clearing condition in the EMA regime. Substituting (9) and (15) into the market-clearing condition (19) and collecting terms gives

\[
P_{t_2}^{E} K = \frac{\beta y Q_t}{(1 + \beta)} + \frac{\beta x_{t_1} V_{t_1}^{E} (P_{t_2}^{E} - P_{t_1}^{E})}{(1 + \beta)}.
\]

\(^6\) The implications for the other inequalities in Proposition 1 will be apparent from the discussion.
Eq. (26) equates the aggregate value of assets with aggregate saving of young agents. In Appendix A it is shown that this expression can be written as

\[ P^E_{t_2} = P^R_{t_2} + \left( \frac{\beta y x_{t_1}}{(1 + \beta)K} \right) F(Q_t), \] (27)

where \( F(Q_t) = [E(G(Q_t)|x_{t_1})/G(Q_t)] - 1. \)

The relative unconditional variances of \( P^E_{t_2} \) and \( P^R_{t_2} \) as functions of the aggregate demand shock, \( Q_t \), can be cast in terms of elasticities of aggregate savings as functions of \( Q_t \). Note that, by definition, aggregate saving in the RMA regime is, in equilibrium, equal to \( P^R_{t_2}K \). Thus, the response of aggregate saving to \( Q_t \) will be larger in the EMA regime if the second term on the right-hand side of (27) is positively correlated with the first term. In that case, \( P^E_{t_2} \) will have higher (unconditional) variance than \( P^R_{t_2} \) – a given aggregate demand shock requires a larger change in the EMA price than the RMA price to clear markets. In contrast, if the second term on the right-hand side of (27) is (sufficiently) negatively correlated with \( Q_t \), then the elasticity of aggregate saving in the EMA regime is lower than in the RMA regime. In this case, (unconditional) asset price volatility at \( t_2 \) will be lower in the EMA regime.

What matters for the relative variances of asset prices at \( t_2 \), therefore, is the covariance between the two terms on the right-hand side of (27). As formalized below, this covariance can, in principle, be positive or negative, and thus there could be either higher or lower unconditional asset price volatility in the EMA regime at \( t_2 \). For this covariance to be negative it would have to be the case that, unconditionally, larger aggregate demand shocks reduce the demand for shares in trading period \( t_2 \) by type-1 agents relative to this demand in the RMA regime. This could occur because, even though asset prices tend to be increasing in aggregate demand shocks, the difference \( (P^E_{t_2} - P^E_{t_1}) \) could actually be decreasing in \( Q_t \), thereby causing the second term on the right-hand side of (26) to be decreasing in \( Q_t \). Nonetheless, the analysis in Section 6 suggests that the only situation that occurs for any reasonable parameterizations of uncertainty is a strongly positive covariance, and thus in practice \( P^E_{t_2} \) is more volatile than \( P^R_{t_2} \).

Finally, before presenting the formal result on unconditional variances, one can write the following expression for the weighted average of asset prices in the EMA regime:

\[ [\mu^{E}_{t_1} P^E_{t_1} + (1 - \mu^{E}_{t_1})P^E_{t_2}] = P^R_{t_2} - \left( \frac{y x_{t_1}}{(1 + \beta)K} \right) F(Q_t). \] (28)

What is noteworthy about this expression is that it implies that weighted average asset prices in the EMA regime tend to have lower variance than \( P^R_{t_2} \) in precisely those circumstances where \( P^E_{t_2} \) is more volatile than \( P^R_{t_2} \). To see why, note that
\[
\left[ \mu_{t_2}^E P_{t_2}^E + (1 - \mu_{t_2}^E) P_{t_2}^R \right] = \left[ P_{t_2}^E + \mu_{t_1}^E (P_{t_2}^E - P_{t_2}^R) \right].
\]

Thus, positive covariance between the two terms on the right-hand sides of (26) or (27) raises the variability of \( P_{t_2}^E \) relative to \( P_{t_2}^R \), but it also tends to produce negative covariation between the two terms in the expression \( P_{t_2}^E + \mu_{t_1}^E (P_{t_2}^E - P_{t_2}^R) \). This consequence tends to reduce the variability of weighted average asset prices in the EMA regime. The variance of this weighted average of asset prices is more important to type-1 agents than simply the variance of asset prices at \( t_2 \) and it is more important to all old agents. The role of this will be paramount in the welfare analysis presented in Section 6.

**Proposition 2.** Let
\[
\gamma = \frac{\text{Cov}(Q_t, x_t F(Q_t))}{\text{Var}(x_t F(Q_t))}.
\]

Then
(a) for \( \gamma \in \left( \frac{1}{2}\beta, \infty \right) \),
\[
\text{Var}(P_{t_2}^E) > \text{Var}(P_{t_2}^R) > \text{Var}\left( \mu_{t_1}^E P_{t_1}^E + (1 - \mu_{t_1}^E) P_{t_2}^E \right);
\]
(b) for \( \gamma \in \left( \frac{1 - \beta}{2\beta}, \frac{1}{2}\beta \right) \),
\[
\text{Var}(P_{t_2}^E) > \text{Var}\left( \mu_{t_1}^E P_{t_1}^E + (1 - \mu_{t_1}^E) P_{t_2}^E \right) > \text{Var}(P_{t_2}^R);
\]
(c) for \( \gamma \in \left( -\frac{1}{2}, \frac{1 - \beta}{2\beta} \right) \),
\[
\text{Var}\left( \mu_{t_1}^E P_{t_1}^E + (1 - \mu_{t_1}^E) P_{t_2}^E \right) > \text{Var}(P_{t_2}^E) > \text{Var}(P_{t_2}^R);
\]
(d) for \( \gamma \in \left( -\infty, -\frac{1}{2} \right) \),
\[
\text{Var}\left( \mu_{t_1}^E P_{t_1}^E + (1 - \mu_{t_1}^E) P_{t_2}^E \right) > \text{Var}(P_{t_2}^R) > \text{Var}(P_{t_2}^E).
\]

Consider next the variability of asset prices conditional on information actually available to traders:

**Proposition 3.** \( \text{Var}[P_{t_2}^R | S_{t_1}] \) may be greater or less than \( \text{Var}[\mu_{t_1}^E P_{t_1}^E + (1 - \mu_{t_1}^E) P_{t_2}^E | x_{t_1}] \) depending on the informational consequences of opening markets more often (i.e. depending on the set \{\( S_{t_1}, x_{t_1} \)\}). However, \( E[\text{Var}(\mu_{t_1}^E P_{t_1}^E + (1 - \mu_{t_1}^E) P_{t_2}^E | x_{t_1})] < E[\text{Var}(P_{t_2}^E | S_{t_1})] \) regardless of the informational consequences of opening markets more often, and thus on average the conditional variance of the weighted average of asset prices is lower when markets are open more often.

To understand this result, note that even if having markets open less often leads to much less information, this does not necessarily have any connection to traders’ perceptions of asset price variability. For example, the information set
$S_t$ could cause traders to place more weight on states with relatively less volatile prices than they would if they knew $x_t$. Nonetheless, on average conditional asset price volatility in the RMA regime is greater than the conditional variance of weighted average asset prices in the EMA regime, regardless of the informational consequences of limiting the amount the market is open for trading.

Finally, consider the consequences of opening markets more often for conditional asset price volatility at a given point in time. In particular, consider the conditional variances of $P^{E}_{t_2}$ and $P^{R}_{t_2}$. A well-known result in the finance literature is that processing orders periodically rather than continuously reduces price volatility at any given point in time that markets are open. In the context of this paper, closing the asset market at $t_1$ necessarily reduces conditional asset price volatility at $t_2$ only if one assumes – which is an implicit assumption in the literature – that there is no informational consequence of closing markets. That is, using (20) and (21) to calculate conditional asset price variances, it is easily verified that $\text{Var}(P^{E}_{t_2}|x_t) > \text{Var}(P^{R}_{t_2}|S_t)$ necessarily only if one assumes $x_t \in S_{t_1}$. When there are informational consequences of closing markets, however, there is no reason to expect this to be the case.

Before turning to the welfare analysis, note that the above asset price expressions and all the formal results hold when $D$ is a random variable. This generalization of uncertainty in the model requires minor cosmetic changes to the analysis presented above. The analysis of conditional moments of asset prices requires simply that the conditioning information in both trading regimes includes the current realization of the dividend (since it is paid at the start of period $t$), and then only to the extent that $D$ contains information about $Q_t$. The formal results regarding unconditional moments are identical; the discussion of Proposition 2 would, however, need to be modified slightly to account for the fact that the stochastic properties of $F(Q_t)$ (and thus $\gamma$) would depend on both $Q_t$ and $D$.

6. Welfare effects of opening markets more often

6.1. Numerical methodology

The approach taken in this section of the paper to evaluate the welfare consequences of opening markets more often is as follows. First, the param-

---

7 It is readily verified that the expressions for optimal asset demands and thus equilibrium asset prices are identical to those presented above except that $D$ refers to the current dividend realization at $t_1$. It is also easily verified by inspecting the technical proofs presented in Appendix A that they are unaffected when $D$ is a random variable realized at $t_1$.

8 This is readily established from (25) and the proof of Proposition 3.
eters \(\{ y, K, \beta, D \}\) are assigned values (the procedure for doing so is discussed below). Second, conditional on the four-tuple \(\{ y, K, \beta, D \}\) the stochastic process \(\tilde{x}\) is parameterized so that the standard deviation of asset prices in the RMA regime is within a range of observed levels of asset price volatility. Third, this second step is repeated for alternative values of the four-tuple \(\{ y, K, \beta, D \}\). Finally, the welfare consequences (unconditionally and conditionally) are determined by computing how much more or less lifetime consumption agents would require in the RMA regime to be indifferent to opening markets more often.

The ranges of values considered for the four-tuple \(\{ y, K, \beta, D \}\) are: \(y \in [1, 1000]\), \(K \in [1, 1000]\), \(\beta \in [0.7, 1.0]\) and \(D \in [1, 100]\). The effects of increasing these intervals is discussed below. The stochastic process \(\tilde{x}\) is assumed to follow a Markov chain with a particular structure to accommodate possibly different state spaces for \(\tilde{x}\) in trading periods 1 and 2. Specifically, \(\tilde{x}\) has two possible realizations in each of trading periods 1 and 2, but these possible realizations may not be the same in trading periods 1 and 2. In other words, the process \(\tilde{x}\) has four possible realizations during each time period, but only two states have positive probability in each trading period. The process \(\tilde{x}\) is therefore technically a four-state Markov chain with a block-diagonal transition matrix – there are two \(2 \times 2\) blocks of zeros in the \(4 \times 4\) transition matrix. With the additional assumption that the transition probabilities are constant, \(\tilde{x}\) necessarily has a unique steady-state distribution (Feller, 1968). This distribution can be used to calculate unconditional moments of asset prices and unconditional welfare functions.

The standard deviation of daily percentage changes in the US Standard and Poor 500 stock price index is 0.92% over 1973–1996, with weekly observations it is about 2.5% over 1928–1996, with monthly data it is about 3.4% over 1957–1996, and with annual postwar data it is 13.7%. Standard deviations of price changes for US. Treasury securities are roughly half as large as the S&P 500 index for similar frequencies. Although relatively high frequencies might be more realistic for many interpretations of the model, it should also be recognized that the standard deviations reported above are from low-risk government bonds and a well-diversified equity index (for the world’s largest equity market). To ensure that the structure imposed on the range of price volatilities produced by the model does not rule out possibly interesting applications, but at the same time does rule out most unrealistic parameterizations, I consider a range of standard deviations of asset price changes in the RMA regime of \((0\%, 20\%)\).

This range of asset price volatilities imposes structure on parameterizing the Markov chain \(\tilde{x}\), but it still leaves considerable degrees of freedom in choosing the transition probabilities and the state space. In particular, there are in general many parameterizations of \(\tilde{x}\) that yield identical asset price volatility, but which may lead to very different welfare consequences of opening markets.
more often. It is, therefore, important to consider a wide variety of parameterizations of \( \tilde{x} \) for each level of asset price volatility. The approach taken to accommodate this is as follows. First, if state 1 (2) denotes the low (high) value of \( \tilde{x} \) in a given trading period, then the four-tuple \( \{ \pi_{11}, \pi_{12}, \pi_{21}, \pi_{22} \} \) characterizes the non-zero elements of the transition matrix. I consider the following possible values for these transition probabilities:

- \( \{ 0.9, 0.1, 0.1, 0.9 \} \),
- \( \{ 0.1, 0.9, 0.9, 0.1 \} \),
- \( \{ 0.3, 0.7, 0.7, 0.3 \} \),
- \( \{ 0.7, 0.3, 0.7, 0.3 \} \),
- \( \{ 0.3, 0.7, 0.3, 0.7 \} \).

Second, conditional on the transition probabilities (and the parameters \( \{ y, K, \beta, D \} \)), I draw from nonoverlapping uniform distributions, 3000 possible values for the low state \( x_t \), and 3000 possible values for the high state \( x_t \). The uniform distributions from which the low and high values of \( x_t \) are drawn have supports \([60, 100]\) and \([100, 140]\). Wider intervals or intervals with a higher or lower mean do not affect the findings, just the fraction of the 6000 draws that yield asset price volatilities in the range reported above. This procedure is repeated by drawing 3000 possible values for each of the two realizations of \( x_t \).

Finally, the unconditional standard deviation of the percentage change in \( P_t \) is calculated for each of the possible realizations of the state variable \( \tilde{x} \). For those combinations that produce asset price standard deviations in the range \((0\%, 20\%)\), the equilibrium in the EMA regime is computed, and the welfare difference between the two regimes is determined.

### 6.2. Welfare findings

Fig. 1 reports the welfare benefits of opening markets more often – both conditional on an agent’s type and unconditionally – for specific values of the parameters \( \{ y, K, \beta, D \} \) and the transition probabilities. The integers on the horizontal axis correspond to bins of standard deviations of asset price changes in the RMA regime: the integers on the axis denote the highest standard deviation in each bin, and the lowest standard deviation in each bin is two percentage points less than the highest standard deviation in the bin. The length of the vertical lines correspond to integers on the horizontal axis. The length of these lines therefore measure the range of the welfare benefits of opening markets more often, and the tick on each vertical line is the median welfare benefit.

There are three main implications of Fig. 1. First, the welfare benefit of opening markets more often is always positive, and it is fairly small – the median unconditional welfare benefit is less than 0.2% of lifetime consumption.

---

9 The unconditional welfare function is a convex combination of type-specific welfare functions using weights implied by the steady state distribution of \( \tilde{x} \). Specifically, the weights are \( E[x_t/(x_t+x_{t-1})], \ i = 1, 2 \), which is interpreted as the unconditional fraction of the population that are of type-\( i \), \( i = 1, 2 \).
Second, the welfare benefit of opening markets more often is generally increasing in asset price volatility. Third, there is considerable variation in the welfare benefit for a given degree of volatility of asset prices – the lowest welfare benefit when there is high volatility in asset markets is below the median welfare benefit for much lower levels of volatility.

In all parameterizations underlying Fig. 1 the unconditional variance of \( \tilde{\mu}_t \tilde{P}_t + (1 - \tilde{\mu}_t) \tilde{P}^R_t \) is lower than the variance of \( P^E_t \). This works to raise unconditional welfare in the EMA regime. Proposition 1 states that the mean of weighted average asset prices in the EMA regime is lower than in the RMA regime. This works to lower mean consumption in agents’ second period of life, and therefore offsets the benefits of lower volatility of asset prices. Nonetheless, the ratio of the mean to the standard deviation of asset prices in the EMA regime is greater than in the RMA regime for all parameterizations underlying Fig. 1. In addition, in the EMA regime type-2 agents on average acquire a larger share of assets when young than type-1 agents, which has the consequence of raising type-two agents’ mean second-period consumption.\(^{10}\)

---

\(^{10}\) This assertion is explained below.
Although this also reduces mean consumption of type-1 agents in their second period of life in the EMA regime, these agents necessarily have higher mean consumption in their first period of life (as implied by Proposition 1). Finally, for a given level of volatility of asset prices in the RMA regime, the welfare benefits of the EMA regime are largest when volatility in asset markets is concentrated in $x_t^2$ rather than $x_t^1$. The reason is that in these circumstances the ability to trade at $t_1$ is most beneficial in terms of sharing risk between the two generations (buyers and sellers of assets).

In light of the fact that type-1 agents have higher savings on average in the EMA regime than in the RMA regime, it might appear odd that type-2 agents generally acquire a larger share of the stock of assets than do type-1 agents. These two observations are consistent because type-1 agents are aggressive buyers of assets only when the asset price is relatively high (see (15)). Because type-2 agents save a constant amount, they therefore are able to acquire a greater number of shares when the asset price is relatively low. Of course, whether type-1 agents acquire a larger share of assets on average depends on the curvature of their saving function (as a function of $P_{t}^{E}$) relative to the price of purchasing assets. It can be verified from (9) and (15) that the difference $(V_{t_2}^{1,E} - V_{t_2}^{2,E})$ is an increasing concave function of the asset price. This implies that type-1 agents are sacrificing a relatively large share of assets when prices are low, which is more important overall in the numerical analysis discussed here than the fact that type-1 agents acquire a larger share of assets than type-2 agents when prices are high. One interpretation of this is that young type-2 agents effectively share in the returns earned (on average) by type-1 agents on their asset-holdings within period $t$ in the EMA regime.

Alternative values of the four-tuple $\{y, K, b, D\}$ or the transition probabilities have the following effects on the welfare results reported in Fig. 1. First, a lower value of $b$ reduces the welfare benefits because this leads to less saving, and it also causes agents to attach less weight to the risk-sharing benefits of EMA in the second period of life. Reducing $b$ from 1.0 to 0.8, say, reduces the welfare benefits on average by about 10% of those reported in Fig. 1. Second, an increase in the parameters $K$ or $D$ reduces the welfare benefits of opening markets more often. The intuition behind this is that agents’ consumption increases – due to higher dividend income – and thus less concavity of the utility function at higher income levels reduces risk diversification benefits of the EMA regime. For instance, a tenfold increase in either of these parameters reduces the welfare benefits by roughly 40% of those reported in Fig. 1. Third, a higher value of $y$ has small positive effects on welfare because the associated higher level of saving tends to dominate the scale effect of higher income discussed above. Finally, compared with Fig. 1, introducing more persistence in the Markov chain tends to reduce welfare benefits: increasing the probability of continuation of the same state from 0.7 to 0.9 reduces the median welfare benefit in Fig. 1 by about 25%.
There are between 1500 and 2600 different combinations of the state variables underlying the above analysis of unconditional welfare (about 1600 underlying Fig. 1). Recall that these combinations are chosen from 3000 possible combinations, based on the criteria that they produce asset price standard deviations no greater than 20%. It is noteworthy that those parameterizations that do not satisfy this criteria lead to precisely the same qualitative findings as reported in the text above, and they do not affect the intuition regarding the factors responsible for the welfare gains from EMA. There are certainly parameterizations where the variance of $\mu_{t_1}^E p_{t_1}^E + (1 - \mu_{t_1}^E) p_{t_2}^E$ is higher than the variance of $P_{t_1}^R$ – notably by choosing very large variability of $x_{t_1}$ relative to $x_{t_2}$ – which would therefore tend to reduce the welfare benefits of the EMA regime. For example, if (following the approach described in Section 6.1.) the state space of $x_{t_1}$ is chosen from the interval $[1, 10^5]$ and the state space of $x_{t_2}$ is chosen from the interval $[1, 10]$, there are many parameterizations for which this is the case (although the volatility of asset prices in these parameterizations is enormous). However, the welfare benefits of EMA are in fact positive for all of these numerical exercises too. Here the welfare benefits derive almost entirely from the higher level of consumption of young type-1 agents and the sharing of these benefits with young type-2 agents (by the latter acquiring a higher fraction of assets). In summary, there may exist parameterizations such that the EMA regime leads to lower unconditional welfare, but these parameterizations appear to be rare, and they are almost surely unrealistic.

Figs. 2 and 3 show the welfare benefits of the EMA regime conditional on agents’ information (and their type) at $t_1$. The two figures correspond to two alternative informational assumptions in the RMA regime: (i) agents in the RMA regime observe $x_{t_1}$ at $t_1$ and (ii) agents in the RMA regime observe only $x_{t_2 - 1}$ at $t_1$ (which can always be inferred by agents from $P_{t_1}^R$ as the four possible values for the state variable are unique in all parameterizations considered). These two informational assumptions provide polar cases: in one case there are no informational consequences of the trading regime and in the other case agents receive no new information at $t_1$ in the RMA regime. Note that there are four possible combinations of states at $t_2 - 1$ and $t_1$; for illustrative purposes Fig. 3 shows just two possible combinations.

When there is no informational consequence of opening markets more often, the welfare implications are very similar to the unconditional welfare findings. In contrast, the welfare implications when there are informational consequences of opening markets more often can be large, and they can be positive or negative. Informational effects on welfare can be large, and either positive or negative, because they cause agents to place possibly very different

---

11 These combinations produce asset price standard deviations up to 95% for the parameterization in Fig. 1.
weights on possibly very different states of the world. For instance, the EMA regime produces welfare losses relative to the RMA regime if, in the RMA regime, agents assign a relatively small probability (based on observing $x_{t-1}$) to a large positive shock to asset markets during $t$ – the ‘bad state’ from traders’

Fig. 2. Conditional welfare benefit of EMA (state at $t_1$ observable in RMA). 
*Notes:* See Fig. 1.

Fig. 3. Conditional welfare benefit of EMA (state at $t_2 - 1$ known to traders in RMA).
*Note:* See Fig. 1.
perspective – whereas traders in the EMA regime might actually observe at $t_1$ that the large shock has been realized.

7. Conclusion

This paper studies formally the desirability of opening asset markets more often. It is found that for reasonable parameter values price volatility tends to be lower when markets are open more often if the measure of volatility includes the additional period of time that markets are open. Further, unconditional welfare is always higher for reasonable parameter values when markets are open more often. However, the amount that markets are open affects the information flow from markets and this can be beneficial or detrimental to conditional welfare depending on the qualitative information that agents receive.

The main aspect of the framework used in the paper that makes it suitable for studying the effects of opening markets more often is that it is a general equilibrium model in which all agents are rational. Thus, one can measure directly the welfare consequences of opening markets more often. It might be of interest to consider more complicated environments. For instance, following the lead of Kyle (1985) some authors have considered environments with asymmetric information among traders. That might be an interesting possibility to explore in future work on the subject of this paper, although the complexity of embedding adverse selection problems into a dynamic general equilibrium framework that incorporates strategic market timing decisions poses significant technical challenges.

Acknowledgements

Financial assistance from SSHRC of Canada is gratefully acknowledged. Seminar participants at the IMF, Henry van Egteren, Gregor Smith, and especially Charlie Thomas, Dan Bernhardt and two referees provided helpful suggestions.

Appendix A. Equilibria in the trading regimes

RMA: For $D > 0$ and any set of beliefs about asset prices in the subsequent time period, the unique asset demand functions are given by (9) and (14). Thus, from the market clearing condition (19), the unique equilibrium asset price function is (20).
EMA: Consider trading period $t_2$, conditional on any pair of nonnegative numbers $\{\mu_{t_i}^E, P_{t_i}^E\}$. Paralleling the above discussion of equilibrium in the RMA regime, there is a unique equilibrium price function given by Eq. (21).

Next, consider trading period $t_1$. Let $\hat{\mu}_{t_1}^E$ denote the solution to (5) and $\hat{V}_{t_1}^E$ the solution to (17). It is easily verified using (5) and (17) that $\hat{V}_{t_1}^E = (1 - \hat{\mu}_{t_1}^E)y/(D + P_{t_1}^E)$. Using this relationship between $\hat{\mu}_{t_1}^E$ and $\hat{V}_{t_1}^E$ in the market-clearing condition (18) gives the unique equilibrium price function

$$p_{t_1}^E = \frac{x_{t_1}(1 - \hat{\mu}_{t_1}^E) - \hat{\mu}_{t_1}^E KD}{\hat{\mu}_{t_1}^E K}. \quad (A.1)$$

Finally, substituting Eq. (A.1) into (5) and simplifying gives the unique expression for $\hat{\mu}_{t_1}^E$, which is (24) in the text.

Proof of Lemma 1. Follows directly from (24) and the assumption that $\hat{x} > 0$.

Proof of Proposition 1. Begin with the first part of the first inequality. Substituting (24) into (23), and rearranging the resulting expression gives

$$P_{t_2}^E = P_{t_2}^R + \left(\frac{\beta y x_{t_1}}{1 + \beta K}\right) F(Q_t), \quad (A.2)$$

where $F(Q_t) = [E(G(Q_t)|x_{t_1})/G(Q_t)] - 1$. Taking unconditional expectations of both sides of Eq. (A.2) and applying the law of iterated expectations gives

$$E_{t_2} P_{t_2}^E = E_{t_2} P_{t_2}^R + E \left(\frac{\beta y x_{t_1}}{1 + \beta K}\right) \left(\frac{E_{t_1} G(Q_t)}{G(E_{t_1} Q_t)} - 1\right), \quad (A.3)$$

where $G[E(Q_t|x_{t_1})] = [1/((1 + \beta)KD + y(x_{t_1} + \beta E(Q_t|x_{t_1}))].$ By Jensen’s inequality, $E(G(Q_t|x_{t_1}) > G(E(Q_t|x_{t_1}))$, which establishes that $E_{t_2} P_{t_2}^E > E_{t_2} P_{t_2}^R$.

To show that $E_{t_1} P_{t_1}^E < E_{t_2} P_{t_2}^R$, use (20), (22), and (24) to write this inequality as

$$E \left[1 - x_{t_1} y E(G(Q_t|x_{t_1}) - KD\right] < E \left[\frac{\beta Q_t y}{1 + \beta}\right]. \quad (A.4)$$

Expressing the left-hand side with a common denominator, collecting terms, and applying the law of iterated expectations, one can write Eq. (A.4) as

$$E(1/E(G(Q_t|x_{t_1})) < 1/E[G(Q_1|x_{t_1})]. \quad (A.5)$$

By Jensen’s inequality note that $E(G(Q_t|x_{t_1}) > G(E(Q_1|x_{t_1})$ for any $x_t > 0$. This establishes the second part of the first inequality.

Next, consider the second inequality. Substituting (24) into (22) and (23) and simplifying yields

$$[\mu_{t_1}^E P_{t_1}^E + (1 - \mu_{t_1}^E) P_{t_2}^E] = P_{t_2}^R - \left(\frac{y x_{t_1}}{1 + \beta K}\right) F(Q_t). \quad (A.6)$$
Note that \( F(Q_t) \) is a random variable that may in general be positive or negative. Taking unconditional expectations of both sides of Eq. (A.6) and applying the law of iterated expectations gives

\[
E \left[ \mu_t^E P_t + (1 - \mu_t^E) P_t^E \right] = EP_t^E - E \left( \frac{y_{X_t}}{(1 + \beta)K} \right) \left( \frac{E(G(Q_t)|x_t)}{G(Q_t|x_t)} \right) - 1.
\]

(A.7)

By Jensen’s inequality, \( E(G(Q_t)|x_t) > G(E(Q_t|x_t)) \), which implies that \( E[\mu_t^E P_t + (1 - \mu_t^E) P_t^E] < EP_t^E \).

**Derivation of Inequality (25).** The expectation of (A.6) conditional on \( x_t \) is less than the expectation of (20) conditional on \( S_t \) so long as

\[
\beta |E(Q_t|S_t) - E(Q_t|x_t)| > x_t \left( 1 - \frac{E[G(Q_t)|x_t]}{G[E(Q_t|x_t)]} \right),
\]

which yields (25). Since \( E[G(Q_t)|x_t] > G[E(Q_t)|x_t] \) by Jensen’s inequality, the right-hand side of (A.8) is negative.

**Proof of Proposition 2.** Using (27) and (28),

\[
\text{Var}(P_t^E) = \text{Var}(P_t^E) + \left( \frac{\beta y}{(1 + \beta)K} \right)^2 \text{Var}(x_t F(Q_t)) + 2 \left( \frac{\beta y}{(1 + \beta)K} \right) \text{Cov}(P_t^E, x_t F(Q_t)),
\]

(A.9)

\[
\text{Var}(\mu_t^E P_t + (1 - \mu_t^E) P_t^E) = \text{Var}(P_t^E) + \left( \frac{y}{(1 + \beta)K} \right)^2 \text{Var}(x_t F(Q_t)) - 2 \left( \frac{y}{(1 + \beta)K} \right) \text{Cov}(P_t^E, x_t F(Q_t)).
\]

(A.10)

Substitute in (A.9) and (A.10) the fact that \( \text{Cov}(P_t^E, x_t F(Q_t)) = (\beta y/(1 + \beta)K) \text{Cov}(Q_t, x_t F(Q_t)) \). Using the resulting expression for Eq. (A.9) implies that \( \text{Var}(P_t^E) > (\cdot) \text{Var}(P_t^E) \) if and only if \( \gamma > (\cdot) - 1/2 \), where

\[
\gamma = \frac{\text{Cov}(Q_t, x_t F(Q_t))}{\text{Var}(x_t F(Q_t))}.
\]

In the same manner, using (A.10) it follows that \( \text{Var}(\mu_t^E P_t + (1 - \mu_t^E) P_t^E) > (\cdot) \text{Var}(P_t^E) \) if and only if \( \gamma < (\cdot)1/2 \). Finally, isolating \( \text{Var}(P_t^E) \) on one side of (A.10), substituting this into (A.9), it is easily shown that \( \text{Var}(P_t^E) > (\cdot) \text{Var}(P_t^E) \) if and only if \( \gamma > (\cdot)(1 - \beta)/2 \beta \). Noting that \( 1/2 \beta > (1 - \beta)/2 \beta > 0 \), the above set of relations establishes the claim.
Proof of Proposition 3. From (21)

\[ \text{Var} \left[ \mu_t^E P_t^E + (1 - \mu_t^E) P_t^E | x_t \right] = \left( \frac{\beta y (1 - \mu_t^E)}{[1 + \beta (1 - \mu_t^E) K]} \right)^2 \text{Var}(Q_t | x_t). \]  

(A.11)

Similarly, using (20)

\[ \text{Var}[P_{t_2}^E | S_t] = \left( \frac{\beta y}{1 + \beta} \right)^2 \text{Var}(Q_t | S_t). \]  

(A.12)

Thus

\[ \text{Var}[P_{t_2}^E | S_t] > \text{Var}[\mu_t^E P_t^E + (1 - \mu_t^E) P_t^E | x_t] \]

if and only if

\[ \text{Var}(Q_t | S_t) > x_t \text{Var}(Q_t | x_t), \]  

(A.13)

where

\[ x_t = \left( \frac{(1 - \mu_t^E)(1 + \beta)/(1 + \beta (1 - \mu_t^E))}{1 - \mu_t^E} \right)^2. \]

It is apparent that inequality (A.13) may or may not hold, depending to a large extent on conditional second moments of the distribution of \( Q_t \).

Next, it must be shown that

\[ E[\text{Var} \left[ \mu_t^E P_t^E + (1 - \mu_t^E) P_t^E | x_t \right] < E[\text{Var}[P_{t_2}^E | S_t]]. \]

Using (A.13), this is true if

\[ E[\text{Var}(Q_t | S_t)] > E[x_t \text{Var}(Q_t | x_t)]. \]  

(A.14)

Define the random variables \( m_t(x_t) = \text{Var}(Q_t | x_t) \) and \( n_t(x_t) = x_t m_t(x_t) \). From Lemma 1 note that \( x_t \in (0, 1) \). Thus \( m_t(x_t) > n_t(x_t) \) \( \forall x_t \), and consequently \( E m_t(x_t) > E n_t(x_t) \). The remainder of the proof shows that

\[ E m_t(x_t) < E[\text{Var}(Q_t | S_t)], \]  

(A.15)

which is sufficient for (A.14) to be satisfied.

Given that \( E \text{Var}[z_1 | z_2] = \text{Var}(z_1) - \text{Var}[E(z_1 | z_2)] \) for random variables \( \{z_1, z_2\} \), we can write Eq. (A.15) as follows:

\[ \text{Var}[E(Q_t | S_t)] < \text{Var}[E(Q_t | x_t)]. \]  

(A.16)

Let \( z_t = E(Q_t | x_t) \), so \( E(Q_t | S_t) = E[E(Q_t | x_t) | S_t] = E(z_t | S_t) \) by the law of iterated expectations. Using this in (A.16) yields the inequality

\[ \text{Var}[E(z_t | S_t)] < \text{Var}(z_t). \]  

(A.17)

Since \( \text{Var}(z_t) = \text{Var}[E(z_t | S_t)] + E \text{Var}[z_t | S_t] \), the claim is true.
References


