Default risk in a market model

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Abstract

This paper presents a theoretical model of default risk in the context of the “market” model approach to interest rate dynamics. We propose a model for finite-interval interest rates (such as LIBOR) which explicitly takes into account the possibility of default through the influence of a point process with deterministic intensity. We relate the defaultable interest rate to the non-defaultable interest rate and to the credit risk characteristics default intensity and recovery rate. We find that the spread between defaultable and non-defaultable rate depends on the non-defaultable rate even when the default intensity is deterministic. Prices of a cap on the defaultable rate and of a credit spread option are derived. We consider swaps with unilateral and bilateral default risk and derive the fair fixed swap rate in both cases. Under the condition that both counterparties are of the same risk class, we show that for a monotonously increasing term structure the swap rate for a defaultable swap will lie below the swap rate for a swap without default risk. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Traditionally, models of credit risk have focused on the valuation of defaultable bonds. However, bonds are not the only fixed income instrument
subject to credit risk. For example, LIBOR, which is the underlying of many interest rates and money market derivatives, is not a rate for a riskless issuer, but is assumed to apply to a party of AA-rated quality. Therefore, this interest rate is influenced not only by the general behaviour of riskless interest rates, but also by the risk level of AA-rated banks.

In addition, every derivative written on this interest rate, such as future, forward or option, is itself subject to default risk by the counterparties involved in the derivative. The situation is especially complicated for an interest rate swap, which can become either an asset or a liability to each counterparty depending on the random evolution of interest rates. Our aim with the present paper is to analyse the effect of default risk on these contingent claims.

For a credit risk model which focuses on LIBOR and its derivatives, the natural choice of the underlying interest rate model is a model of the simple forward rate. In the recent past, such models have been proposed in the default-free term structure literature. They directly model simple forward rates (LIBOR) as the primary process (cf. Jamshidian, 1997; Miltersen et al., 1997; Musiela and Rutkowski, 1997), as opposed to deriving them from instantaneous rates. One motivation for this, which is also useful in the credit risk setting, is to show how the market practice of using Black's formula to price caps can be placed in the context of an arbitrage-free interest rate model. Black's formula assumes that the interest rate underlying the option contract is lognormally distributed, and then values the option similar to the well-known Black–Scholes formula for options on stocks. Because models of the simple forward rate justify using Black's formula, they are termed 'Market Models'. They also have the desirable property of generating non-negative interest rates while avoiding the non-integrability problems associated with lognormal instantaneous rates. Contrary to the assumptions in these interest-rate models, however, the LIBOR we observe in the real world is not a default-free rate, but incorporates default risk, typically comparable to that of an AA-rated bank.

Existing credit risk models show us two ways in which default risk can be introduced into an interest rate model: On the one hand, there is the classical firm-value based approach, initiated by Black and Scholes (1973) and Merton (1974) and later adopted by Longstaff and Schwartz (1995) and Zhou (1997). Here default occurs when the firm value process hits a possibly stochastic boundary, which may be prespecified or endogenous.

On the other hand, there is the intensity approach, where the time of default is modelled in more reduced form as the first jump time of a point process with deterministic or stochastic intensity, and this jump time is totally unpredictable. This approach was adopted by Jarrow and Turnbull (1995), Duffie and Singleton (1997), and others. In most models of this type, the intensity of the point process as well as the payout ratio after default are imposed exogenously.
Duffee (1996) shows how the default intensity can be fitted to an observed term structure of corporate bond prices. The main advantage of the intensity approach is its tractability, and for this reason we will also use it in the present paper.

The aim of this paper is to study default risk in the context of LIBOR models. We propose an extension of a market model for LIBOR which explicitly takes into account the possibility of default. In our model, the time of default is determined by the first jump time of a point process with deterministic intensity, and the payoff after default is assumed to be a constant fraction of the value of a non-defaultable, but otherwise equivalent asset. By considering defaultable forward agreements on credit, we derive the arbitrage-free term structure of the defaultable simple rate and find prices for several types of options on this rate. Also, we consider interest rate swaps where one or both counterparties are subject to default risk. The paper is structured as follows. In Section 2, we describe the market model used to characterize the evolution of non-defaultable simple forward rates and the mechanism governing the occurrence of default and the corresponding payoffs. Section 3, we derive the arbitrage-free value of the simple defaultable forward rate and the prices of some related derivatives. Section 4 deals with interest rate swaps. Section 5 concludes and some of the proofs have been relegated to Appendix A.

2. The basic model

This section outlines the general principles and building blocks of our credit risk model. We give an intuitive explanation of the construction of the term structure of interest rates in a market model and show how the risk of default enters our model.

Throughout this paper, we will confine our discussion to a discrete tenor model, that is we consider only contracts with maturities in the set

$$\mathcal{T} = \{T_0, T_1 = T_0 + \delta, T_2 = T_0 + 2\delta, \ldots, T_{N-1} = T_0 + (N-1)\delta\},$$

where $\delta$ is a fixed accrual period. The final date is denoted by $T' = T_N = T_0 + N\delta$. For each $T \in \mathcal{T}$ there exists one simple spot interest rate $L(T, T)$ for the interval $[T, T + \delta]$ which is not susceptible to default risk, and which we will call the non-defaultable spot LIBOR. Consequently, we have a forward LIBOR process $(L(t, T))_{t \in [0,T]}$ for every $T \in \mathcal{T}$. We assume that non-defaultable zero-coupon bonds are traded for each maturity $T \in \mathcal{T}$ and $T'$, the price at time $t$ of the bond with maturity $T$ is denoted by $B(t, T)$. Bond prices and forward rates are related via the usual formula

$$1 + \delta L(t, T) = \frac{B(t, T)}{B(t, T + \delta)}.$$
We assume a LIBOR market model, i.e. the stochastic set-up is as follows. We are given a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, P)$ which satisfies the usual hypotheses and supports a $d$-dimensional Brownian motion $W$. For each $T \in \mathcal{F}$, there exists the $(T + \varepsilon)$-forward measure $P^{T+\varepsilon}$ and the dynamics of $L(\cdot, T)$ are

$$
dL(t, T) = \sigma^L(t, T) L(t, T) dW_{T+\varepsilon}^{T+\varepsilon},$$

where $\sigma^L(\cdot, T)$ is a deterministic function and $W_{T+\varepsilon}^{T+\varepsilon}$ is a $d$-dimensional Brownian motion under $P^{T+\varepsilon}$. For a given set of volatilities $\sigma^L$ such a model can be constructed by, for example, backward induction (cf. Musiela and Rutkowski, 1997; Jamshidian, 1997), and we can assume that $P$ is the terminal forward measure. When dealing with swaps, we also use a swap market model, i.e., assume that non-defaultable forward swap rates are lognormal martingales under the appropriate martingale measure.

Default is described by the first jump time of a point process. We assume the jump processes involved are Cox processes with a deterministic intensity under the terminal forward measure $P$. In particular, this implies that the intensity is invariant under changes from one forward measure to another: Under every forward measure, the jump process has the same intensity. The generalization of the results of the paper to a stochastic intensity which is independent of the non-defaultable interest rates is straightforward.

We distinguish between the cases of unilateral and of bilateral default. In the first case, we use:

**Assumption 1 (Unilateral default).** There exists on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ a jump process $N(t)$ with deterministic intensity $\lambda(t)$. Default occurs at the first jump time of $N(t)$, formally

$$\tau := \inf\{t \geq 0 \mid N(t) = 1\}.$$

The recovery rate $D$ is constant. The probability of no default event occurring between time $t$ and $T$ is given by

$$P[N(T) - N(t) = 0] = \exp\left\{ - \int_t^T \lambda(s) \, ds \right\}. \quad (1)$$

This implies that the distribution of the default time is given by

$$P[\tau \leq T] = 1 - \exp\left\{ - \int_0^T \lambda(s) \, ds \right\}$$

and its density is

$$f(t) = \exp\left\{ - \int_0^t \lambda(s) \, ds \right\} \lambda(t).$$

In the case where both counterparties are subject to default risk, we use the following assumption instead:
Assumption 2 (Bilateral default). For $i = 1, 2$, there exists a jump process $N_i(t)$ with deterministic intensity $\lambda_i(t)$. The two jump processes are assumed to be independent. Default of counterparty $i$ occurs at the first jump time $\tau_i$ of $N_i(t)$, so that

$$P[\tau_i > t] = P[N_i(t) = 0] = \exp \left\{- \int_0^t \lambda_i(s) \, ds \right\}, \quad i = 1, 2.$$ 

There exist two constant recovery rates $A_1$ and $A_2$, so that $A_i$ determines the payoff after default of counterparty $i$.

Due to Brémaud (1981, Ch. VIII.2, T7, p. 238), we have the following proposition (see also Last and Brand, 1997, Ex. A5.19, p. 441).

Proposition 1. Consider the marked point process associated with the two jump processes $N_1(t)$ and $N_2(t)$ and the mark space $\{z_1, z_2\}$. This has the compensator

$$\tilde{\lambda}(t, z) = \lambda(t) \left( \frac{\lambda_1(t)}{\lambda(t)} 1_{\{z = z_1\}} + \frac{\lambda_2(t)}{\lambda(t)} 1_{\{z = z_2\}} \right),$$

where

$$\lambda(t) = \lambda_1(t) + \lambda_2(t).$$

This proposition means that the time $\tau = \tau_1 \land \tau_2$, at which the first default occurs, is the first jump time of a point process with the intensity $\lambda(t) = \lambda_1(t) + \lambda_2(t)$. Conditional on there having been a default event at time $t$, the probability that it was counterparty $i$ who defaulted is given by

$$P[z = z_i | \tau = t] = \frac{\lambda_i(t)}{\lambda(t)}. \quad (2)$$

Within the class of intensity-based credit risk models, different specifications of the payoff after default have been used. Duffie and Singleton (1997) assume that the asset pays back a certain percentage of its immediate pre-default value. Jarrow and Turnbull (1995) and also Hull and White (1995) follow the convention that the payoff is in terms of a non-defaultable, but otherwise equivalent security. For example, a defaultable zero-coupon bond would pay off a fraction $A$ of a non-defaultable zero-coupon bond immediately in the event of default. This is equivalent with assuming that the face value of the defaultable bond drops from 1 to $A$, which is then paid at maturity of the bond. It is quite clear that the second convention simplifies the model significantly, because a security’s payoff after default does not depend on the default process any more. In this paper, we will use the second method because of its advantages in terms of tractability. With this convention, the occurrence of a default becomes equivalent to a drop in the face value of the security, for example a zero-
coupon bond, from 1 to $\Delta$. Therefore, the following shorthand will simplify notation later on.

**Definition 2.** Consider a security with maturity $T + \alpha$ which pays its nominal face value 1 if no default has occurred and $\Delta$ in the case of default. The expected payoff of this security, given that no default has occurred before time $T$, is

$$\pi(T, T + \alpha) := E^{T+\alpha}_T \left[ 1_{\{\tau > T+\alpha\}} + \Delta 1_{\{\tau \leq T+\alpha\}} \mid \mathcal{F}_T \right].$$

Using Eq. (1), it is easy to see that

$$\pi(T, T + \alpha) = \Delta + (1 - \Delta) \exp \left\{ - \int_T^{T+\alpha} \lambda(s) \, ds \right\}.$$

Here, as throughout the paper, the superscript $T + \alpha$ on the conditional expectation denotes that it is to be taken under the $(T + \alpha)$-forward measure.

### 3. Defaultable credit agreements

We will use credit agreements (CAs) as central building blocks to generalize the notion of forward LIBOR to a setting with default risk. As we will see, it is quite easy to calculate the simple interest rate at time $T$ applying to a loan from $T$ to $T + \alpha$ to a borrower subject to default risk. The definition of forward defaultable LIBOR is based on this spot defaultable LIBOR and is analogous to the usual definition of forward LIBOR. The forward defaultable LIBOR will be the answer to the question: *What is the “fair” simply compounded fixed rate a default-risky borrower can contract at time $t$ for a loan during the time interval $[T, T + \alpha]$?*

The fundamental contract underlying the notion of a forward LIBOR is the Forward Rate Agreement (FRA), and before we start with the default-risky case, we want to make some general remarks on FRAs when there is no default risk. It is important to note that the cash flow in an FRA is not uniquely defined, but may depend on the individual contract. Below, three different cash flow patterns will be described. If there is no default risk, all patterns lead to the same value of the FRA and consequently to the same forward LIBOR. This is no longer the case in the presence of default risk, which is the reason for our discussion.

With FRAs, it is usual to speak about the buyer and the seller of the contract, where the buyer is the one who pays the predetermined interest rate, and the seller receives the predetermined interest rate. Because we will consider swaps later on, where the cash flows are very similar, we have chosen to use the swap convention (payer and receiver) throughout the whole paper: therefore,
we will call the FRA buyer the payer side (it pays the predetermined and fixed interest rate), and the seller of the FRA is called the receiver.

Assume an FRA which is signed at time 0 for the interval \([T, T + \alpha]\), with the fixed rate \(\kappa\). In this case, three cash flow patterns are possible:

1. All payments may be made at time \(T\): in this case, the receiver side will at time \(T\) receive the amount

\[
\frac{1 + \alpha \kappa}{1 + \alpha L(T, T)} - 1.
\]

Of course, if the amount is negative, the receiver side will have to pay something to the payer side.

2. Another possibility is that the exchange of payments is scheduled for time \(T + \alpha\). In this case, the receiver side will at time \(T + \alpha\) receive the amount

\[
(1 + \alpha \kappa) - (1 + \alpha L(T, T)).
\]

Here, the FRA can also be interpreted as a one-period swap.

3. Finally, the payments can be split up between the dates \(T\) and \(T + \alpha\): at time \(T\), the receiver side will pay 1, and at time \(T + \alpha\), it will receive \(1 + \alpha \kappa\). In this situation, the payer side takes out a loan from the receiver side at time \(T\) and pays back the loan with interest at time \(T + \alpha\).

In the last case, we can interpret the fixed rate \(\kappa\) in the following way: it is the interest rate which can be fixed at time 0 in a contract for a loan for the time period \([T, T + \alpha]\). It is this interpretation we rely upon when we define the notion of “defaultable LIBOR”, and therefore we will define FRAs in terms of cash flow number 3).

Although traded FRAs are usually settled in cash at time \(T\), the underlying interest rate applies to default-risky borrowers. Our specification allows us to capture in one contract the risk between \(T\) and \(T + \alpha\), which is usually incorporated in real-world LIBOR, and the counterparty default risk before \(T\). From now on, we will use the term “Credit Agreement” (CA) to emphasize that we use an FRA with cash flow number 3).

**Definition 3.** The non-defaultable CA with the fixed interest rate \(\kappa\) is an agreement to the following cash flows:

- At time \(T\), the receiver pays 1.
- At time \(T + \alpha\), the receiver gets \(1 + \alpha \kappa\).

Assume first that a set of zero-coupon bonds, \(B(t, T)\) is given for all maturities \(T \in \mathcal{T}\) and for \(T^*\).

**Proposition 4.** The price of CA at time \(t\) for the period \([T, T + \alpha]\) with the fixed interest rate \(\kappa\) is given by
Definition 5. The non-defaultable forward LIBOR at time $t$ for the interval $[T, T + \alpha]$ is the fixed interest rate $\kappa$ for which a new, non-defaultable CA at time $t$ for the interval $[T, T + \alpha]$ has zero value. This particular fixed interest rate is denoted by $L(t, T)$ and is determined via

$$1 + \alpha L(t, T) = \frac{B(t, T)}{B(t, T + \alpha)},$$

$L(T, T)$ is then the spot LIBOR at time $T$ for the interval $[T, T + \alpha]$.

3.1. The unilateral case

Throughout this section, we will always assume that only the payer of the fixed rate (the borrower in the contract) is in danger of default. All contract values will be given from the perspective of the receiver of the fixed rate.

First, we deal with the simplest case of a spot interest rate on a credit with default risk in the interval $[T, T + \alpha]$. To this end, we consider the following contract:

Definition 6. An agreement to a defaultable credit with fixed rate $\kappa$, starting date $T$, and accrual period $\alpha$ is specified by the following cash flows:

- At time $T$, the receiver of the fixed rate pays 1.
- At time $T + \alpha$, two things can happen:
  - Either there was no default in $[T, T + \alpha]$. Then the receiver gets $1 + \alpha \kappa$.
  - Or a default occurred. Then the receiver gets $D(1 + \alpha \kappa)$.

We denote the value of this agreement at time $T$ by $CA(T, T, \kappa)$.

Proposition 7. The value at time $T$ of the credit agreement described in the previous definition is given by

$$CA(T, T, \kappa) = B(T, T + \alpha)\{(1 + \alpha \kappa)\pi(T, T + \alpha) - (1 + \alpha L(T, T))\}$$

with $\pi(T, T + \alpha)$ as given in Definition 2.

Proof. The value of the credit agreement at time $T$ is determined by the following equation:

$$CA(T, T, \kappa) = (1 + \alpha \kappa)B(T, T + \alpha)E^{T+\alpha}\left[\mathbb{1}_{\{\tau>T+\alpha\}} + \Delta 1_{\{\tau\leq T+\alpha\}\mathcal{F}_T}\right] - 1.$$

By inserting the definitions of $\pi(T, T + \alpha)$ and of spot LIBOR $L(T, T)$ we obtain
Just as in the non-defaultable case, the defaultable spot LIBOR rate $L^d(T, T)$ is the value of the fixed rate $\kappa$ which leads to a value of zero for the credit agreement in Definition 6. From Eq. (3) we see:

**Proposition 8.** The defaultable LIBOR rate $L^d(T, T)$ at time $T$ for the interval $[T, T + \alpha]$ is determined by the relation

$$1 + \alpha L^d(T, T) = \frac{1 + \alpha L(T, T)}{\pi(T, T + \alpha)}.$$  

Eq. (3) also lets us easily determine the value at time $t < T$ of a security with no default risk and a terminal payoff of $CA(T, T, \kappa)$ at time $T$. We denote this value by $CA(t, T, \kappa)$ and obtain

$$CA(t, T, \kappa) = B(t, T + \alpha)\{(1 + \alpha \kappa)\pi(T, T + \alpha) - (1 + \alpha L(t, T))\}. \quad (6)$$

By the construction of $CA(t, T, \kappa)$, the process

$$\left( \frac{CA(t, T, \kappa)}{B(t, T + \alpha)} \right)_{t \in [0, T]}$$

is a martingale under the $(T + \alpha)$-forward measure.

It is important to note that $CA(t, T, \kappa)$ would not be the correct value for a forward agreement on a credit starting at time $T$ with a default-risky payer, as no allowance is made for the risk of default between time $t$ and $T$. Instead we need to consider the following contract:

**Definition 9.** Suppose that $t < T$. A defaultable forward agreement to a credit with fixed rate $\kappa$, start date $T$, and accrual period $\alpha$, is specified by the following cash flows:

- If no default occurs up to time $T$, the contract then becomes an agreement to a defaultable credit as described in Definition 6.
- If default occurs at time $\tau \leq T$, the position is closed at time $\tau$ according to a mark-to-market procedure as follows: if the value $CA(\tau, T, \kappa)$ of the equivalent non-defaultable agreement is negative to the (non-defaulting) receiver, she must meet her obligation in full. If the value to the receiver is positive at time $\tau$, then she receives only a fraction $\Delta$ of this value from the (defaulting) payer.

We denote the value at time $t$ of this contract to the receiver by $CA^d(t, T, \kappa)$.

The cash flows to the receiver resulting from this agreement can be summarized as follows:

$$CA(T, T, \kappa) = (1 + \alpha \kappa)B(T, T + \alpha)\pi(T, T + \alpha) - 1 = B(T, T + \alpha)\{(1 + \alpha \kappa)\pi(T, T + \alpha) - (1 + \alpha L(T, T))\}. \quad \Box$$

If no default occurs until time $T$, receive $CA(T, T, \kappa)$ at time $T$.

If default occurs at a time $\tau \leq T$, receive $CA(\tau, T, \kappa) - (1 - \Delta)[CA(\tau, T, \kappa)]^+$ at the random time $\tau$.

We will now calculate $CA_d(t, T, \kappa)$. We need the following lemma, which tells us how to value payments made at the random time $s$. Its proof can be found in Appendix A.

**Lemma 10.** Let $Z$ be a predictable process on $(\Omega, \mathcal{F}, \mathbb{F}, P)$. Suppose that a payment of $Z_s$ will be made at time $s$ if $s \leq T$. Let $t < T$ and suppose that default has not occurred up to time $t$. Then, the value $V_t$ at time $t$ of this random payment is given by

$$V_t = B(t, T + x) \int_t^T \lambda(u) e^{-\int_u^T \lambda(v) dv} E^{T+u} \left[ \frac{Z_u}{B(u, T + x)} \right] \mathcal{F}_t \, du.$$

**Proposition 11.** The value at time $t$ of the defaultable forward agreement is

$$CA_d(t, T, \kappa) = CA(t, T, \kappa) - (1 - \Delta)B(t, T + x) \times \int_t^T \lambda(u) e^{-\int_u^T \lambda(v) dv} E^{T+u} \left[ \frac{CA(u, T, \kappa)}{B(u, T + x)} \right] \mathcal{F}_t \, du.$$

**Proof.** The formula can be proved by calculation while applying Lemma 10 to the process $\tilde{Z}$, where

$$\tilde{Z}_u = CA(u, T, \kappa) - (1 - \Delta)[CA(u, T, \kappa)]^+.$$

However, the following approach is more intuitive. As we have mentioned above, the value of $CA_d(t, T, \kappa)$ is generated by a payment of $CA(T, T, \kappa)$ at time $T$ if default does not occur before $T$, and a payment of $CA(\tau, T, \kappa) - (1 - \Delta)[CA(\tau, T, \kappa)]^+$ at time $\tau$ if $\tau \leq T$. We recall that $(CA(u, T, \kappa))_{u \in [0, T]}$ is the price process of a non-defaultable claim with maturity $T$ and terminal payoff $CA(T, T, \kappa)$. The receiver gets this claim in any case at the time $\min(\tau, T)$. Since this is a European claim without any interim payments, the time at which the claim is actually received is irrelevant. Therefore, this component of the price has the value $CA(t, T, \kappa)$. The remaining payment of $-(1 - \Delta)[CA(\tau, T, \kappa)]^+$ at time $\tau$ if $\tau \leq T$ is valued by applying Lemma 10 to $Z$, where

$$Z_u = -(1 - \Delta)[CA(u, T, \kappa)]^+.$$

We note that the expectation appearing inside the integral in Eq. (8) has the form of a put option on non-defaultable LIBOR, i.e., of a European floorlet. We define
and using Eq. (6), we see that

\[ C_A(t, T, \kappa) = zB(t, T + z)(K(\kappa) - L(t, T)). \]  

\[ \text{(10)} \]

This implies that, for any \( u \geq t \), we have

\[ E^{T+z}\left[ \frac{[C_A(u, T, \kappa)]}{B(u, T + z)} | \mathcal{F}_t \right] = zE^{T+z}\left[ [K(\kappa) - L(u, T)]^+ | \mathcal{F}_t \right]. \]  

\[ \text{(11)} \]

The advantage of using a market model for forward LIBOR is that we can price this floorlet by Black's formula (see, for example, Hull, 1997; Musiela and Rutkowski, 1997). Let us first define the defualtable simple forward rate in this setting. In analogy to the non-defualtable case, it is defined as the value of the fixed rate, which gives the corresponding forward agreement a value of zero.

**Definition 12.** The simple defualtable forward rate \( \kappa^d \) is defined as the value of the fixed rate \( \kappa \), which leads to a value of zero for the defualtable forward agreement, i.e., is defined by the relation

\[ C_A^d(t, T, \kappa^d) = 0. \]

**Proposition 13.** The value of \( \kappa^d \) is determined by the equation

\[ (1 + z\kappa^d)\pi(T, T + z) - (1 + zL(t, T)) \]

\[ = z(1 - \Delta) \int_t^T \lambda(u)e^{-\int_u^T \lambda(v)dv}E^{T+z}\left[ [K(\kappa^d) - L(u, T)]^+ | \mathcal{F}_t \right] du. \]  

\[ \text{(12)} \]

**Proof.** By using Eq. (8) in conjunction with (9)–(11), we see that

\[ C_A^d(t, T, \kappa) = zB(t, T + z)(K(\kappa) - L(t, T)) \]

\[ - zB(t, T + z)(1 - \Delta) \int_t^T \lambda(u)e^{-\int_u^T \lambda(v)dv}E^{T+z}\left[ [K(\kappa) - L(u, T)]^+ | \mathcal{F}_t \right] du. \]

Setting the value of \( C_A^d(t, T, \kappa) \) equal to zero gives the formula that we claimed. \( \Box \)

With this proposition and our assumption about forward LIBOR we can state the following result:

**Corollary 14.** Due to our assumption that non-defualtable forward LIBOR \( L(\cdot, T) \) is lognormally distributed under the \( T + z \)-forward measure, we can simplify the equation to
\[(1 + \pi^d)\pi(T, T + \pi) - (1 + L(t, T)) = \pi(1 - A) \int_t^T e^{-\int_t^u \frac{1}{2} \sigma^d(v) \, dv} (K(\kappa^d)) \mathcal{N}(d_2(u)) - L(t, T)\mathcal{N}(d_1(u)) \ \delta(t) \ du,\]

where

\[d_1,2(u) := \frac{\log(L(t, T)/K(\kappa^d)) + 1/2 \sigma^d(u, T)^2}{\tilde{\sigma}^d(u, T)}\]

and

\[\tilde{\sigma}^d(u, T)^2 = \int_u^T \sigma^d(s, T)^2 \, ds.\]

**Proof.** This is a simple application of Black’s formula to our context. \[\square\]

The numerical implementation of this result is straightforward. In order to find the correct interest rate for a forward credit agreement, which takes into account the differential credit standings of lender and borrower, an equation has to be solved, which involves only a one-dimensional integral over normal distributions.

### 3.2. The bilateral case

Until now we have only considered the default risk of one party, namely the payer party which takes out the loan implicit in the CA. We will now generalize the assumptions and add the possibility of default by the receiver side of the contract. This will not change anything during the second part of the CA between \(T\) and \(T + \pi\) (the loan period). During the first part of the CA, however, the payer is now at risk to a default of the receiver, should interest rates have moved in a way favourable to the payer side.

To allow for independent defaults of both counterparties, two point processes are needed. Therefore, in this section we will use Assumption 2 and the receiver will be referred to as counterparty 1.

**Definition 15.** Again, suppose that \(t < T\). A forward agreement credit agreement with bilateral default risk, fixed rate \(\kappa\), start date \(T\), and accrual period \(\pi\), is specified by the following cash flows:

- If no default occurs up to time \(T\), the contract then becomes an agreement to a defaultable credit as described in Definition 6.
- If default occurs at time \(\tau \leq T\), the position is closed at \(\tau\) using a mark-to-market procedure. If the value \(\text{CA}(\tau, T, \kappa)\) at time \(\tau\) of the equivalent non-
defaultable agreement is negative to the non-defaulting party, she must meet
her obligation in full. If it is positive, then she receives only a fraction \( \Delta_j \) of
this value from the defaulting party.

We denote the value at time \( t \) of this contract to the receiver by \( CA^{bd}(t, T, \kappa) \).

Once again, we can summarize the cash flows to the receiver as follows:

- If no default occurs until time \( T \), receive \( CA(T, T, \kappa) \) at time \( T \).
- If \( s \leq T \), then
  - receive \( \Delta_1 [CA(\tau, T, \kappa)]^+ - \Delta_1 [CA(\tau, T, \kappa)]^- \) at the random time \( \tau \) if \( z = z_1 \).
  - receive \( \Delta_2 [CA(\tau, T, \kappa)]^+ - [CA(\tau, T, \kappa)]^- \) at \( \tau \) if \( z = z_2 \).

From this we see that:

**Proposition 16.** The price of a CA with bilateral default risk is given by

\[
CA^{bd}(t, T, \kappa) = B(t, T + \alpha) \left\{ \mathbb{E}^{T+\alpha} \left[ \frac{CA(T, T, \kappa)}{B(T, T + \alpha)} 1_{\{\tau > T\}} \bigg| \mathcal{F}_t \right] + \mathbb{E}^{T+\alpha} \left[ \frac{1}{B(T, T + \alpha)} 1_{\{\tau = z = z_1\}} \left( [CA(\tau, T, \kappa)]^+ - \Delta_1 [CA(\tau, T, \kappa)]^- \right) \bigg| \mathcal{F}_t \right] + \mathbb{E}^{T+\alpha} \left[ \frac{1}{B(T, T + \alpha)} 1_{\{\tau = z = z_2\}} \left( \Delta_2 [CA(\tau, T, \kappa)]^+ - [CA(\tau, T, \kappa)]^- \right) \bigg| \mathcal{F}_t \right] \right\}.
\]  

\[ (13) \]

In the special case that both counterparties are of the same credit quality, the
optionalities involved in the contract cancel out and the formula simplifies in
the following way.

**Proposition 17.** Assume that both counterparties are of the same credit quality,
i.e., that \( \lambda_1 = \lambda_2 \) and \( \Delta_1 = \Delta_2 = \Delta \), then, given that there has been no default up
to time \( t \), the price of the CA with bilateral default risk is given by

\[
CA^{bd}(t, T, \kappa) = CA(t, T, \kappa) \left( 1 - \frac{1}{2} \left( 1 - \Delta \right) \left( 1 - e^{-\int_t^T \lambda(s) ds} \right) \right). \]

\[ (14) \]

**Proof.** We can give the following intuitive argument. Since \( \lambda_1 = \lambda_2 \), Eq. (2)
implies that

\[
P^{T+\alpha}[z = z_1|\tau = u] = P^{T+\alpha}[z = z_1|\tau = u] = \frac{1}{2}.
\]

\[ (15) \]

It follows from our summary of the payoffs that, if \( \tau \leq T \), the receiver gets a
payoff with the expected value \( \frac{1}{2} (1 + \Delta) CA(\tau, T, \kappa) \) at time \( \tau \). However, just as
in the proof of Proposition 11, the actual timing of the payoff is irrelevant. Therefore, we have
Algebraic manipulation of the expression in Eq. (16) gives the formula claimed. □

Formula (14) leads directly to the following result for the implied forward rate in the case of bilateral and equal default risk:

**Corollary 18.** The fixed rate $j^{bd}$ which makes the price of a CA with bilateral and equal default risk equal to zero is determined by

\[
CA^{bd}(t, T, \kappa) = CA(t, T, \kappa) \left( P^{T+\bar{\tau}}[\tau > T] + \frac{1}{2} (1 + \Delta)P^{T+\bar{\tau}}[\tau \leq T] \right)
= CA(t, T, \kappa) \left( e^{-\int_t^T \tilde{\lambda}(u) \, du} + \frac{1}{2} (1 + \Delta) \left( 1 - e^{-\int_t^T \tilde{\lambda}(u) \, du} \right) \right).
\]

(16)

This result can be used in two ways. First, in the way it is stated here, it allows the calculation of risky forward rates if non-defaultable forward rates are

\[
CA(t, T, \kappa^{bd}) = 0
\]

and is equal to

\[
k^{bd} = \frac{1}{\alpha} \left( \frac{1 + \alpha L(t, T)}{\pi(T, T + \alpha)} - 1 \right).
\]

(18)

**Proof.** The fixed rate $k^{bd}$ is defined by the relation $CA^{bd}(t, T, k^{bd}) = 0$. By Eq. (14), this is equivalent to $CA(t, T, k^{bd}) = 0$, and by Eq. (6), this in turn is equivalent to

\[
1 + \alpha k^{bd} = \frac{1 + \alpha L(t, T)}{\pi(T, T + \alpha)}.
\]

The previous results have shown that $k^{bd}$ is the “fair” simple forward rate underlying a credit agreement involving two counterparties which are subject to default and whose risk characteristics are equal. In first approximation, a market participant’s risk characteristics are described by her credit rating. In this sense, bilateral and equal default risk is the basic assumption which underlies all the usually $AA$-adjusted LIBOR quotes. This justifies the following definition.

**Definition 19.** The forward defaultable LIBOR rate $L^d(t, T)$ is defined by the relation

\[
1 + \alpha L^d(t, T) = \frac{1 + \alpha L(t, T)}{\pi(T, T + \alpha)}.
\]

(19)

This result can be used in two ways. First, in the way it is stated here, it allows the calculation of risky forward rates if non-defaultable forward rates are
known, for example from the government bond curve. The other possibility is that risky forward rates are known, for example in the form of forward LIBOR. Then, by rewriting the above result, riskless forward rates can be computed. In the case of LIBOR, they would have very short term to maturities and could be used to stabilize the short end of the riskless government zero-coupon bond curve.

In instantaneous short- or forward-rate models, the defaultable interest rate is obtained by adding a default premium to a risk-free interest rate (see, for example, Jarrow and Turnbull, 1995; Duffie and Singleton, 1997). In both cases, nice representations of the defaultable rates are available in terms of non-defaultable rates. For Duffie and Singleton (1997), the instantaneous short rate for defaultable assets is given by the interest rate for non-defaultable assets plus the product of loss rate and hazard rate:

\[ r^d(t) = r^s(t) + (1 - \Phi(t))h(t). \]

Here \( \Phi(t) \) is the recovery rate, and \( h(t) \) the hazard rate (equivalent to our jump intensity). In the case of Jarrow and Turnbull (1995), the defaultable instantaneous forward rate is given by

\[ f^d(t, T) = f^s(t, T) + A(t, T) \lambda, \]

where \( A \) is a function depending on loss rate and jump intensity. The previous results suggest that in models of finite-period interest rates, it is more appropriate to define the risky interest rate as the product of a “risk-factor” times the non-defaultable rate.

Note that simple discounting with the risky interest rate does not yield the prices of risky zero-coupon bonds. Each defaultable interest rate is adjusted for default risk in the associated interval. Discounting with the product of these risky interest rates adjusts for the possibility of several defaults (one in each interval), whereas a defaultable bond can only default once. Consequently, a yield constructed from defaultable LIBOR will be higher than the yield of a traded bond belonging to the same risk class.

The result can also be applied to the valuation of derivatives dependent on forward defaultable LIBOR. Below, to show the simplicity of the procedure involved, we present two examples: one is a cap on defaultable LIBOR, the other is a Credit Spread Option (CSO) on the difference between defaultable and non-defaultable LIBOR.

**Proposition 20.** (1) Consider a caplet on defaultable LIBOR with strike \( K \), reset date \( T \) and settlement date \( T + \alpha \). Its value \( C_{pl}^d(t, K) \) at time \( t \) is given by

\[ C_{pl}^d(t, K) = \frac{1}{\pi(T, T + \alpha)} C_{pl}(t, K'), \]

where \( C_{pl}(t, K') \) denotes the price of a caplet on non-defaultable LIBOR corresponding to the same dates and with a modified strike of
Consequently, the price of a cap on defaultable LIBOR with strike \( K \) and reset dates \( T_0, \ldots, T_{n-1} \) is given by

\[
\text{Cap}^d(t, K) = \sum_{j=1}^{n} \frac{1}{\pi(T_{j-1}, T_j)} \text{Cpl}(t, K'_{j-1}),
\]

where

\[
K'_{j-1} = \pi(T_{j-1}, T_j)K - \frac{1}{\alpha} (1 - \pi(T_{j-1}, T_j)).
\]

(2) The value of a CSO with reset date \( T \), settlement date \( T + \alpha \), and strike \( K \) is given by

\[
\text{CSO}(t, T) = \frac{1 - \pi(T, T + \alpha)}{\pi(T, T + \alpha)} \text{Cpl}(t, K'),
\]

where the modified strike is now

\[
K' = \frac{\pi(T, T + \alpha)}{1 - \pi(T, T + \alpha)} K - \frac{1}{\alpha}.
\]

Note that in the cap valuation, the modified strike depends on the reset date of each caplet, and is not fixed as in a usual cap without default risk. Furthermore, the credit spread in our setup is not deterministic (although we have a deterministic hazard rate), it is an increasing function of the non-defaultable forward LIBOR \( L(t, T) \).

**Proof.** (1) The payoff \( \Pi \) at time \( T + \alpha \) of the caplet on defaultable LIBOR is \( \alpha[L^d(T, T) - K]^+ \). Using Eq. (19), and by some algebra, we get

\[
\Pi = \alpha \left[ \frac{1}{\alpha} \left( \frac{1 + \alpha L(T, T)}{\pi(T, T + \alpha)} - 1 \right) - K \right]^+
\]

\[
= \frac{\alpha}{\pi(T, T + \alpha)} [L(T, T) - K']^+.
\]

The caplet formula follows, since \( \alpha[L(T, T) - K']^+ \) is the payoff of a caplet on non-defaultable LIBOR. The formula for the cap is obtained by summing over all the caplets.

(2) The formula for the CSO is proved in the same way, its payoff \( \Pi_{\text{CSO}} \) at \( T + \alpha \) is

\[
\Pi_{\text{CSO}} = [L^d(T, T) - L(T, T) - K]^+.
\]
4. Defaultable swaps

In this section, we will concentrate on the valuation of defaultable interest rate swaps. The default risk of swaps is very different from that in bonds. In a swap, the participants periodically exchange notional amounts. There is, however, no lump-sum exchange of principal at the beginning or at the end of the term of a swap, so that traditional default risk models for bonds cannot be used for the valuation of defaultable swaps. For example, a default-free swap can be written as a portfolio of zero-coupon bonds:

\[ FS(0, \kappa) = B(0, T_0) - \kappa \sum_{j=1}^{N} B(0, T_j) - B(0, T_N). \]

This formula cannot generally be applied to defaultable swaps by just exchanging the non-defaultable bonds \( B(0, T) \) for defaultable bonds. Similarly, a default-free swap can be written as a portfolio of FRAs:

\[ FS(0, \kappa) = - \sum_{j=1}^{N} \text{FRA}(0, T_j, \kappa). \]

Again, exchanging the default-free FRAs for defaultable FRAs does not in general give an expression for a defaultable swap.

Swaps with default risk have been considered before in the literature. Cooper and Mello (1991) used a structural firm-value based model to value interest-rate and currency swaps with unilateral default risk. Duffee and Huang (1996) analyse swaps with bilateral default risk. They derive a backward stochastic differential equation for the value of a defaultable swap. Via the Feynman–Kac theorem, the associated non-linear partial differential equation is calculated, which is then solved numerically. We use a different methodology to calculate the price of a defaultable swap, and we do not need to solve a partial differential equation. The idea was already outlined by Sorensen and Bollier (1994) in a setting where default could happen at only one point in time. They proposed to model the danger of default of one counterparty as an option. Because both counterparties are subject to default risk, each counterparty holds two options. Note that the exercise time of these options cannot be determined by the parties, but is given exogenously by the time of default. Sorensen and Bollier (1994) also argued that the value of a default option is given by the product of the value of a European swaption to replace the swap and the probability of default. To value the swaption, they proposed to use a market model, which was not yet formally developed at that time. In this section, we will formalize these ideas and use a similar setup as before to value a defaultable swap and determine the fair swap rate. A very good introduction to swaps, swap derivatives and the use of different models for valuation purposes is given in Musiela and Rutkowski (1997b), from which we will also use some results.
4.1. Unilateral default risk

We start off by limiting the possibility to default to only one counterparty, in this case the counterparty which receives the fixed rate. However, the method shown is very general and can easily be extended to the case where both counterparties are in danger to default.

**Definition 21.** The swap with receiver default risk is an agreement to the following cash flows:
- Before default occurs, on the dates $T_j$ the payer side of the contract pays the predetermined fixed interest $\kappa_j$, whereas the receiver side pays the floating rate $L(T_{j-1}, T_{j-1})$.
- At the time when default of the receiver party occurs, the value of a non-defaultable swap contract with the same fixed interest rate $\kappa$ to the payer side is determined. If this value is positive, the receiver side pays a certain fraction of it to the payer side. However, if the value is negative, it is paid in full by the payer side.

In order to obtain a nice representation of the value of a defaultable swap, we introduce some notation.

**Definition 22.** We denote the index of the last time of exchange of interest payments before default with

$$[\tau] := \max \{ j \geq 0 : T_j < \tau \}.$$  

In other words, the time of default is in the interval $\tau \in ]T_{[\tau]}, \tau_{[\tau]+1}$. $T_{[\tau]}$ is the last time of an exchange of interest payments before default occurs.

We define the price of a swap contract as its value to the fixed-rate paying party, and as a direct result of Definition 21 we have the following proposition, the proof of which is contained in Appendix A.

**Proposition 23.** The price of a defaultable swap contract is given by the following expression:

$$\begin{align*}
FS_d^{d}(0, \kappa) &= AE \left[ \sum_{j=1}^{[\tau]} E \left[ \frac{1}{B(T_j)} (L(T_{j-1}, T_{j-1}) - \kappa) \right] 
+ A \left( \sum_{j=[\tau]+1}^{N} E \left[ \frac{1}{B(T_j)} (L(T_{j-1}, T_{j-1}) - \kappa) | \mathcal{F}_\tau \right] \right)^+ 
- \left( \sum_{j=[\tau]+1}^{N} E \left[ \frac{1}{B(T_j)} (L(T_{j-1}, T_{j-1}) - \kappa) | \mathcal{F}_\tau \right] \right)^- \right].
\end{align*}\tag{21}$$
For ease of notation, here and in the following propositions expectations are taken under the spot measure.

Observe that in our setup, the fixed rate \( \kappa \) will be lower than in a swap with no default risk, because the payments of the default-risky floating-rate counterparty are insecure, so that it can obtain only a smaller fixed interest rate in return than a floating-rate paying counterparty without default risk.

The value of a defaultable swap consists of two parts: The first is a default-free swap with the same rate as the defaultable swap, the second is a forward payer swaption, but with a random exercise time. This can be interpreted as follows: The writer of the swap earns the amount given by the price of a default-free swap, if no default occurs. However, if default occurs, he loses a fraction \( \Lambda - 1 \) of this amount, which is represented in the payer swaption part of the formula.

To find the fixed swap rate for the defaultable swap, its price at time 0 must be set to zero, \( \text{FS}^d(0, \kappa) = 0 \), and this equation has to be solved for the fixed swap rate. Unfortunately, under the lognormal model of forward LIBOR, no closed form solution is available for the swaption price which occurs in this formula, although, as Brace et al. (1997) show, a closed form approximation still exists.

However, if forward swap rates instead of forward LIBOR are assumed to be lognormally distributed, as is common when pricing swaptions, the problem can again be reduced to the one encountered in the case of defaultable LIBOR. We obtain the following result, also proven in Appendix A.

**Proposition 24.** The fair fixed interest rate \( \kappa \) for the defaultable swap contract can be approximated by solving the following equation:

\[
1 - \varsigma \kappa \sum_{j=1}^{N} B(0, T_j) - B(0, T_N) \\
= \varsigma \int_{0}^{T_N} \left( \sum_{j=|u|+1}^{N} B(0, T_j)(\kappa(0, u, [u] + 1, N))N(h_{1}) \\ - \kappa N(h_{2})\right) \lambda(u) \, du,
\]

\[
\sum_{j=|u|+1}^{N} B(0, T_j)(\kappa(0, u, [u] + 1, N))N(h_{1}) \\
- \kappa N(h_{2})\right) \lambda(u) \, du,
\]
As in the case of credit agreements with unilateral default risk, this formula contains only an integral over normal distributions and is therefore easy to implement numerically.

4.2. Bilateral default risk

We generalize the valuation formulas from the previous subsection to a setting where both parties are subject to default risk. We assume that both default premia are incorporated in the fixed rate, and that the floating rate paid is the riskless LIBOR $L(T_j, T_j)$. Similar to the part on CAs with bilateral default risk, we make the assumption 2. With the aid of Proposition 1, we can state the price of a swap contract with two-sided default risk.

Proposition 25. The price of a swap contract with two-sided default risk is given by

$$FS^{bd}(0, \kappa) = xE \left[ \sum_{j=1}^{[\tau]} E \left[ \frac{1}{B(T_j)} (L(T_{j-1}, T_{j-1}) - \kappa) \right] + \left( \sum_{j=[\tau]+1}^{N} E \left[ \frac{1}{B(T_j)} (L(T_{j-1}, T_{j-1}) - \kappa) \right] \right)^+ \right]$$

where $\{z = z_1\}$ indicates default of the floating-rate party.

Again, this swap contract can be rewritten in the following form:
FS^{bd}(0, \kappa)
\begin{align*}
&= \alpha \sum_{j=1}^{N} E \left[ \frac{1}{B(T_j)} (L(T_{j-1}, T_{j-1}) - \kappa) \right] \\
&+ \alpha (d_1 - 1) E \left[ 1_{\{z=z_1\}} \left( \sum_{|i|+1}^{N} E \left[ \frac{1}{B(T_i)} (L(T_{j-1}, T_{j-1}) - \kappa)|\mathcal{F}_r \right] \right)^+ \right] \\
&- \alpha (d_2 - 1) E \left[ 1_{\{z=z_2\}} \left( \sum_{|i|+1}^{N} E \left[ \frac{1}{B(T_i)} (L(T_{j-1}, T_{j-1}) - \kappa)|\mathcal{F}_r \right] \right)^- \right].
\end{align*}

(23)

Intuitively, this formula is easy to explain: The first line represents the value of the swap without default risk. The second line represents the loss to the fixed rate payer side due to default of the floating rate party, if the value of the swap to the payer was positive. The third line represents the gain of the fixed rate payer side if she defaults and does not have to pay the full value of the swap which was negative to her.

The problems in finding the fixed swap rate in this case are the same as in the case where only one party is subject to default risk, and they can be treated similarly.

Let us assume now that both counterparties are of the same risk class, i.e., $A_1 = A_2 = \Delta$ and $\lambda_1 = \lambda_2$. In this case, we have the following result (for the proof see Appendix A):

**Proposition 26.** The price of a swap contract with bilateral and equal default risk is given by

$$FS^{bd}(0, \kappa) = \alpha \sum_{j=1}^{N} B(0, T_j) (L(0, T_{j-1}) - \kappa) \left( 1 - \frac{1}{2} (1 - \Delta) \left( 1 - e^{-\int_0^{T_j} \lambda(s) \, ds} \right) \right).$$

There is an intuitive reason for this result: because both counterparties are of the same risk class, the swaption parts, which represent the option to default, cancel out. Therefore, this result is independent of our assumptions of log-normally distributed forward LIBOR or swap rates.

From the formula it can be seen that future interest payments $(L(T_{j-1}, T_{j-1}) - \kappa)$ are discounted with the expression

$$B^{d}(0, T_j) := B(0, T_j) \left( 1 - \frac{1}{2} (1 - \Delta) \left( 1 - e^{-\int_0^{T_j} \lambda(s) \, ds} \right) \right).$$

This is the value of a default-risky zero-coupon bond with maturity $T_j$, default intensity $\lambda$ and loss ratio $\frac{1}{2} (1 - \Delta)$. This result can be explained intuitively: Because now two counterparties instead of one are subject to default risk, the default intensity $\lambda = \lambda_1 + \lambda_2$ is double that of the unilateral default risk case.
However, the expected loss due to a default is halved because the probability that it was a default of each counterparty is exactly 50%.

We are now interested in the relationship between the swap rate of a swap with bilateral and equal default risk, $\kappa^{bd}$, and the swap rate of a swap without default risk, $\kappa$.

We have the following result:

**Corollary 27.** Assume that the forward LIBOR curve is monotone. Then

- for a monotonously increasing forward LIBOR curve, the defaultable swap rate is smaller than the non-defaultable swap rate.
- for a monotonously decreasing forward LIBOR curve, the defaultable swap rate is greater than the non-defaultable swap rate.

Minton (1997) and Sun et al. (1994) compare swap rates and par bond yields based on LIBOR with OTC swap rates. In particular, Monton finds that swap rates derived form Eurodollar futures are higher than those observed in the OTC market. Two possible explanations for this result are given: firstly, the practice of daily resettlement in the futures market leads to credit enhancements absent in plain vanilla swaps. Secondly, Burghardt and Hoskins (1994) have shown numerically that, even without default risk, Eurodollar futures prices introduce a bias towards higher swap rates when the yield curve is upward-sloping. In our setting, default risk has the same effects as was shown in the preceding corollary.

Similar to Minton (1997), Sun et al. (1994) find that par bond yields based on LIBOR are higher than OTC swap rates. As was noticed before (cf. the discussion after Definition 19), constructing the yield of a defaultable par bond from LIBOR introduces an overcompensation for default risk, leading to a higher yield. Therefore, our model is able to explain these empirical phenomena through no-arbitrage arguments.

5. Conclusion

By analysing the impact of default risk on credit agreements and swaps in a market model, we have derived equations determining the arbitrage-free values of simple defaultable forward rates and swap rates. The time of default is given by the first jump of a time-inhomogenous Poisson process with deterministic intensity, and the payoff after default is a constant percentage of the value of a non-defaultable, but otherwise equivalent asset. Under the assumption of log-normality of non-defaultable forward LIBOR, we derive equations for the fixed rate in forward credit agreements which compensates for (unilateral or bilateral) default risk. These equations involve nothing more complicated than a one-dimensional integral over normal distributions and therefore numerical
implementation is straightforward. In addition, we are able to obtain a rigorous definition of LIBOR which incorporates default risk in analogy to the non-defaultable case. Furthermore, we value swaps with unilateral and bilateral default risk. We show that a defaultable swap can be written as the sum of a non-defaultable swap and a swaption, and we present an approximation for the fair swap rate in the presence of unilateral and bilateral default risk. Finally, we analyse the impact of the term structure of forward LIBOR on the defaultable swap rate. The results allow lenders and money market dealers to find the correct interest rate for forward credit agreements and swaps which take into account the differential credit standings of lender and borrower. This allows them to incorporate the credit standing into the price of the contract. A further application of the results presented here would be the valuation of credit facilities or credit lines, which correspond to the option of entering a credit agreement and, therefore, can be valued in a similar way.

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Appendix A

Proof of Lemma 10. We denote by $H$ the process given by $H_u := 1_{\{\tau \leq u\}}$. By stopping, we see that the process $(1_{\{\tau \leq t\}} \lambda(u))_{u \geq 0}$ is an intensity for $H$. By definition of the $(T + \alpha)$-forward measure we have

$$V_t = B(t, T + \alpha) E^{T+\alpha} \left[ \frac{Z_t}{B(\tau, T + \alpha)} 1_{\{\tau \leq T\}} \left| \mathcal{F}_t \right. \right]$$

$$= B(t, T + \alpha) E^{T+\alpha} \left[ \int_t^T \frac{Z_u}{B(u, T + \alpha)} \lambda(u) 1_{\{\tau \leq u\}} du \left| \mathcal{F}_t \right. \right].$$

Since $Z$ and the bond price process are predictable, we can rewrite this conditional expectation using the intensity of $H$:

$$V_t = B(t, T + \alpha) E^{T+\alpha} \left[ \int_t^T \frac{Z_u}{B(u, T + \alpha)} \lambda(u) 1_{\{\tau \leq u\}} du \left| \mathcal{F}_t \right. \right]$$

$$= B(t, T + \alpha) \int_t^T \lambda(u) E^{T+\alpha} \left[ \frac{Z_u}{B(u, T + \alpha)} 1_{\{\tau \leq u\}} \left| \mathcal{F}_t \right. \right] du.$$  \hspace{1cm} (24)
Since $N$ is a Cox process (cf. Lando, 1998), we can use iterated conditional expectations to obtain

$$E^{T+\tau}
\left[
\frac{Z_u}{B(u, T + \alpha)} I_{u \leq \tau}
\right]_{\mathcal{F}_t}
= E^{T+\tau}
\left[
\frac{Z_u}{B(u, T + \alpha)} e^{-\int_t^\tau \lambda(v) dv}
\right]_{\mathcal{F}_t}
= e^{-\int_t^\tau \lambda(v) dv} E^{T+\tau}
\left[
\frac{Z_u}{B(u, T + \alpha)}
\right]_{\mathcal{F}_t}.$$  

By inserting this relation into Eq. (24), we obtain the formula that was claimed. \hfill \Box

**Proof of Proposition 23.** To prove the first equation, we begin with a fixed default time $\tau$. For all exchange dates $T_j$ up to and including $T_{\tau}$, interest rate payments are exchanged normally. For all payments occurring after $\tau$, it has to be decided if their time $\tau$ discounted expected value is positive or negative to the payer. If it is positive, then a fraction $\Delta$ is transferred at time $\tau$ to the payer. This settlement value has to be discounted to time zero, so in effect the discounting to date $\tau$ cancels out. A similar argument holds if the time $\tau$ discounted expected value is negative to the payer. Finally, the expectation over all possible default times $\tau$ has to be taken. The second equality follows directly from the fact that $X^- = X^+ - X$ for some variable $X$, and by conditioning on $\mathcal{F}_t$. \hfill \Box

**Proof of Proposition 24.** For this result, we have rewritten the swap part of Eq. (22) as a weighted sum of zero-coupon bond prices:

$$\sum_{j=1}^N E \left[ \frac{1}{B(T_j)} (L(T_{j-1}, T_{j-1}) - \kappa) \right]$$

$$= \sum_{j=1}^N B(0, T_j) E^{T_j} [L(T_{j-1}, T_{j-1}) - \kappa]$$

$$= \sum_{j=1}^N \frac{1}{B(T_{j-1}, T_j)} - 1 - \alpha \kappa$$

$$= \sum_{j=1}^N \left( B(0, T_{j-1}) - (1 + \alpha \kappa) B(0, T_j) \right)$$

$$= 1 - \alpha \kappa \sum_{j=1}^N B(0, T_j) - B(0, T_N).$$
For the swaption part of Eq. (22), we use first the default intensity as the distribution of default times

$$\alpha E\left[\left(\sum_{|j|=1}^{N} E\left[\frac{1}{B(T_j)} (L(T_{j-1}, T_{j-1}) - \kappa) |\mathcal{F}_t]\right]\right)^+\right]$$

$$= \alpha \int_0^{T_N} e^{-\int_0^u \lambda(v) dv} E\left[\left(\sum_{|j|=1}^{N} E\left[\frac{1}{B(T_j)} (L(T_{j-1}, T_{j-1}) - \kappa) |\mathcal{F}_u]\right]\right)^+ \lambda(u) du,$$

and now the expectation inside the integral is similar to the value at time 0 of a standard European swaption with exercise time $\tau$. In terms of the forward swap rate, this can be written as (see Musiela and Rutkowski, 1997b)

$$= \alpha \int_0^{T_N} e^{-\int_0^u \lambda(v) dv} \sum_{j=|u|+1}^{N} (B(0, T_j) E T_j \left[(\kappa(u, u, [u] + 1, N) - \kappa)^+\right]) \lambda(u) du,$$

where $\kappa(t, T, m, N)$ is the forward swap rate for at time $t$ for a swap contract which is signed at $T$, with payment dates $T_m$ to $T_N$. Under the assumption that these forward swap rates are lognormally distributed, we can finally write

$$= \alpha \int_0^{T_N} e^{-\int_0^u \lambda(v) dv} \sum_{j=|u|+1}^{N} B(0, T_j) \left(\kappa(0, u, [u] + 1, N) \mathcal{N}(h_1) \right.$$  

$$- \kappa \mathcal{N}(h_2)) \lambda(u) du,$$

where

$$h_{1,2} = \frac{\log(\kappa(0, u, [u] + 1, N)/\kappa) \pm \frac{1}{2} \sigma^2(u, [u] + 1, N)^2 u}{\sigma^2(u, [u] + 1, N) u}$$

for an appropriate choice of $\sigma^2$. To conclude, we have finally transformed Eq. (22) into an equation very similar to that for the defaultable LIBOR, which can easily be solved numerically for the defaultable swap rate $\kappa$. However, this is not rigorous because of the following: the first payment date of the swaption in these expressions is the date $T_{|u|+1}$. The payment due at this date is $L(T_{|u|}, T_{|u|}) - \kappa$, and the value of this payment is known already at the time the swaption is “exercised” due to default, because we have $T_{|u|} < \tau$. \[\square\]

**Proof of Proposition 26.** The last two lines of Eq. (23) can be rewritten in the following way:
This gives the desired result. □

Proof of Corollary 27. The two swap rates are determined by the expressions

\[
\kappa = \frac{\sum_{j=1}^{N} B(0, T_j) L(0, T_{j-1})}{\sum_{j=1}^{N} B(0, T_j)},
\]

\[
\kappa^{bd} = \frac{\sum_{j=1}^{N} B^{d}(0, T_j) L(0, T_{j-1})}{\sum_{j=1}^{N} B^{d}(0, T_j)}.
\]

It can be seen that both swap rates are weighted averages of the forward LIBORs \(L(0, T_j)\), where the weights are given by the non-defaultable zero coupon bond prices \(B(0, T_j)\) and the default-risky zero-coupon bond prices

\[
B^{d}(0, T_j) = B(0, T_j) \left(1 - \left(1 - e^{-\int_0^{T_j} \lambda(s) ds}\right) \frac{1}{2} (1 - A)\right),
\]

respectively. Due to the default risk, the weights in the defaultable case are shifted towards the short end of the curve. Therefore, if forward LIBORs are increasing, the defaultable swap rate will lie below the non-defaultable swap rate, and vice versa. This confirms the corollary. □

References


