After-tax term structures of real interest rates: Inferences from the UK linked and non-linked gilt markets

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Abstract

This study estimates the after-tax term structure of real interest rates using the prices of UK linked and non-linked gilts over the period from 25 January 1986 until 25 October 1993. The impact of differential taxation and the existence of “noise” in observed market prices is found to produce a significant impact on the parameter estimation of term structure models when compared to methods, such as Brown and Schaefer (Brown, R., Schaefer, S., 1994. Journal of Financial Economics 35, 1–42), that did not. Two major observations can be made regarding the estimates for spot real interest rates. Firstly, the volatility of the short-term rate is much lower than that found by Brown and Schaefer (1994) which provides a better fit with the predictions of the CIR single factor model for interest rates. Secondly, consistent with Rumsey (Rumsey, J., 1993. An impact of the assumptions about taxes on the estimation of the properties of interest rates. Working Paper), there appears to be some evidence to suggest that single factor interest rate models produce a better fit to interest rates on an after-tax basis than on a before tax basis. © 2000 Published by Elsevier Science B.V. All rights reserved.

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1. Introduction

The behaviour of real interest rates is a core issue for financial economics in terms of both the dynamics of the short-term rate and the characteristics of the overall term structure. Empirical investigations of the relationship between nominal interest rates, real interest rates and expected inflation have traditionally been hindered by the fact that, of the three variables, only nominal interest rates are able to be directly extracted from market prices. Without an observable measure of real interest rates, it has often been convenient to characterize this fundamental variable as both constant and exhibiting a flat term structure. This assumption, however, has been disputed in several studies including Nelson and Schwert (1977), Walsh (1987) and Rose (1988) who conclude, by various indirect techniques, that real rates do indeed vary over time. With the introduction of “real return” or “index linked” gilts (ILGs) in the UK during the early 1980s, however, there now exists a means of directly extracting real interest rates from prevailing market prices. Furthermore, the existence of relatively liquid markets in both linked and non-linked gilts provides a means of examining the relationship between nominal interest rates, real interest rates and inflation.

Although there have been a number of studies which have investigated the properties of real interest rates by indirect methods, 2 none have attempted to characterize a zero coupon term structure of after-tax real interest rates in a manner such as Litzenberger and Rolfo (1984) describe for nominal interest rates. Under this approach, the complication of differential taxation across investors is addressed through the identification of the marginal tax rate associated with a representative investor. The marginal tax rate of this proxy investor is determined on the basis of the specific after-tax cashflows which price the universe of bonds with minimum “noise”.

Brown and Schaefer (1994) were the first to estimate the before-tax real-term structure of interest rates from a cross section of ILGs. Using the approach of Schaefer (1981), they cast real-term structure estimation as an optimization problem under the condition that the present value of each ILG cannot exceed its observed price. Later in the same paper, Brown and Schaefer (1994) use these observed ILG prices to also estimate the parameters of the CIR model for the interest rate process.

The approach of Schaefer (1981), however, does not allow for observation error as it treats observed market prices as the intrinsic or “true” prices of the securities under consideration. Treating observed prices as “true” prices in a

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2 Studies of this nature include Walsh (1987), Rose (1988) and Evans et al. (1993). Studies which have directly incorporated the yields of ILGs include Pitman (1991) and Woodward (1990, 1993).
market that is actually segmented, subjects the study of Brown and Schaefer
(1994) to systematic bias.

Observed prices are typically used in studies which estimate a linear valu-
ation operator consisting of a set of stochastic discount factors (SDFs) for
equity markets. In one of a series of papers, Hansen and Jagannathan (1994)
apply the diagnostic of asset model pricing errors and recognize that there is a
set of true valuation operators which contain more than one element. Their
paper seeks to obtain some measure of the mispricing produced when a proxy
valuation operator (one not belonging to the set of true operators) is used.
They measure the mispricing by minimizing the difference between the proxy
and the set of true valuation operators. However, as the true set is not directly
observable, duality theory is utilized to measure the mispricing in terms of the
security prices and the portfolios (shadow prices) which are directly observ-
able.

A related issue covered in a series of papers by Stutzer (1993, 1994) con-
cerns the multiplicity of stochastic discount factors. In incomplete markets,
or in cases of incomplete information, many sets of SDFs will be possible. In
such cases, the question of how to choose amongst the many SDFs is of sig-
nificant interest, with Stutzer (1994) advocating minimization of the entropy
function.

The recent literature which has sprung from the papers of Hansen and
Jagannathan (1994, 1993), hinges on the premise that observed prices do not
allow for arbitrage opportunities. As stated by Hansen and Jagannathan
(1994), “Except possibly when there are arbitrage opportunities present in the
data set used in the empirical investigation, the set of correctly specified dis-
count factors is non-empty and typically large”.

The question of how SDFs can be estimated in incomplete markets where
observed prices do allow for arbitrage has been largely overlooked. This is
undoubtedly due to the fact that this line of research has, thus far, primarily
focused on equity markets. For equity markets it is a very difficult task to prove
that observed prices do, indeed, allow for arbitrage. In order to do so, it would
be necessary to demonstrate that no risk neutral probability (a set of SDFs)
exists that could equate present values to observed prices.

In contrast, for fixed income markets it is generally quite easy to
demonstrate that observed prices allow for arbitrage opportunities. For
example, consider a market for government bonds. Let $A$ be a matrix of
payoffs if the securities are held to maturity such that $a_{ij}$ is the payoff
from bond $i$ in state $j$. In a complete market, the number of independent
securities must be equal to the number of payment periods provided the
system is consistent. For a typical bond market, however, the number of
payment periods is at least twice that of the number of bonds. Thus, a
typical bond market is incomplete and, assuming the system is consistent,
consists of many valuation operators. Every valuation operator with
d > 0, such that \( d_1 > d_2 > \cdots > d_n \) and \( Ad = P \), is theoretically a possible estimator of the term structure.

Empirically, however, it is typically impossible to find even one viable valuation operator as the system is generally inconsistent due to the fact that observed prices contain noise. This is true even if the spread between the bid price, \( P_b \), and the ask price, \( P_a \), is considered. Ronn (1987) concludes that it is still generally impossible to find a valuation operator, \( d \), even when the constraint is broadened to \( P_b \leq Ad \leq P_a \).

These findings call into question the practice of utilizing observed prices in order to estimate the parameters of interest rate process models whose very essence is that of no arbitrage. Brown and Schaefer (1994) estimate the parameters of the no arbitrage CIR model by using observed ILG prices which have been shown to contain buy and hold arbitrage opportunities. This paper suggests an improvement over this method of parameter estimation by utilizing a set of “true” prices that do not allow for arbitrage opportunities. It is shown that the difference between the parameters estimated in this paper compared with that of Brown and Schaefer (1994) is significant for the estimation of interest rate derivative securities. Interest rate derivatives are of interest because the price sensitivity of these securities to errors in parameter estimation is significant.

Moreover, as another issue, Brown and Schaefer (1994) ignore tax implications even though the method of Schaefer (1981) implicitly assumes a segmented market. They assume that all ILGs are priced correctly for all tax brackets which may not be a reasonable assumption for the gilt market. As pointed out by Woodward (1990), the absence of capital gains taxes on gilts in the UK will very likely lead to segmented market equilibria with distinct tax clientele effects.

The purpose of this paper may be summarized as follows:

- To generate after-tax real-term structures from a cross section of ILGs using an alternative estimation procedure to Brown and Schaefer (1994) which allows for observation error and assumes that a marginal tax bracket exists for which all securities are priced correctly.
- To estimate the parameters of the CIR one factor model for the after-tax interest rate process with an approach similar to Brown and Schaefer (1994) but by using the “true” prices of ILGs which are implied by the above term structure estimation. These “true” prices do not allow for buy and hold arbitrage opportunities.

Despite the existence of both linked and non-linked gilts, the relationship between nominal interest rates and real interest rates cannot be determined in a straightforward manner. This is due to the non-contemporaneous indexation of inflation which, in the case of the ILG, involves a lag of eight months. The existence of this lag implies that an ILG is effectively non-linked over the eight-month period just prior to its maturity. Suitable real
interest rates may still be inferred, however, by adjusting the empirically estimated real-term structure by a proxy for the inflation expected over the eight-month lag period just prior to ILG maturity. The approach used by Brown and Schaefer (1994) to correct for the lag is to estimate expected inflation with an ARIMA model. In contrast, this study uses a procedure whereby inflation expectations are extracted directly from the relationship between the nominal and real after-tax term structures. This method provides greater flexibility as it is able to incorporate non-seasonal price shocks into the estimate of the relationship between nominal and real-term structures.

The outline of the paper is as follows. Section 2 describes the market and the tax treatment for UK index linked gilts. Section 3 illustrates the construction of after-tax real and nominal term structures while Section 4 provides estimates for the parameters of the CIR one factor model. Section 5 provides conclusions.

2. The UK index linked gilt market

Index linked gilts (ILGs) were first issued in the UK in 1981 as part of the governments anti-inflation program. Both the coupon and the principal of these securities are linked to the UK retail price index (RPI). The basis for indexation is the RPI for the period eight months prior to gilt issue; two months lag to enable measurement of the RPI and six months lag to enable exact calculation of accrued interest based on a known forthcoming coupon payment. The existence of the lag results in imperfect inflation protection except in the case where the inflation experienced eight months prior to the ILG issue is identical to the inflation experienced during the last eight months of the ILGs life. The magnitude of this imperfection diminishes as the holding period for the security increases.

As of 1993, there had been a total of seventeen unrestricted ILG issues in the UK (excluding an initial 1981 issue limited to pension funds), beginning in 1982 with maturity dates ranging from 1988 to 2030. ILGs were issued on a consistent basis throughout the 1980s with coupon rates varying from 2% to 4.625%. Volumes have ranged in issue size from 300 million pounds to 1 billion pounds (see Table 1).

The total volume of ILGs has risen from 8.4% (9.5 billion pounds) of the overall volume of outstanding government issued gilts at 31 March 1985 to 16.2% (17.5 billion pounds) at 31 March 1990. The liquidity of the market is relatively thin, however, with a turnover of only 2.9% of the total gilt market turnover in 1990 (de Kock, 1991). Holdings of ILGs are concentrated within a small number of investors including, predominantly, pensions funds and
insurance companies. Individual investors are largely confined to the short end of the market.

Since 1986, both linked and unlinked gilts have been exempt from capital gains taxes with the effect that lower coupon securities sold at a discount are taxed preferentially in comparison to those with higher coupons. In general, ILGs, whose inflation compensation accrues to both coupon and principal, possess a tax advantage over non-linked gilts, whose inflation compensation accrues only to the coupon which is taxed as ordinary income. Therefore, the effects of differential taxation must be considered in any comparison of respective returns and subsequent measurement of expected inflation (Woodward, 1990).

3. After-tax term structures

The impact of taxes on the estimation of the term structure of interest rates has been well documented. Pioneered by McColloch (1975), this line of research has been extended by Schaefer (1982), Jordan (1984) and Litzenberger and Rolfo (1984) amongst others. The existence of environments where ordinary income and capital gains are taxed differentially (even in the case where capital gains are non-taxed) may make it impossible for investors in different tax brackets to agree on the relative pricing of bonds. In such a case, the difference between the observed price of a bond and its appropriate present value may include a tax effect over and above simple noise. The implication of this for the bond market is that a non-segmented equilibrium may be unobtainable due to the existence of buy and hold tax arbitrage opportunities. A segmented equilibrium characterised by tax clientele effects would be possible, however, if frictions exist to limit arbitrage profit. An example of such a fric-

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3 Katz and Prisman (1991) provide a summary of the literature in this field.
tion would be a restriction on the ability to sell short. In such an environment, the resulting equilibrium is segmented in terms of investors only holding bonds which are priced correctly for their own particular tax bracket.

Brown and Schaefer (1994) were the first to estimate the before-tax real-term structure of interest rates using cross sections of weekly ILG prices. Using the approach of Schaefer (1981), they estimate the real-term structure as an optimization problem with the condition that the present value of each ILG cannot exceed its observed price. In a market characterized by a ban on short sales and a segmented equilibrium caused by differential taxation, this criterion would be satisfied by an infinite number of possible real-term structures.

The specific term structure selected by the Schaefer (1981) approach is the one that maximizes the present value of an arbitrarily chosen stream of cash-flows $c_i$ subject to the constraint that the present value of each of the $j$ outstanding ILGs cannot exceed their market price. This approach can be formulated as follows:

$$\max_{d_i} \sum_i d_i c_i,$$

subject to:

$$A d \leq P,$$

where $d$ and $d_i$ represent the vector of discount factors and its corresponding period $i$ element, and where $A$ and $P$ represent the payoff matrix of ILG cashflows and the vector of market prices, respectively.

The use of this approach for term structure estimation may be problematic, however. No allowance is made for observation error as it treats observed prices as “true prices”. This assumption seems unrealistic given the non-synchronous trading of securities in the bond market. In addition, given that the choice of the cashflow stream in the objective function is arbitrarily chosen then, as pointed out by Katz and Prisman (1991), so also may be the term structure determined by the optimization. Consequently, the estimation will be biased in terms of the specific cashflow stream chosen and cannot be used to identify mispriced bonds or the marginal tax rate associated with the representative investor.

In their study, Brown and Schaefer (1994) assume that no significant tax clientele effect exists in the ILG market and, thus, choose to ignore taxes. If this is indeed the case, then there is no a priori reason to expect that the error between market price and the present value contains anything beyond simple noise. Nonetheless, their method actually estimates the term structure based on the assumption that the prices of bonds only contain positive error terms. It is,

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4 This issue is referred to in Dybvig (1989).
therefore, not clear why the Schaefer (1981) approach is preferable to well
known term structure estimation approaches that are based upon the mini-
mization of the sum of squared error terms, particularly given that Brown and
Schaefer (1994) later appeal to random price “observation error” in their esti-
mation of the CIR model parameters.

As pointed out by Woodward (1990), the absence of capital gains taxes on
gilts since 1986 in the UK will very likely lead to segmented market equilibria
with distinct tax clientele effects. Thus, rather than assuming the absence of tax
clientele effects, this study utilizes the approach of McColloch (1975) to gen-
erate an after-tax term structure for ILGs based on the assumption that a given
marginal tax bracket exists for which all bonds are priced correctly for a
representative investor.

The marginal tax rate of the representative investor is determined as the
specific rate which produces the lowest sum of squared errors from amongst
each of the after-tax term structures estimated. This approach is based on the
Litzenberger and Rolfo (1984) study which assumes that, on a particular date,
the representative investors marginal tax rate is the same for all maturities of
gilts. In the UK gilt market, the regression is simplified by the absence of
capital gains taxes which eliminates the need for the instrumental variable
adjustment utilized in McColloch (1975).

The data employed in this study are that of Woodward (1990, 1993) and
includes non-linked and ILG prices as well as the monthly RPI collected over
ninety-four periods from 25 January 1986 until 26 October 1993. Eighty-five
non-linked gilts and seventeen ILGs were used throughout the period with
callable and convertible gilts excluded from the sample.

The specific methodology used is that of Litzenberger and Rolfo (1984)
whereby the time interval to the longest gilt maturity date is partitioned into \( k \)
subintervals, each containing approximately an equal number of payment
dates, by placing \( k + 1 \) knot points at \( t = 0, 1, \ldots, T \). Given that no additional
explanatory significance is found beyond the third power, the spot real after-
tax (marginal tax rate \( \tau \)) discount factor, \( z_{0,t}^{0,t} \), for term of length \( t \), thus, takes
the form

\[
z_{0,t}^{0,t} = 1 + \alpha t + \beta t^2 + \sum_{i=1}^{k} \gamma_i t_i^3 D_i(t),
\]

where if \( t < t_{i-1} \), then

\[ D_i(t) = 0 \]

and if \( t > t_{i-1} \), then

\[ D_i(t) = 1. \]

The price of an ILG can be expressed as
\[ P_{n,0} = \sum_{j}^{n} c_{t,j}x_{t}^{0,j} + \epsilon, \]  

(3)

where \( c_{t,j} \) represent the period \( j \) real after-tax cash flows and where, \( \epsilon \) captures noise and tax effects.

The substitution of Eq. (2) into Eq. (3) produces the following regression equation:

\[ P_{n,0} - \sum_{j}^{n} c_{t,j} = \alpha_{0} + \alpha \sum_{j}^{n} c_{t,j}t_{j} + \beta \sum_{j}^{n} c_{t,j}t_{j}^2 + \sum_{i=1}^{k} \gamma_{i} \sum_{j}^{n} c_{t,j}(t_{j} - t_{i})^{3} + \epsilon. \]  

(4)

The discount function is, thus, determined by substituting the resultant regression coefficients into Eq. (2).

Regressions are run over a range of marginal tax rates with the rate which minimizes the sum of squared error terms presumed to be that of the representative investor. For the representative investor, the \( \epsilon \) term is assumed to be free of tax effects. An identical methodology is utilized to determine nominal term structures of interest rates. The regression equation corresponding to the determination of the nominal term structure is as follows:

\[ P_{n,0} - \sum_{j}^{n} C_{t,j} = \alpha_{0} + \alpha \sum_{j}^{n} C_{t,j}t_{j} + \beta \sum_{j}^{n} C_{t,j}t_{j}^2 + \sum_{i=1}^{k} \gamma_{i} \sum_{j}^{n} C_{t,j}(t_{j} - t_{i})^{3} + \epsilon, \]  

(5)

where in this case, \( C_{t,j} \) represent the period \( j \) nominal after-tax cash flows. Unlike Woodward (1990), however, it is not assumed that the same representative investor is appropriate for both the ILG and the non-linked gilt markets. Given that government tax laws differed in their treatment of linked and non-linked gilts over this period argues for an approach that explicitly distinguishes between the respective marginal tax rates of each class of security.

The marginal tax rates of the representative investor calculated for both nominal gilts and ILGs are provided in Appendix A. The representative investor clearly differs across the linked and non-linked markets which further underscores the importance of considering differential after-tax effects when determining the relationship between interest rates and inflation. The representative investors marginal tax rate appropriate for the non-linked market exhibits a declining trend from approximately 20% in 1986 to approximately 10% in 1993 which is consistent (although slightly lower in magnitude) with Woodward (1990). In contrast, the tax rate appropriate for the ILG market increases from essentially 0% in 1986 to approximately 10% in 1993.

The method of real-term structure estimation described above ignores the eight-month lag of inflation indexation on ILGs. Without further refinement,
this approach will provide a true estimation of the real term structure only if the expectation of inflation over the last eight months of the ILGs life is equal to actual inflation eight months prior to ILG issue and that no corresponding inflation risk premium exists. If this is not the case, an inflation bias will be inherent in the estimate of real spot rates which diminishes proportionately with the length of the term. To correct for this bias, a method of estimating inflation expectations for the non-indexed eight-month period is required.

The approach used by Brown and Schaefer (1994) to correct for this bias is to estimate expected inflation through the use of an ARIMA model which accommodates the seasonality of inflation expectations. In contrast, this study uses an alternative procedure to correct for the lag by extracting inflation expectations directly from the relationship between the estimates of the nominal and real after-tax term structures. This method provides greater flexibility as it is able to incorporate non-seasonal price shocks into the estimate of inflation expectations.

The continuously compounded real, $n$ month spot rate, $k_0^n$, extracted from the estimated ILG term structure when $n \geq 8$ can be expressed as

$$k_0^n = r_0^n + \left( \frac{8}{n} \right) (\hat{\pi}_{n-8} - \hat{\pi}_0^n) \tag{6}$$

and when $n < 8$,

$$k_0^n = r_0^n + (\pi_0^n - \hat{\pi}_{n-8}^n), \tag{7}$$

where $r_0^n$ is the true $n$ period real spot rate and where, $\hat{\pi}$ and $\pi$ represent actual inflation experienced prior to ILG issue and expected inflation over the last months of the ILGs life, respectively. Without loss of generality it can be assumed that $\pi$ also contains the inflation risk premium appropriate for the period in question. For $n \geq 8$, the lag bias in the real spot rate estimate is reflected by the difference between $\pi_{n-8}^n$ and $\pi_0^n$. The magnitude of this bias is clearly proportional to the term of the spot rate and at long terms to maturity, becomes negligible. Note that for terms of eight months or less, the real rate essentially possesses the characteristics of a nominal rate.

The continuously compounded nominal, $n$ month spot rate, $K_0^n$, extracted from the estimated non-linked gilt term structure is free of any lag bias and can be simply expressed as

$$K_0^n = r_0^n + \pi_0^n. \tag{8}$$

Under the assumption that an “expectations hypothesis” for the term structure of inflation compensation is valid, the following relationship holds:

$$\pi_0^{n+8} = \left( \frac{n}{n+8} \right) \pi_0^n + \left( \frac{8}{n+8} \right) \pi_{n+8}^n. \tag{9}$$
Given the following $n + 8$ and $n$ month nominal spot rates:

\[
K_0^{n+8} = r_0^{n+8} + \pi_0^{n+8} = r_0^{n+8} + \left( \frac{n}{n+8} \right) \pi_0^n + \left( \frac{8}{n+8} \right) \pi_n^{n+8},
\]  

(10)

\[
K_0^n = r_0^n + \pi_0^n,
\]  

(11)

and the following $n + 8$ month real spot rate:

\[
k_0^{n+8} = r_0^{n+8} + \left( \frac{8}{n+8} \right) (\pi_n^{n+8} - \hat{\pi}_{-8}),
\]  

(12)

then the true $n$ spot real rate derived from the estimated nominal and real-term structures can be expressed as

\[
r_0^n = K_0^n - \left( \frac{n+8}{n} \right) (K_0^{n+8} - k_0^{n+8}) + \left( \frac{8}{n} \right) \hat{\pi}_{-8}.
\]  

(13)

This approach estimates expected future inflation by using the information directly contained in the market prices of linked and non-linked gilts. The previously noted difference between the representative investors marginal tax rate appropriate for the ILG and non-linked gilt markets is critical for the estimation and underscores the need to consider nominal and real rates on an after-tax basis.

After-tax real and nominal term structures are estimated over monthly periods from 25 January 1986 until 25 October 1993 and are illustrated in Figs. 1 and 2.

Table 2 contains before- and after-tax mean spot rates and standard deviations over the sample period.

Several observations can be made from the estimates. First, as noted by Brown and Schaefer (1994), real-term structures are neither flat nor constant over time. The shapes of the real-term structures, while predominantly “humped” shaped, can be either upwardly or downwardly sloping. Most of the volatility, however, is at the short end as spot rates for longer terms become remarkably constant.

The impact of the eight-month lag for inflation indexation is apparent as the lag adjusted real spot rates are much less variable than the unadjusted rates at the short end of the term structure. This is not an unexpected result given that unadjusted spot rates are highly influenced by the inflation expected over the eight-month period just prior to maturity.

A comparison of before-tax and after-tax rates reveals only a small reduction in the means and a negligible change in the volatilities for ILGs due to the fact that the marginal tax bracket for these securities over much of the period was 0%. This is not surprising given that ILGs are held predominantly by pension funds and insurance companies, institutions which are quite likely to
be tax-exempt. Means of after-tax spot rates were much lower for non-linked gilts with a non-negligible decline in volatility.

Although the January 1986 until December 1989 sample period of the Brown and Schaefer (1994) study is not identical to that of this study, some
comparisons can still be made. Their results indicate significantly greater volatility at the short end of the term structure which is likely due to the different approaches used to correct for the lag in indexation. It is also noteworthy that the Schaefer (1981) approach to term structure estimation is likely to produce an upward bias in the estimation of spot rates. Brown and Schaefer (1994) spot rate estimates in Table 2 show some indication of this.

4. CIR model for the real-term structure

Brown and Schaefer (1994) fit the Cox et al. (1985) term structure model to market prices of ILGs in order to assess the extent to which the single factor model captures the main features highlighted by their empirical approach. They find results that are both consistent and inconsistent with the theoretical model. On the one hand, the term structures estimated by Brown and Schaefer (1994) consistently demonstrate the general shapes, the high correlation of crosssectional spot yields, and the low volatility of long-term yields predicted by the CIR model. On the other hand, intertemporal stability of the CIR parameters is firmly rejected.

This study endeavors to estimate CIR model parameters in a manner similar to Brown and Schaefer (1994) but to determine the theoretical term structure
utilizing the “true” prices implied by the term structure estimation procedure in contrast to their approach which utilizes directly observed market prices. Given that observed prices may not be free from buy and hold arbitrage opportunities for all investors, then their use cannot be consistent with an equilibrium model for the term structure of interest rates. As pointed out by Hull (1993), a model of the term structure which is comfortably within a 1% error margin for pricing a bond could produce an error as large as 50% for pricing a derivative security on that bond. In order to adjust for this inconsistency, Brown and Schaefer (1994) appeal to observation error and fit the CIR model by minimizing the sum of squared errors between the market and “model” prices. This approach produces a bias, however, if a tax clientele effect, such as that demonstrated in the previous section, were to exist over and above simple noise. In this study, the theoretical after-tax prices are determined from the term structure estimation approach of this study which, given the assumption of a representative investor, are free from clientele effects.

As a first step, Table 3 illustrates the correlation of cross-sectional spot rates for both before- and after-tax real spot rates using the term structure estimation approach of this study.

Consistent with Brown and Schaefer (1994), a fairly high degree of correlation is found for changes in real spot rates which suggests some promise for the one factor approach to modelling real interest rates. Note that the correlation for after-tax real rates is higher than that for before tax rates which is consistent with Rumsey (1993) who finds that interest rate models, in general, provide a better fit to after-tax interest rates than to before tax interest rates.

This study estimates the parameters of the CIR square root process in a manner similar to Brown and Dybvig (1986) and Brown and Schaefer (1994) by assuming the following process for the instantaneous short-term rate:

\[ \dot{r} = \vartheta(\theta - r) dt + \sigma \sqrt{r} dz, \]

where \( \theta \) is the mean, \( \vartheta \) the mean reversion coefficient and \( \sigma \) is the volatility. Given the above process and the arbitrage pricing condition, it can be shown that the price, \( P(r, t, T) \), of a default free, zero coupon bond must satisfy the following partial differential equation:

\[ \vartheta(\theta - r)P_r + P_t + \frac{1}{2} \sigma^2 rP_{rr} - rP = \lambda rP, \]

5 The after-tax results demonstrate higher correlation than both the before tax results of this study and the before tax results of Brown and Schaefer (1994) although it must be noted that the sample periods of the two studies are not identical.
where $\lambda$ represents the market price of risk. The terminal condition
\[ P(r, T, T) = 1 \]  
implies that, for any $\tau$,
\[ P(r, t, t + \tau) = A(\tau)e^{-B(\tau)r}, \]  
where
\[ A(\tau) = \left( \frac{2\gamma e^{\left(2(\theta + \lambda + \gamma)\tau\right)}}{\left(\theta + \lambda + \gamma\right)\left(e^{\gamma\tau} - 1\right) + 2\gamma} \right)^{2\theta\theta/\sigma^2}, \]
\[ B(\tau) = \left( \frac{2\left(e^{\gamma\tau} - 1\right)}{\left(\theta + \lambda + \gamma\right)\left(e^{\gamma\tau} - 1\right) + 2\gamma} \right), \]  
and
\[ \gamma = \sqrt{(\theta + \lambda)^2 + 2\sigma^2}. \]  
The $\tau$ period zero coupon spot rate defined by this process is
\[ r_0^\tau = \frac{B(\tau)r - \ln(A(\tau))}{\tau}, \]  
where
\[ \lim_{\tau \to \infty} r_0^\tau = \frac{2\theta\theta}{\theta + \lambda + \gamma}. \]
Given that prices are dependent only on the risk adjusted interest rate process implies that the parameters to be estimated are, $r$, $\sigma^2$, $\theta\theta$ and $\vartheta + \lambda$.

The parameter values are estimated in an approach similar to Brown and Dybvig (1986) and Brown and Schaefer (1994) by solving the following optimization problem across $j$ ILGs with respect to the risk adjusted CIR parameters:

$$\min \sum_j (\bar{P}_j - P_j)^2. \quad (22)$$

This approach differs from the above studies in that $P_j$ represents the theoretical price for the CIR model and $\bar{P}_j$ represents the “true” price determined from the empirical estimation procedure in the previous section rather than the observed market price. The “true” price represents the observed price less the observation error.

The CIR after-tax term structure estimates over the sample period are illustrated in Fig. 3, with mean spot rates and standard deviations provided in Table 4.

Similar to the conclusions reached by Brown and Schaefer (1994), the CIR model is quite able to accommodate the general shapes of the observed term structures. The agreement between the mean and the standard deviations of the CIR model and the observed term structures for each term is good and is particularly strong for longer maturities. The understatement of the real rate
standard deviation that the CIR model suggests for the shorter end is significantly less than that found by Brown and Schaefer (1994) which is again likely due to the different approaches used for measuring inflation expectations.

The estimates of the CIR parameters are shown in Table 5. While stability of these parameters is rejected, there appears to be less variation in the estimates

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<th>Term</th>
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*a The table provides estimates for the CIR one factor interest rate model for both before and after tax interest rates over the period from 25 January 1986 until 25 October 1993.
for after-tax interest rates over time than for before tax rates in this study or in that of Brown and Schaefer (1994). This is again consistent with Rumsey (1993) and provides further support for fitting one factor interest rate models to after-tax interest rates.

The appropriate choice and calibration of term structure models is critical to the accurate pricing of derivative securities. The valuation of these securities is highly sensitive to the estimation procedures utilized. As a framework for comparison, the parameter estimates of Brown and Schaefer (1994) are compared with the results of this paper in order to value a fifteen year zero coupon bond and hypothetical five year call options on that bond. Three European call options are considered with strike prices of $40, $50 and $60, respectively. The modified binomial approach of Tian (1993) is used to construct a lattice of the CIR interest rate process. Under this approach, a path-independent binomial tree is created from a transformation of the assumed interest rate process to one which has constant volatility. In general, for an interest rate process described as

$$dr = \mu(r,t) dt + \sigma(r,t) dz$$

the transformed constant volatility process can be defined as

$$d\Theta = q(r,t) dt + \Theta, \sigma dz,$$  \hspace{1cm} (24)

where

$$q(r,t) = \Theta t + (\mu - \lambda t) \Theta r + \frac{1}{2} \sigma^2 \Theta rr.$$  \hspace{1cm} (25)

For the CIR interest rate process an appropriate transformation is simply

$$\Theta = \sqrt{r},$$  \hspace{1cm} (26)

which implies a transformed stochastic process of

$$d\Theta = q d\tau + v dz,$$  \hspace{1cm} (27)

where

$$q = [\vartheta (\Theta - r) - \lambda \Theta r] \Theta r + \frac{1}{2} \sigma^2 \Theta rr = \frac{\sigma_1}{\Theta} - \sigma_2 \Theta,$$

$$v = \frac{\sigma}{2},$$

and where
Tian (1993) finds the speed of convergence to the closed form solution to be within approximately 0.05% for a zero coupon bond with only a ten-step lattice. The values of the bond and call options estimated by both the Brown and Schaefer (1994) approach and that of this paper are shown in Table 6.

The results demonstrate the magnitude of error inherent in a misspecification of parameter estimates for an interest rate process. Although there is a fair degree of consistency across both sets of CIR parameter estimates for the values of a bond, the results obtained for derivatives can vary widely. The proportional differences in the estimated values for the zero coupon bond range from 0.97–1.27% compared with a range of 2.68–400.00% for values of the call.

\[ \begin{align*}
\alpha_1 &= \frac{4\bar{\theta} \theta - \sigma^2}{8}, \\
\alpha_2 &= \frac{\bar{\theta} + \lambda \sigma}{2}.
\end{align*} \]

Tian (1993) finds the speed of convergence to the closed form solution to be within approximately 0.05% for a zero coupon bond with only a ten-step lattice.\(^6\)

The values of the bond and call options estimated by both the Brown and Schaefer (1994) approach and that of this paper are shown in Table 6.

The results demonstrate the magnitude of error inherent in a misspecification of parameter estimates for an interest rate process. Although there is a fair degree of consistency across both sets of CIR parameter estimates for the values of a bond, the results obtained for derivatives can vary widely. The proportional differences in the estimated values for the zero coupon bond range from 0.97–1.27% compared with a range of 2.68–400.00% for values of the call.

\(^6\) This is strictly true only when \(\alpha_1 > 1\) (Tian, 1993).
option. This clearly illustrates the magnitude of the pricing errors possible in interest rate derivative securities which stem from the misspecification of CIR parameter estimates. Hence, if observed market prices do not preclude buy and hold arbitrage opportunities, then to use these to estimate the parameters of an interest rate process can lead to large errors when valuing interest rate derivative securities.

5. Conclusions

Utilizing an alternative approach to Brown and Schaefer (1994), this study estimates the after-tax term structure of real interest rates from information contained in both the UK linked and non-linked gilt markets over the period from 25 January 1986 until 25 October 1993.

Two major observations can be made regarding the estimates for spot real interest rates. Firstly, the volatility of the short-term rate is much lower than that found by Brown and Schaefer (1994) which provides a better fit with the predictions of the CIR single factor model for interest rates. This is likely due to the different approaches used by the two studies to correct for the lag in indexation.

Secondly, consistent with Rumsey (1993), there appears to be some evidence to suggest that single factor interest rate models produce a better fit to interest rates on an after-tax basis than on a before-tax basis.

It is clear from a comparison of the results of this study and Brown and Schaefer (1994) that the pricing of interest rate derivatives is highly sensitive to the procedures employed to estimate the parameters embodied in interest rate models such as CIR. It is shown that the differences in the techniques used to construct term structures alone can hugely impact these estimates. This study addressed the impact of differential taxation and the existence of “noise” in observed market prices on term structure estimation and found significant impact on parameter estimation when compared to methods that did not. It is clear that further research is required in order to determine the impact of a host of other market frictions such as the effect of liquidity on real-term structure estimation.

Acknowledgements

The authors would like to thank Zvi Alfasi, Alan Kimche and Uri Passy. Special thanks are due to Eli Katz and Thomas Woodward. The usual caveat applies. Prisman would like to thank the SSHRC of Canada and the York University Research Authority for their financial support.
### Appendix A

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