Binomial valuation of lookback options

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Abstract

We present a straightforward and computationally efficient binomial approximation scheme for the valuation of lookback options. This enables us to value American lookback options. Previous research on lookback options has assumed that the contracts are based on the extrema of the continuously observed price of the underlying security; in practice, however, contracts are often based on the extrema of prices sampled at a finite set of fixed dates. We adapt our binomial scheme to investigate the impact on the value of the options. © 2000 Elsevier Science B.V. All rights reserved.

1. Introduction

Goldman et al. (1979a) provided closed-form valuation formulae for a European-style option to buy (sell) a non-dividend paying security at the lowest (highest) price achieved by that security over the life of the option. When their paper was published, such options were not traded. Subsequently, such
‘lookback’ or ‘hindsight’ options have gained a modest but enduring place among the range of derivative securities available over-the-counter from a number of financial institutions. This provides perhaps the most notable instance to date of a theoretical valuation model prompting the commercial introduction of new types of contingent claim.

From a commercial point of view, one might have expected American-style lookback options to have appeared alongside their European-style counterparts. Yet American lookback options are almost unheard-of. This contrasts markedly with the case of ordinary puts and calls, where American options, both over-the-counter and exchange-traded, are commonplace. We believe that this situation is due, in no small measure, to the unavailability of an efficient valuation technique for American lookback options. In this paper, we remedy that situation by presenting a straightforward and computationally efficient binomial valuation scheme. Our proof that the resulting option values converge to the correct levels as the time step length is reduced to zero has the attractions of being direct and of using only basic mathematics; the approach it adopts appears to transfer readily to other contexts, and may therefore be a useful contribution.

Previous research on lookback options has assumed that the contracts are based on the extrema of the continuously observed price of the underlying security; in practice, however, contracts are often based on the extrema of prices sampled at a finite set of fixed dates, typically at daily intervals. We adapt our binomial scheme to investigate the impact on the value of the options. It emerges that even a switch from continuous to daily sampling has a significant impact on option values, even when options are far from expiry; our results therefore provide opportunities for improved pricing and, hence also, risk management.

The remainder of the paper is organized as follows. Section 2 rehearses the assumptions of the model and the resulting abstract general valuation formula for contingent claims. We generalize slightly the assumptions of Goldman et al. (1979a) by allowing the underlying security to pay a constant continuous proportional dividend. This extension accommodates, inter alia, the foreign interest rate when the security is one unit of a foreign currency. Our exposition makes use of the modern ‘martingale’ approach to contingent claims analysis stemming from Harrison and Kreps (1979). Section 3 formalises the

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2 Conze and Viswanathan (1991) rederived Goldman, Sosin and Gatto’s results, together with valuations of various related European-style contingent claims. They also present upper bounds for the value of the early exercise right in American lookback options. However, the authors’ claim that these bounds are ‘tight’ must be interpreted with caution, since their own illustrative results (Table 1, p. 1902) show that their upper bound for the value of a new 6-month American lookback put can exceed 160% of the value of its European counterpart.
Until Section 8, we implicitly follow the existing literature in assuming that the contracts are based on the extrema of the continuously observed security price process. I.e. maximum (minimum) in the case of a put (call) option. Numerous technical details are omitted in the interest of brevity. The concerned reader familiar with the martingale approach to contingent claims analysis should be able to rectify the omissions without difficulty.

Section 4 discusses why the valuation techniques for American-style ordinary puts and calls do not carry over readily to lookback options. In Section 5, we overcome these problems by means of a change of numeraire and a binomial lattice scheme in a new variable. Section 6 provides illustrative results to assess the accuracy of our binomial scheme as applied to European lookback options, against the closed-form formulae obtained in Section 3, and to illustrate the value of the early exercise right in American lookback options. Combining these with further results tends to refute a conjecture by Garman [1987] that the ‘early exercise premium’ of an American lookback option (i.e. the excess of the value of the American option over that of an otherwise identical option) is well approximated by the early exercise premium of an American ordinary option with strike equal to the extreme security price so far achieved. Section 7 considers the convergence properties of our binomial scheme as the length of each discrete time step tends to zero.

In Section 8 we modify the binomial scheme to investigate the impact of basing lookback contracts on the extrema of a finite sample of prices of the underlying security, and present illustrative results. Section 9 concludes the paper.

2. Assumptions and abstract valuations results

We assume a frictionless competitive market is open continuously in a security whose price follows as Ito process:

\[ \frac{dS}{S} = \mu \, dt + \sigma \, dZ \]  

(1)

where \( Z(\cdot) \) is a standard Brownian motion; \( \sigma > 0 \) is the constant price volatility; and the instantaneous expected return \( \mu \) is an essentially arbitrary stochastic process adapted to the Brownian filtration. The security pays a continuous proportional dividend at a constant yield \( y \). Interpretations of the security include that of a stock, and of a unit of a ‘foreign’ currency. In the latter case, \( S \) corresponds to the exchange rate, expressed in the form of the ‘domestic’ currency price of one unit of the ‘foreign’ currency, and \( y \) corresponds to the

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3 Until Section 8, we implicitly follow the existing literature in assuming that the contracts are based on the extrema of the continuously observed security price process.

4 I.e. maximum (minimum) in the case of a put (call) option.

5 Numerous technical details are omitted in the interest of brevity. The concerned reader familiar with the ‘martingale’ approach to contingent claims analysis should be able to rectify the omissions without difficulty.
‘foreign’ interest rate (Garman and Kohlhagen, 1983). Goldman et al. (1979a) assume \( y = 0 \).

We also assume that riskless borrowing and lending are available at the same constant interest rate \( r \geq 0 \).

It is now well-known, using seminal results of Harrison and Kreps (1979), that, given the above assumptions, there exists a unique pricing operator for claims contingent on the security price process. Moreover, this pricing operator can be represented in terms of the expected value of the discounted payoff\(^6\) from the contingent claim, where the expectation is taken under an ‘equivalent martingale measure’ (EMM), a reassignment of probabilities, under which the security price process (1) can be rewritten as:

\[
dS/S = (r - y) \, dt + \sigma \, dZ^* \tag{2}
\]

where \( Z^*(\cdot) \) is a standard Brownian motion under the EMM. In particular, if a contingent claim which gives a payoff \( V(t') \) at some — possibly stochastic stopping — time, \( t' \), is traded at an earlier time \( t \), than the value of the claim at \( t \) is given by:

\[
V(t) = E_t^*[e^{-r(t'-t)} \, V(t')] \tag{3}
\]

where \( E_t^*[\cdot] \) denotes the conditional expectation operator at \( t \) under the EMM.\(^7\)

3. Lookback pay-offs and valuation of European options

A European (American) lookback call option contracted at time zero affords its holder the right to purchase one unit of the security, on an exercise date equal to (at or before) some specified fixed expiry date \( T \), at a price equal to the lowest value of \( S \) attained between time zero and the exercise date. Hence, as Goldman et al. (1979a) show, the payoff upon exercise at time \( t' \) is

\[
V_{\text{call}}(t') = S(t') - S_{\text{inf}}(t') \tag{4}
\]

where

\[
S_{\text{inf}}(t) = \inf\{S(u): 0 \leq u \leq t\} \tag{5}
\]

\(^6\) It suffices for this paper for us to consider claims with a single payoff only.

\(^7\) Harrison and Kreps’ analysis required that pay-offs be square integrable under the original probability measure, but also goes through under the alternative requirement that pay-offs be square integrable under the EMM. Standard results in stochastic calculus can be used to verify that this latter requirement is satisfied for all contingent claims considered in this paper.
The pay-off to a lookback put option is analogously given as

\[ V_{\text{put}}(t') = S_{\sup}(t') - S(t') \]  

(6)

where

\[ S_{\sup}(t) \equiv \sup\{S(u): 0 \leq u \leq t\}. \]  

(7)

To value the European put\(^8\) we note that \( t' = T \) and that

\[ S_{\sup}(T) = \exp\{X_{\sup}(T)\} \]

where

\[ X_{\sup}(t) \equiv \sup\{\ln S(u): 0 \leq u \leq t\}. \]

Applying Ito’s lemma to (2) yields that, under the EMM, \( \ln S() \) follows a straightforward Brownian motion with drift:

\[ d \ln S = m^* \, dt + \sigma \, dZ^* \]

where

\[ m^* = r - y - \frac{1}{2} \sigma^2. \]

We can then obtain the probability distribution function of \( X_{\sup}(T) \), conditional on \( X(t) \) and \( X_{\sup}(t) \), from e.g. Harrison (1985, Section 1.9, pp. 14–15), as

\[ F(x) = \begin{cases} \Phi(b_1(x)) - e^{2(x - \overline{X(t)})m^*/\sigma^2} \Phi(-b_2(x)) & \text{if } x \geq X_{\sup}(t), \\ 0, & \text{otherwise}, \end{cases} \]

where \( \Phi() \) denotes the standard Normal distribution function, and

\[ b_1(x) = \frac{x - \overline{X(t)} - (T - t)m^*}{\sigma \sqrt{T - t}}, \]

\[ b_2(x) = \frac{x - \overline{X(t)} + (T - t)m^*}{\sigma \sqrt{T - t}}. \]

Hence, by substituting into (3),

\[ V_{\text{put}}(t) = e^{-(T-t)y} \int_{\#} e^{x} \, dF(x) \, dx - e^{-(T-t)y}S(t) \]

---

\(^8\) We commence with puts rather than calls because, by doing so, the ensuing analysis will link most neatly into the requisite mathematics in Harrison (1985).
which, upon evaluating the integral, reveals that \( V_{\text{put}}(t) \) can be expressed as the sum of two components:

\[
V_{\text{put}}(t) = P(t) + B_{\text{put}}(t)
\]

where \( P(t) \) is the value of an ordinary European put option with strike equal to \( S_{\sup}(t) \), i.e.

\[
P(t) = -S(t) e^{-\sigma(T-t)\sqrt{T-t} - \sigma} + S_{\sup}(t) e^{-\sigma(T-t)\Phi(\sigma(T-t))}
\]

with

\[
z_1 = b_1(X_{\sup}(t))
\]

and \( B_{\text{put}}(t) \) is the value of what Garman (1987) calls the ‘strike bonus option’, given for the general case \( y \neq r \) by

\[
B_{\text{put}}(t) = \frac{S(t)}{a} \left\{ e^{-(T-t)y\Phi(\sigma(T-t) - z_1)} - e^{-(T-t)y\Phi(\sigma(T-t))} \right\}
\]

(9a)

where \( a = 2(r - y)\sigma^2 \) and \( z_2 = b_2(X_{\sup}(t)) \) while, for \( y = r \)

\[
B_{\text{put}}(t) = S(t)e^{-\sigma(T-t)\sqrt{T-t} \Phi(\sigma(T-t)) - \Phi(z_2)}
\]

(9b)

where \( \phi(\cdot) \) is the standard Normal density function.

To value the otherwise identical call option, we exploit the relation

\[
X_{\inf}(t) \equiv \inf \{ \ln S(u) : 0 \leq u \leq t \}
\]

\[
= -\sup \{ -\ln S(u) : 0 \leq u \leq t \}
\]

and apply the same machinery, obtaining

\[
V_{\text{call}}(t) = C(t) + B_{\text{call}}(t)
\]

(10)

where \( C(t) \) is the value of an ordinary European call option with strike equal to \( S_{\inf}(t) \), i.e.

\[
C(t) = S(t) e^{-(T-t)\Phi(\sigma(T-t) - z_3)} - S_{\inf}(t) e^{-(T-t)\Phi(\sigma(T-t))}
\]

\footnote{The term ‘strike bonus option’ arises from the potential increase in the price at which the option allows its holder to sell the security, if and when the security price attains a new supremum.}
with $z_3 = b_1(X_{\text{inf}}(t))$ and where, for $y \neq r$

$$B_{\text{call}}(t) = \frac{S(t)}{a} \left\{ e^{-(T-t)y} \left( \frac{S_{\text{inf}}(t)}{S(t)} \right)^a \Phi(z_4) - e^{-(T-t)y} \Phi(z_3 - \sigma \sqrt{T-t}) \right\} \quad (11a)$$

with $z_4 = b_2(X_{\text{inf}}(t))$ while, for $y = r$

$$B_{\text{call}}(t) = S(t) e^{-(T-t)y} \sigma \sqrt{T-t} \{ \phi(z_4) + z_4 \Phi(z_4) \}. \quad (11b)$$

**Remark 1.** The valuation formulae of Goldman et al. (1979a, b) correspond to the case $y = 0 < r$. The formulae for the general $y \neq r$ case were independently derived by Babbs (1986) and Garman (1987). Babbs (1986) also covers the $y = r$ case, for which the formulae given for $y \neq r$ would involve division by zero.\(^{10}\) The observation that the valuation formulae could be so arranged that certain of the terms correspond to the values of ordinary European options is due to Garman (1987).

**Remark 2.** Goldman et al. (1979a, b) derived their results using the approach of Cox and Ross (1976a, b) which entails establishing the possibility of a perfect hedge before moving to the artificial ‘risk neutral’ probabilities we nowadays term the EMM. This forced them to undertake a lengthy technical discussion to establish that the sensitivity of the option value to $S$ vanishes when $S$ achieves a new extremum. As Goldman et al. (1979b, p. 405) indicated, it is a testimony to the power of Harrison and Kreps’ (1979) results that they enable us to obtain valuations without any such discussion. See also footnote 11.

### 4. Difficulties posed by American-style exercise

As in the more familiar case of ordinary puts and calls, the valuation of American lookback options is complicated by the need to determine simultaneously the value of the option and the optimal exercise rule. Experience with ordinary options suggests a number of approaches, based either on a partial differential equation satisfied by the option value, or upon a binomial approximation scheme. As we will now show, these approaches do not readily transfer to the lookback case.

It is trivial to verify that the value processes of European lookback options, as given by (8)–(11b), satisfy

$$-rV + \frac{\partial V}{\partial t} + (r - y) S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = 0 \quad (12)$$

\(^{10}\) The necessary formulae could also be obtained, via L’Hopital’s rule, from those for the $y \neq r$ case.
subject to the respective boundary conditions (4) and (6) for \( t' \equiv T \). It can be shown that the American options also satisfy (12)\(^\text{11}\) subject to the additional respective early exercise boundary conditions:

\[
V_{\text{call}}(t) \geq S(t) - S_{\text{inf}}(t), \quad 0 \leq t \leq T;
\]

\[
V_{\text{put}}(t) \geq S_{\text{sup}}(t) - S(t), \quad 0 \leq t \leq T.
\]

By analogy with techniques developed for ordinary options, we might then hope either to construct a finite difference scheme for these partial differential equations (cf. e.g. Brennan and Schwartz, 1977) or obtain an approximate solution by largely analytical methods (cf. e.g. Barone-Adesi and Whaley, 1987). Unfortunately, it is hard to see how the finite difference or analytic approximation approaches could accommodate the presence of the path-dependent term \( S_{\text{sup}}(t) \) (or \( S_{\text{inf}}(t) \) as appropriate) as an argument of \( V \) and its derivatives, or in the boundary conditions. (In the case of ordinary options, by contrast, this term is replaced by the fixed exercise price.)

The same difficulty besets the application to lookback options of the binomial approximation developed for ordinary options by Cox et al. (1979) (CRR). Because of the Markovian nature of the security price under the EMM, the value of an ordinary option depends only on time and the current price of the binomial branching process over a recombinant lattice, therefore, CRR limit the number of calculations involved in valuing an ordinary option, to being governed by the number of nodes in the lattice — i.e. \((n + 1)(n + 2)/2\) where \( n \) is the number of discrete time steps into which the outstanding life of the option is divided — rather than by the total number of price paths — i.e. \( 2^n \); this affords great computational efficiency. Unfortunately, in the case of lookback options, the need to keep track of current extreme values and not just current spot values means that not all the necessary information would be carried by individual nodes of the lattice. If the lattice structure for \( S \) is retained, therefore, the need to keep track of the additional, path-dependent, extreme price information will

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\(^{11}\) Briefly, the argument runs as follows. The discounted option value process must be a martingale under the EMM. Since the information filtration is Brownian, the martingale is continuous, and hence predictable (see, e.g. Elliott, 1982, Corollary 6.34, p. 57). It must therefore have zero drift (see, e.g. Elliott, 1982, Lemma 11.39, p. 121). By the relevant extension of Ito’s lemma (see, e.g. Harrison, 1985, Proposition 6.3, p. 67), the drift consists of an absolutely continuous term corresponding to the LHS of (12), plus a term due to the sensitivity of the option value to \( S \) when \( S \) achieves a new extremum. But such times constitute, pathwise, a set of Lebesgue measure zero (see, e.g. Harrison, 1985, Proposition 1.6, p. 15); the two terms can therefore sum to zero only if each separately vanishes.

Note the contrast between the ease with which martingale methods enable this result to be obtained, and the lengths to which Goldman et al. (1979a, pp. 1115, 1116, 1124, 1125) had to go using the methods available to them. It is unclear how to extend Goldman et al.’s line of argument to the American case.
cause the number of calculations to grow substantially faster than the number of nodes — leading to a significant loss of computational efficiency.\(^\text{12}\)

In the next section, we introduce a change of numeraire, following which we present a binomial scheme in a new variable, that achieves for lookback options levels of computational efficiency similar to those which CRR attained for ordinary options.

5. A simple binomial scheme for lookbacks

So far, we have been implicitly content to use as numeraire the units of account in which the security price is expressed. We now adopt units of the security as our numeraire. This is most intuitively grasped when the security is a ‘foreign’ currency — our change of numeraire then consists simply of working in ‘foreign’ rather than ‘domestic’ currency terms. A mere change of numeraire is economically neutral — in particular (8)–(11b) still hold and require only division by \(S(t)\) to express them in our new units. However, the shift simplifies the analysis considerably since the payoffs of lookback options each become functions of single variables rather than of two: thus (4) becomes

\[
V^f_{\text{call}}(t') = 1 - \exp\{M(t')\}
\]

(13)

where the superscript reminds us we are working in units of the security, and

\[
M(T) \equiv \ln \frac{\inf\{S(u); 0 \leq u \leq t\}}{S(t)}
\]

while (5) becomes

\[
V^f_{\text{put}}(t') = \exp\{N(t')\} - 1
\]

(14)

where

\[
N(t) \equiv \ln \frac{\sup\{S(u); 0 \leq u \leq t\}}{S(t)}.
\]

It is natural to re-express all price processes in terms of our new numeraire. We therefore work not with \(S\), but with

\[
R(\cdot) \equiv 1/S(\cdot).
\]

\(^{12}\) Extensive combinatoric analysis would be required to see how far short of \(O(2^n)\) the growth could be contained. The ‘reduced lattice’ in Rubinstein (1990) suggests that \(O(n^3)\) is attainable. (Contrast the \(O(n^2)\) in this paper.)
The key to (17) is that, under the new EMM, the forward price of ‘domestic’ currency, expressed in units of ‘foreign’ currency (i.e. $\exp(M(t) - r t)$ in units of foreign currency), must be a martingale.

See Harrison (1985) for the requisite mathematics.

$R$ represents the price of cash in terms of units of the security; when $S$ is an exchange rate, $R$ expresses the same exchange rate in the reciprocal form of the ‘foreign’ currency price of one unit of ‘domestic’ currency. With this change, and simple manipulations,

$$M(t) = \ln R(t) - \sup\{\ln R(u); 0 \leq u \leq t\} \leq 0$$

and

$$N(t) = \ln R(t) - \inf\{\ln R(u); 0 \leq u \leq t\} \geq 0.$$

A change of numeraire entails a change of EMM. Under the new EMM,

$$d \ln R = (y - r - \frac{1}{2} \sigma^2) dt + \sigma d\tilde{Z}$$

where $\tilde{Z}(t)$ is, once more, a standard Brownian motion.

As in Section 3, we focus on lookback put options. A precisely analogous treatment can be applied to lookback calls.

Under the EMM, the process for $N$ differs from that of $R$ solely by the presence of a reflecting barrier at zero:

$$dN = (y - r - \frac{1}{2} \sigma^2) dt + \sigma d\tilde{Z} + dL$$

where

$$L(t) = \sup\left\{ \max\left\{ -\ln \frac{R(u)}{R(0)}, 0 \right\} : 0 \leq u \leq t \right\}.$$

Note that this process for $N$ has constant parameters, and that the put pay-off (14) depends solely on $N$. To value the lookback put, therefore, we may use $N$ as our sole state variable. We now proceed to construct a binomial approximation scheme based on $N$.

Consider the evaluation at time $t$ of a lookback put expiring at $T$. We divide the period from $t$ to $T$ into $n$ discrete time steps of length $h \equiv (T - t)/n$.

The CRR approximation to (17) would consist of a binomial lattice in which, for each time step, $\ln R(t)$ rises or falls by an amount $\sigma \sqrt{h}$ with respective ‘risk neutral’ probabilities $p$ and $1 - p$ where

$$p = \frac{\exp\{y - r h\} - \exp\{-\frac{\sigma \sqrt{h}}{2}\}}{\exp\{\frac{\sigma \sqrt{h}}{2}\} - \exp\{-\frac{\sigma \sqrt{h}}{2}\}}.$$  

Since (18) differs from (17) only by the presence of a reflecting barrier at zero, we are prompted to form our binomial approximation scheme for $N$ by modifying

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13 The key to (17) is that, under the new EMM, the forward price of ‘domestic’ currency, expressed in units of ‘foreign’ currency (i.e. $\exp\{y - r (T - t)\} R(t)$) must be a martingale.

14 See Harrison (1985) for the requisite mathematics.
the CRR scheme to incorporate such a barrier. Specifically, we require

\[ N(t + (j + 1)h) = \begin{cases} 
N(t + jh) + \sigma \sqrt{h} & \text{with probability } p, \\
|N(t + jh) - \sigma \sqrt{h}| & \text{with probability } 1 - p.
\end{cases} \]  

(20)

This generates a recombinant lattice, with a reflecting barrier at zero, as illustrated in Fig. 1. The nodes shown with unfilled discs result from an odd number of reflections at the barrier. We label the nodes occurring at the end of the \( j \)th time step as \((0, j), (1, j), \ldots, (j, j)\) with node \((0, j)\) farthest from the barrier at zero and moving monotonically closer (see Fig. 1). The overall number of nodes for an \( n \)-time step lattice is precisely the same as that in the CRR lattice for \( R \).

Although there is no one-to-one correspondence between nodes in our lattice for \( N \) and the CRR lattice for \( R \), each movement in \( N \) nevertheless mirrors a movement in \( R \). Hence the probabilities \( p \) and \( 1 - p \) are the correct ‘risk neutral’ probabilities in the lattice for \( N \). (This is easily confirmed directly.) Thus

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\[ \text{Fig. 1. Lattice scheme with a reflecting barrier.} \]
the valuation of the lookback put takes place by backwards recursion over the
lattice as follows. We deal with the case of European exercise first, and then
indicate the modifications for American exercise.

The transformed payoff equation (14) gives us

\[ V(i, n) = \exp\{N(i, n)\} - 1 \quad \forall i \]

(21)

the terminal values of \( N \) being related to the node index \( i \) by

\[
N(i, n) = \begin{cases} 
(N(t) + (n - 2i)\sigma\sqrt{h}), & 0 \leq i \leq k, \\
(n - i + 1 - f)\sigma\sqrt{h}, & \text{odd } (i - k) > 0, \\
(n - i + f)\sigma\sqrt{h}, & \text{even } (i - k) > 0,
\end{cases}
\]

(22)

where

\[ k \quad (f) = \text{integer (fractional) part of } N(t)/\sigma\sqrt{h}. \]

(23)

For time steps \( j = n, n - 1, \ldots, 1 \), we apply the backwards recursion:

\[
V(i, j - 1) = e^{-y(h)}\left\{ pV(i, j) + (1 - p)V(i_d, j) \right\} \quad i = 0, \ldots, j - 1
\]

(24)

where \((i_d, j)\) is the node such that

\[ N(i_d, j) = |N(i, j - 1) - \sigma\sqrt{h}| \]

i.e. where

\[
i_d = \begin{cases} 
i + 2 & \text{if } k \leq i < j - 1, \\
i + 1 & \text{otherwise.}
\end{cases}
\]

The value of the option (in units of our numeraire) is given by \( V(0, 0) \). For an
American lookback put, we replace the RHS of (24) by the immediate exercise
value: \( \exp\{N(i, j - 1)\} - 1 \) at any node at which that value is greater. \( N(i, j - 1) \)
is given by (22), with \( j - 1 \) replacing \( n \).

6. Numerical examples

Table 1 shows results of using our binomial scheme, with various numbers of
time steps, to value new 6-month European and American call and put options.
The row for infinite time steps contains the analytic solution, where available.

Note that all the results for 150 time steps are within two pennies of the most
accurate results quoted. In proportion to the initial security price, this is the

\(^{16}\) We use node references as function arguments in the obvious way, and drop put subscripts and
superscripts.
Table 1
Lookback option values under continuous sampling

<table>
<thead>
<tr>
<th>Time steps</th>
<th>Calls</th>
<th>Puts</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>European or American (^*)</td>
<td>European</td>
</tr>
<tr>
<td>10</td>
<td>12.88</td>
<td>8.97</td>
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<tr>
<td>50</td>
<td>13.13</td>
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</tr>
<tr>
<td>6000</td>
<td>13.19</td>
<td>9.29</td>
</tr>
<tr>
<td>( \infty )</td>
<td>13.19</td>
<td>9.29</td>
</tr>
</tbody>
</table>

\(^*\)We assume a non-dividend paying stock with initial price 100 and volatility 20\%. The interest rate is 10\%. (For European options, this permits comparison with Table 1 on p. 1902 of Conze and Viswanathan (1991)). Conze and Viswanathan (1991) have shown that an American lookback call on a non-paying stock is worth the same as its European counterpart. Our binomial scheme also produces identical values.

The same degree of accuracy as that reported for the CRR scheme for valuing ordinary options, as reported by Cox et al. (1979, pp. 258, 259). Further results (not reported here) indicate that the relative accuracy of our scheme holds up well for away-from-the-money options or those of shorter maturity, but falls off slightly for higher levels of volatility. Nevertheless, bearing in mind that the number of nodes in our binomial scheme grows with the number of time steps in precisely the same way as in the CRR scheme, we feel fully justified in claiming essentially the same high level of computational efficiency.

Garman (1987) has conjectured that the value of an American lookback option is well approximated by the sum of the values of an American ordinary option with strike equal to the extreme security price so far achieved, and of what Garman terms the ‘strike bonus option’ — a European option whose pay-off is the increment between the current and final extreme security prices, or between the final spot price and final extreme price, whichever is the smaller. This conjecture can be more simply expressed as the surmise that the early exercise premium on an American lookback option is well approximated by the early exercise premium on an American ordinary option with strike equal to the extreme security price so far achieved. From the last two columns of Table 1, we deduce that the early exercise premium on the six-month at-the-money option is well approximated by the sum of the values of an American ordinary option with strike equal to the extreme security price so far achieved, and of what Garman terms the ‘strike bonus option’.

\(^{17}\) I.e. maximum (minimum) in the case of a put (call) option.

\(^{18}\) A lookback option is said to be at-the-money if the current security price represents the extremum observed so far.
lookback put is $10.17 - 9.29 = 0.88$; the early exercise premium on the corresponding ordinary put is 0.52. We feel that the quality of approximation in this particular case is rather poor. To be fair, however, Garman’s conjecture was advanced in the context of a paper on lookback options on foreign exchange. More typical values of interest rates and volatility in a foreign exchange context might be, say, $r = 10\%$, $y = 8\%$, $\sigma = 11\%$; for these parameter values the early exercise premia on the six-month at-the-money lookback and ordinary puts are 0.20 and 0.12, respectively. As in the previous case, the early exercise premium on ordinary options is only roughly 60% of the corresponding figure for lookback options. While ‘well approximated’ allows a certain latitude, our view is that Garman’s conjecture must be rejected.

7. Convergence

The burgeoning literature on the convergence of discrete-time valuation methods for contingent claims to their continuous-time counterparts (see, e.g. Cox et al., 1979; Cutland et al., 1991; He, 1990; Hull and White, 1990; Nelson and Ramaswamy, 1990, does not cover path-dependent payoffs such as those on lookbacks. Our reformulation of the lookback valuation problem, in Section 5, removed the path dependency at the expense of basing the analysis on processes subject to a reflecting barrier at zero. The only kind of barriers considered in the extant convergence literature are where the diffusion coefficient vanishes at the barrier (see especially Nelson and Ramaswamy, 1990). The question of the convergence of our binomial scheme for lookbacks, therefore, is novel. Our method of resolving it appears to be novel also.

An exhaustive treatment of convergence issues is beyond the scope of the present paper. We will confine ourselves to what we regard as the key issue: as the number of time steps increases to infinity, does the resulting value of a European lookback put, as computed under our binomial scheme, converge to its continuous-time counterpart as given by the analytical results of Section 3? The demonstration we will present carries over, *mutatis mutandis*, to lookback calls, and extends to American lookbacks by arguments such as those in Carverhill and Webber (1990).

The mathematical machinery used to establish convergence has often been quite elaborate. Thus, for example, Cutland et al. (1991) use non-standard analysis; He (1990) uses martingale convergence theory and requires stringent piecewise smoothness and growth conditions on the pay-off of the contingent claim; Babbs (1990) makes extensive use of moment generating functions and a novel style of dominated convergence. By contrast, the method of resolving convergence questions in this paper uses only basic mathematics. It can certainly be applied to the binomial scheme of Cox et al. (1979) for ordinary options, and we conjecture that it is of widespread applicability.
We present here an intuitive version of our argument; and indicate the technical details requiring additional attention. That attention is relegated to the Appendix.

Placing hats over values of the option, according to our binomial scheme, to emphasise the distinction from the true value at the corresponding values of \( N \) and of time, we may define the error at node \((i, j)\) by

\[
\eta(i, j) \equiv \hat{V}(i, j) - V(N(i, j), t + jh).
\]  

(25)

Substituting (25) into the fundamental recursion (24), i.e.

\[
V(i, j - 1) = e^{-yh} \{(pV(i, j) + (1 - p)V(i_d, j))\},
\]

we obtain

\[
\eta(i, j - 1) = e^{-yh} \{pV(N(i, j), t + jh) + (1 - p)V(N(i_d, j), t + jh)\}
- V(N(i, j - 1), t + (j - 1)h) + e^{-yh} \{p\eta(i, j) + (1 - p)\eta(i_d, j)\}
\]  

(26)

Taking Taylor’s expansions of the \( V \) terms on the RHS of (26) around \((N(i, j - 1), t + (j - 1)h)\), we find that

\[
V(N(i, j), t + jh) = V + h \frac{\partial V}{\partial t} + \sigma \sqrt{h} \frac{\partial V}{\partial N} + \frac{1}{2} \sigma^2 h \frac{\partial^2 V}{\partial N^2} + O(h^{3/2})
\]  

(27)

where all functions on the RHS of (27) are evaluated at \((N(i, j - 1), t + (j - 1)h)\); likewise,

\[
V(N(i_d, j), t + jh) = V + h \frac{\partial V}{\partial t} - \sigma \sqrt{h} \frac{\partial V}{\partial N} + \frac{1}{2} \sigma^2 h \frac{\partial^2 V}{\partial N^2} + O(h^{3/2}).
\]  

(28)

Substituting (27) and (28) into (26), and applying a first-order Taylor’s expansion to the first instance of \(e^{-yh}\), and to \( p \) as given by (19),

\[
\eta(i, j - 1) = h \left\{ -yV + \frac{\partial V}{\partial t} + \left( y - r - \frac{1}{2} \sigma^2 \right) \frac{\partial V}{\partial N} + \frac{1}{2} \sigma^2 h \frac{\partial^2 V}{\partial N^2} \right\} + O(h^{3/2})
+ e^{-yh} \{p\eta(i, j) + (1 - p)\eta(i_d, j)\}.\]

(29)

We show in the Appendix that \( V \) satisfies the PDE:

\[
-yV + \frac{\partial V}{\partial t} + \left( y - r - \frac{1}{2} \sigma^2 \right) \frac{\partial V}{\partial N} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial N^2} = 0.
\]  

(30)

\[\text{19 We show in the Appendix that (28) holds even at the reflecting barrier, where } N(i_d, j) \neq N(i, j - 1) - \sigma \sqrt{h}.\]
Hence,

$$|\eta(i, j - 1)| \leq O(h^{3/2}) + \max\{|\eta(\ell, j)|: \ell = 0, \ldots, j - 1\}. \quad (31)$$

The interpretation of (31) is that the backwards valuation recursion adds an error of at most \(O(h^{3/2})\) at each step; since the number of steps is inversely proportional to \(h\), the accumulated error at the base node \((0, 0)\) is therefore \(O(h^{1/2})\) and so tends to zero.

The validity of the final step in the above argument depends on the various errors being uniformly of the stated orders (i.e. in the sense of being uniformly bounded upon division by the specified powers of \(h\)). This in turn depends on the derivatives of \(V\) being sufficiently well-behaved over the region of the binomial approximation. Unfortunately, the time derivative of \(V\) is unbounded as we approach the expiry date \(T\), and the space derivative is unbounded in \(N\).\(^{20}\) In the Appendix, we detail a solution to this problem, based on a purely technical modification of the binomial scheme so that it operates not over the time interval \([t, T]\) but over \([t, T - \delta]\) for arbitrarily small \(\delta\), and so that the option value is replaced by its 'discounted intrinsic value' for very large \(N\).

**Remark.** The PDE (30) could, of course, be used as the basis of an alternative numerical technique for valuing lookbacks. We prefer our binomial scheme, however, for two reasons. Firstly, binomial methods maximize intuitive clarity. Secondly, we can adapt our binomial scheme (see the next Section) to cope with basing the security price extremum upon finite sampling; by contrast, the validity of the PDE is confined to the continuous sampling case.

### 8. Finite sampling

So far, this paper has maintained the assumption, made by the existing literature, that lookback contracts are based on the extrema of the entire continuous sample path of the underlying security price. In practice, continuous and undisputable records of securities prices are generally unavailable,\(^{21}\) and lookback contracts are often based on the extrema of a finite sample of prices,

---

\(^{20}\) This problem — among others — is probably why Cox and Rubinstein (1985, pp. 208–209) refer to an argument based on Taylor’s expansions as offering only ‘intuitive confirmation’ that the CRR binomial technique for valuing ordinary options converges to the Black–Scholes PDE.

\(^{21}\) The closest approximations to continuous availability are provided by a modest number of currencies and securities traded on exchanges. Certain foreign exchange rates are monitored by Reuters on a more or less continuous basis, as currency trading activity moves between successive financial centres around the world. Even these data, however, are interrupted by weekends and bank holidays. Various futures and other exchanges monitor the prices of all transactions throughout trading hours, and even publish daily high and low prices.
taken at specified intervals, typically daily. Obviously, limiting the set of prices of which the extremum is taken reduces the payoffs — and thus earlier values — of the options, but the magnitude of the reduction has not previously been investigated.

As we shall see below, a European lookback option under finite sampling can be regarded as a specialised variant of an option on the maximum of several risky assets, one for each outstanding sampling date. Where the number of sampling dates is large, however, this insight is not of great computational benefit. In fortunate contrast, our binomial approximation scheme can be adapted quite readily; moreover, it can be applied to American lookbacks also.

Consider a European lookback put\(^{22}\) option to sell one unit of the security at time \(T\) at the maximum of the security prices observed at times \(T_1 < T_2 < \cdots < T_Q\). The payoff is

\[
V(T) = \max\{S(T_1) - S(T), \ldots, S(T_Q) - S(T), 0\}. \tag{32}
\]

Making the change of variables introduced in Section 5 enables us to transform (32) into

\[
V'(T) = \max\left\{ \frac{R(T)}{R(T_1)} - 1, \ldots, \frac{R(T)}{R(T_Q)} - 1, 0 \right\}. \tag{33}
\]

Thinking of the ratios \(R(T)/R(T_1), \ldots, R(T)/R(T_Q)\) as ‘asset prices’, the form of (33) is precisely that of the pay-off on a call option on the maximum of \(Q\) risky assets, with unit strike price. This is not just an idle thought, since, under the EMM, the logarithms of these ‘asset prices’ are jointly Normally distributed with conditional (i.e. given information available at \(t\)) means, for \(q = 1, \ldots, Q\):

\[
E_t\left[ \ln \frac{R(T)}{R(T_q)} \right] = \begin{cases} 
\ln \frac{R(t)}{R(T_q)} + \left( y - r - \frac{1}{2} \sigma^2 \right)(T - t), & T_q \leq t, \\
\left( y - r - \frac{1}{2} \sigma^2 \right)(T - t_q), & T_q > t,
\end{cases} \tag{34a}
\]

and (co-)variances, for \(q_1, q_2 = 1, \ldots, Q\):

\[
\text{cov}_t\left[ \ln \frac{R(T)}{R(T_{q_1})}, \ln \frac{R(T)}{R(T_{q_2})} \right] = (T - \max\{t, T_{q_1}, T_{q_2}\})\sigma^2. \tag{34b}
\]

We may therefore apply the analytical valuation results for options on the maximum of several risky assets, due to Johnson (1987) and Babbs and Salkin

\(^{22}\)Once more we focus on puts; analogous arguments apply to calls.
Unfortunately, those results involve multidimensional Normal integrals, and thus become computationally cumbersome in high dimensions, i.e. when the number of outstanding sampling dates is large. Nevertheless, for small numbers of sampling dates, the above insight will provide us with a check on the accuracy of our modified binomial approximation scheme, to which we now turn.

Fresh consideration of the pay-off of the lookback put reveals that we may still write (allowing for exercise of American options before $T$):

$$V^I(t') = \exp \{ N(t') \} - 1$$

as long as we make the modified definition:

$$N(t) \equiv \ln R(t) - \min \{ \ln R(T_q): T_q \leq t \}.$$  \hfill (35)

From (35), we can see that, between sampling dates, the increments of $N$ are identical to those of $\ln R(t)$. At sampling dates, if $N$ has become negative, it is reset to zero. As in the continuous sampling case, $N$ is the sole state variable required.

The modifications required to our binomial approximation scheme are intuitively obvious: rather than reflecting $N$ upwards whenever it would otherwise become negative, we allocate sampling dates to time points (i.e. the ends of time steps) and simply reset to zero any negative values of $N$ occurring at those times.

Fig. 2. Modified lattice scheme for finite sampling.
Fig. 2 relates to the ‘normal’ case where $k \geq 0$. The formal description is complicated by the need to cover also the ‘inverted’ case where $k < 0$ which can arise when the valuation date $t$ lies between sampling dates.

Clearly, this entire allocation procedure involves an element of approximation, but whose impact will tend to zero as the time step length is reduced.

Let $j^*$ be the number of the time step to whose end we allocate the first sampling date (if any) to cause some values of $N$ to be reset to zero. More formally, define 

$$A \equiv \{ j > \min\{k, 0\}; n - j \text{ even}, \text{ and } \{T_1, \ldots, T_Q\} \cap (t, t+jh] \neq \emptyset \}$$

then

$$j^* = \begin{cases} \min\{ j \in A \}, & A \neq \emptyset, \\ \infty, & A = \emptyset. \end{cases}$$

We say that ‘sampling occurs at’ $\ell \geq j^*$ if $n - \ell$ is even and $\{T_1, \ldots, T_Q\} \cap [t + (\ell - 1)h, t + \ell h] \neq \emptyset$.

Clearly, the number, $I_j$, of nodes at the end of the $j$th time step, and the corresponding values of $N$, depend on whether the time step is before or after the $j^*$th. In the latter case, the number and values of the nodes depend on the length of time since the most recent sampling date and upon whether all or only some of the nodal values of $N$ were reset to zero at $j^*$. Thus

$$I_j = \begin{cases} j+1, & \text{if } j < j^*, \\ \frac{2j - j^* + 2 - j}{2}, & \text{if } j \geq j^* \text{ and } k + j^* < 0, \\ \frac{2j - j^* + 2 - j}{2} + \text{integer part of } \frac{2j - j^* + 2 + k}{2}, & \text{if } j \geq j^* \text{ and } k + j^* \geq 0. \end{cases}$$

---

23 Fig. 2 relates to the ‘normal’ case where $k \geq 0$. The formal description is complicated by the need to cover also the ‘inverted’ case where $k < 0$ which can arise when the valuation date $t$ lies between sampling dates.

24 Clearly, this entire allocation procedure involves an element of approximation, but whose impact will tend to zero as the time step length is reduced.

25 The first case of this node count is obvious. The second case follows from the fact that all nodes are reset to zero at $j^*$. The third case adds the additional nodes which arise at $j$, arising from nodes not reset to zero at $j^*$. The fact that we have forced both $j^*$ and $j$ to be even provides important simplifications.
where

\[ j' = \max\{\ell : j^* \leq \ell \leq j\} \text{ and } '\text{'sampling occurs at' } \ell', j = j^*, \ldots, n. \]

The values of \( N \) at the end of the \( j \)th time step, indexed \((0, j), \ldots, (I_j - 1, j)\), are given for \( j < j^* \), by

\[ N(i, j) = N(t) + (j - 2i)\sigma \sqrt{h}, \quad i = 0, \ldots, I_j - 1 \]

while, for \( j \geq j^* \), if \( k + j^* < 0 \),

\[ N(i, j) = (j - j^* - 2i)\sigma \sqrt{h}, \quad i = 0, \ldots, I_j - 1 \]

and, if \( k + j^* \geq 0 \),

\[
N(i, j) = \begin{cases} 
N(t) + (j - 2i)\sigma \sqrt{h}, & i = 0, \ldots, I_{j^*} - 2, \\
(j - j^* - i + I_{j^*} - 1)\sigma \sqrt{h}, & \text{odd } (i - I_{j^*} + 2) > 0, \\
N(t) + (j - i - I_{j^*} + 2)\sigma \sqrt{h}, & \text{even } (i - I_{j^*} + 2) > 0.
\end{cases}
\]

The evolution of \( N \) over the lattice is described, in terms of risk neutral probabilities, by the relation that, if \( N(t + (j - 1)h) = N(i, j - 1) \), then

\[ N(t + jh) = \begin{cases} 
N(i_u, j) \text{ with probability } p, \\
N(i_d, j) \text{ with probability } 1 - p,
\end{cases}
\]

where for \( i \geq I_j - 1 \) (i.e. for the lowest nodes just prior to sampling), \( i_u = i_d = I_j - 1 \) whereas, for \( i < I_j - 1 \), \( i_u = i \) and

\[ i_d = \begin{cases} 
i + 1, & \text{otherwise}. \\
i + 2, & j > j^* \geq -k \text{ and } I_{j^*} - 2 \leq i < I_j - 2,
\end{cases} \]

We evaluate the option over this lattice by the terminal condition\(^{26}\)

\[ V(i, n) = \max\{e^{N(i, n)} - 1, 0\}, \quad i = 0, \ldots, I_n - 1 \]

and the backwards recursion over \( j = n, n - 1, \ldots, 1 \):

\[ V(i, j - 1) = e^{-yh}(pV(i_u, j) + (1 - p)V(i_d, j)), \quad i = 0, \ldots, I_j - 1. \]

The option value (in units of our numeraire) is given by \( V(0, 0) \). The results presented in Table 2 and Fig. 3 illustrate the impact of finite sampling upon the values of lookback options.

The most striking feature to emerge from Table 2 is that alterations in sampling frequency have a significant impact on option value. Even a switch

\(^{26}\) We allow here for the case that the expiry date \( T \) is not a sampling date, creating a possibility of a lookback option expiring out-of-the-money.
Table 2
Lookback option values under finite sampling

<table>
<thead>
<tr>
<th>Sampling</th>
<th>European puts</th>
<th></th>
<th>American puts</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Binomial time steps</td>
<td>Numerical integration</td>
<td>Binomial time steps</td>
<td>Numerical integration</td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>2000</td>
<td>6000</td>
<td>200</td>
</tr>
<tr>
<td>Quarterly</td>
<td>4.55</td>
<td>4.57</td>
<td>4.57</td>
<td>4.57</td>
</tr>
<tr>
<td>Weekly</td>
<td>7.57</td>
<td>7.65</td>
<td>7.66</td>
<td>na</td>
</tr>
<tr>
<td>Daily</td>
<td>8.25</td>
<td>8.62</td>
<td>8.64</td>
<td>na</td>
</tr>
<tr>
<td>Continuous</td>
<td>9.28</td>
<td>9.29</td>
<td>9.29</td>
<td>9.29</td>
</tr>
</tbody>
</table>

This table employs the same stock price, interest rate and volatility assumptions as Table 1, and values the same European and American put options, save that, except in the final line of Table 2, the contract is based on the maximum stock price observed at various regular intervals — corresponding roughly to the frequencies indicated — rather than continuously. The fourth column of values is based upon the insight that a lookback put option based on finite sampling at Q dates can be viewed as an option on the maximum of Q assets, and uses numerical integration to compute the expected payoff (under the EMM of Sections 2 and 3). The integration scheme is an adaptation of that of Haselgrove (1961) and required at most 3500 computations of the integrand to converge to two decimal places.

This explanation does, however, beg the question of why, under daily sampling, a binomial valuation using 200 time steps does not produce a value closer to the result under continuous sampling. Intuitively expressed, the answer is that, whereas under continuous sampling, N is reflected upwards as soon as it reaches zero, under finite sampling N is merely reset to zero at the nearest convenient time point; for the parameter values in the table, the cumulative effect of these latter more ‘feeble’ reflections is appreciable. Moreover, we decided that ‘convenient’ time points occur only at even numbers of time steps from expiry; thus, for example, with only 200 time steps in all, only 100 provide ‘convenient’ time points — coming closer to sampling every two days rather than each day, let alone continuously.

from continuous to daily sampling reduces option values by about 0.65. Throughout the table, changes in sampling frequency affects European and American option values more or less equally. The results obtained by numerical integration confirm the validity of the adapted binomial scheme. For finite but frequent sampling, larger numbers of time steps (approximately 2000 for weekly or daily sampling) are required to obtain the degree of accuracy obtained with 150 steps under continuous sampling; this should not surprise us since the finer detail of a finite sampling scheme is inevitably lost unless the number of time steps is several times the number of sampling dates.27

27 This explanation does, however, beg the question of why, under daily sampling, a binomial valuation using 200 time steps does not produce a value closer to the result under continuous sampling. Intuitively expressed, the answer is that, whereas under continuous sampling, N is reflected upwards as soon as it reaches zero, under finite sampling N is merely reset to zero at the nearest convenient time point; for the parameter values in the table, the cumulative effect of these latter more ‘feeble’ reflections is appreciable. Moreover, we decided that ‘convenient’ time points occur only at even numbers of time steps from expiry; thus, for example, with only 200 time steps in all, only 100 provide ‘convenient’ time points — coming closer to sampling every two days rather than each day, let alone continuously.
Fig. 3 illustrates the varying impact of finite sampling as the maturity of an option shortens by plotting against time the values of an at-the-money European put under continuous and daily sampling, using the same stock price, interest rate and volatility data as in the tables. The striking feature of the graph is the way the difference between the value of the option under continuous and daily sampling holds up until just before expiration: from 0.66 at six months, the difference declines only very gradually until a few days before expiry; with one day to expiry the difference has only fallen to 0.42.

9. Summary and concluding remarks

The paper by Goldman et al. (1979a) on European lookback options prompted the commercial introduction of the contingent claims they analysed. The virtual absence of American lookback options may be due in part to the unavailability of a straightforward and computationally efficient valuation technique. We have now provided just such a technique in the form of a binomial approximation scheme. Moreover, the method by which we established convergence was novel, direct, and straightforward; it may well be of wider use.

Existing research on lookback contracts has assumed that they are based on continuous sampling of the price of the underlying security. We have adapted our binomial scheme to cover contracts under which sampling occurs at a fixed finite set of dates; the impact on option values was significant. We note that the nature of the sampling regime may also have a significant impact on other types of path-dependent contingent claims, such as the related claims analysed by
Conze and Viswanathan (1991), and — especially when the underlying security price is close to the barrier/dropout level — on ‘barrier’ and ‘dropout’ options.

This paper has been restricted to binomial approximation, largely for ease of exposition. A number of authors (see e.g. Kamrad and Ritchken, 1991) have reported gains in computational efficiency from replacing binomial approximation schemes for conventional options by multinomial ones. We would expect similar gains to be achievable for lookback options.

We hope that our paper will stimulate attempts to develop convenient approximation techniques for other types of contingent claims, e.g. ‘average price’ and ‘average strike’ options, whose payoff functions contain path-dependent terms.

10. Disclaimer

Views expressed in this paper are those of the author, and not necessarily those of any organisation to which he is or has been affiliated.

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Appendix A

The treatment of convergence in Section 7 deferred a number of matters for further attention in this Appendix; we deal with them in turn. We first establish that (28) remains valid where \((i, j - 1)\) is a node adjacent to the reflecting barrier. For such a node, as noted in footnote 19, \(N(i_d, j) \neq N(i, j - 1) - \sigma \sqrt{h};\) instead, we have \(N(i, j - 1) = \lambda \sigma \sqrt{h}\) where \(\lambda\) equals either \(f\) or \(1 - f\) [as defined in (23)], and \(N(i_d, j) = (1 - \lambda)\sigma \sqrt{h}\). The equivalent expansion to (28) is

\[
V(N(i_d, j), t + jh) = V + h \frac{\partial V}{\partial t} + (1 - 2\lambda)\sigma \sqrt{h} \frac{\partial V}{\partial N} + \frac{1}{2} (1 - 2\lambda)^2 \sigma^2 h \frac{\partial^2 V}{\partial N^2} + O(h^{3/2})
\]

(A.1)

where all functions on the RHS are evaluated at \((\lambda \sigma \sqrt{h}, t + (j - 1)h)\).
Now, if we make a Taylor’s expansion of $\partial V/\partial N$ about $(\lambda \sigma \sqrt{h}, t + (j - 1)h)$ and exploit the fact that $(\partial V/\partial N)(0, u) = 0, \forall u$ we obtain

$$0 = \frac{\partial V}{\partial N}(0, t + (j - 1)h) = \frac{\partial V}{\partial N}(\lambda \sigma \sqrt{h}, t + (j - 1)h)$$

$$- \lambda \sigma \sqrt{h} \frac{\partial^2 V}{\partial N^2}(\lambda \sigma \sqrt{h}, t + (j - 1)h) + O(h). \quad (A.2)$$

Substituting (A.2) into (A.1), we can re-arrange to obtain (28) as required.

Secondly, we establish that the option value $V$ (as measured in units of the security price) satisfies the PDE (30).

As noted in Section 5, following our change of numeraire, $N$ serves as the sole state variable for $\leq$, and the process for $N$ is given, under the EMM, by (18). Hence, applying the relevant extension of Ito’s lemma (see, e.g. Harrison, 1985, Proposition 6.3, p. 67) to the discounted option value $e^{-yt}V$ as a function of $N$ and time, we obtain

$$d(e^{-yt}V) = e^{-yt} \left\{ -yV + \frac{\partial V}{\partial t} + \left(y - r - \frac{1}{2} \sigma^2 \right) \frac{\partial V}{\partial N} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial N^2} \right\} dt$$

$$+ e^{-yt} \frac{\partial V}{\partial N} \bigg|_{N=0} dL + e^{-yt} \sigma \frac{\partial V}{\partial N} dZ. \quad (A.3)$$

Now, the discounted option value must be a martingale under the EMM. By arguments identical to those set out in footnote 11, therefore, the first and second terms on the RHS of (A.3) must separately vanish. The coefficient of $dt$ in the former of those terms is precisely the LHS of (30).

The third, and perhaps most important task of this Appendix is to modify our binomial scheme so that the error terms arising at each time step, as expressed in (27)–(31), are uniformly of the orders stated, while ensuring that the errors propagated from the values of $\eta$ at the boundaries of the binomial scheme can simultaneously be made arbitrarily small. As indicated in Section 7, this will complete our proof of convergence.

Consideration of (8)–(9a) reveals that, from the expiry date, $T$, where the time derivative of $V$ does not exist, $V$ is infinitely differentiable, and thus the Taylor’s expansions reflected in (27)–(31) hold uniformly on compact sets. We will seek to exploit this.

Close to the expiry date, $T$, or for large values of $N$ at any time, the uncertainty as to the eventual pay-off of the option shrinks towards zero. Formally, we define $H$ as the difference between the option value $V$ and the
option’s ‘discounted forward intrinsic value’:

\[ H(N, u) \equiv V(N, u) - e^{-(T-u)\gamma} \left( e^{(T-u)(y-r)} - 1 \right) \].

\[ \text{(A.4)} \]

If we re-express (8)-(9a)\(^{28}\) in units of the security price, and in terms of \( N \), and substitute in (A.4), elementary manipulations yield

\[
H(N, u) = \left( 1 + \frac{1}{a} \right) e^{-(T-u)y} \Phi(\sigma\sqrt{T-u-z_1}) - e^{N-(T-u)r} \Phi(-z_1)
\]

\[
- \frac{e^{aN-(T-u)r}}{a} \Phi(-z_2)
\]

\[ \rightarrow 0 \text{ as } N \rightarrow \infty, \text{ uniformly on } [t, T]. \]

Moreover, we can deduce from (A.4) that \( H(\cdot) \) is a continuous function which vanishes on \( u = T \). Over any bounded range of values of \( N \), therefore, \( H(\cdot) \) tends uniformly to zero as \( u \rightarrow T \).

We are now ready to present our modified binomial scheme. For any \( \varepsilon > 0 \), choose \( N^*, \delta > 0 \) such that:

\[ |H(\cdot)| < \frac{\varepsilon}{2} \text{ outside } [0, N^*] \times [t, T-\delta]. \]

Run the binomial scheme over the interval \([t, T-\delta]\) instead \([t, T]\) as previously, and, at the final time step \((j=n)\) and for any node \((i,j)\) at which \( N(i,j) > N^* \), set the option value equal to the ‘discounted forward intrinsic value’ i.e. set

\[
\hat{V}(i,j) = \exp\{N(i,j) - (T-t-jh)r\} - \exp\{ - (T-t-jh)y \}.
\]

Then, assuming without loss of generality that we confine attention to \( h \leq \text{some } h_0 \), we know that all our Taylor’s expansions are taking place within the compact set \( [0,N^*+\sigma\sqrt{h_0}] \times [T-\delta] \) and thus hold uniformly. Moreover, on any boundary of our binomial scheme — by which we mean at \( j = n \) or for \( N(i,j) > N^* \), the error term \( \eta(i,j) \) is bounded by \( \varepsilon/2 \). We deduce that (31) holds uniformly in \( h \), and by recursion across reducing

\[^{28}\text{The same convergence properties emerge in the } y = r \text{ case, for which (9b) applies in place of (9a).} \]
values of \( j \) gives us:

\[
|\eta(i, j)| = O(h^{1/2}) + \varepsilon/2 < \varepsilon \text{ as } h \to 0
\]

thus establishing our desired convergence result.

References

Babbs, S.H., 1986. FX Hindsight Options, manuscript memorandum.