Optimal portfolio policies with borrowing and shortsale constraints

Lucie Teplá*

INSEAD, Boulevard de Constance, 77305 Fontainebleau cedex, France
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Abstract

We characterize optimal intertemporal portfolio policies for investors with CRRA utility facing either a borrowing constraint, or shortsale restrictions, or both. The optimal constrained portfolios are identified as optimal unconstrained portfolios for a higher riskless rate, or for a subset of the risky assets, or for a combination of the two settings. Our characterization is based on duality results in Cvitanić and Karatzas (1992, Annals of Applied Probability 2, 767–818) for optimal portfolio investment when portfolio values are more generally constrained to a closed, convex, nonempty subset of $\mathbb{R}^n$. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

This paper characterizes optimal intertemporal portfolio policies for CRRA-utility investors facing either a borrowing limit on the total wealth invested in

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*Tel.: +33-1-60-72-44-85; fax: +33-1-60-72-40-45.
E-mail address: lucie.tepla@insead.fr (L. Teplá).

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the risky assets, or shortsale restrictions on all risky assets, or both. The characterization is based on the first-order conditions to a minimization problem identified by Cvitanić and Karatzas (1992) as underlying the dual formulation of the optimal portfolio investment problem when portfolio values are more generally constrained to a closed, convex, nonempty subset of $\mathbb{R}^n$ (when there are $n$ risky assets). In each setting, the optimal constrained portfolio is identified as an optimal ‘unconstrained’ portfolio. Specifically, with borrowing constraints only, CRRA-utility investors act as if unconstrained but facing a higher interest rate. With shortsale constraints only, these investors act as if unconstrained when investing only in a subset of the risky assets. With borrowing and shortsale constraints, both effects obtain. Specifically, the optimal portfolio is equivalent to the optimal borrowing-constrained-only portfolio for a subset of the risky assets, and thus to the optimal unconstrained investment in these assets at a higher interest rate.

Results closely related to a number of those derived here in a dynamic setting have previously been identified as holding in a one-period, mean-variance or Markowitz framework. Black (1972) establishes that an investor who cannot borrow at all chooses a tangency portfolio corresponding to a higher interest rate. Brennan (1971) considers the setting in which the investor can borrow without limit, but faces a borrowing rate which is greater than the lending rate. The optimal portfolio is again equivalent to a tangency portfolio, in this case corresponding to one of three possible ‘risk-free’ rates.1 Separately, Lintner (1965) identifies the optimal shortsale-constrained Markowitz portfolio as the optimal unconstrained portfolio for a subset of the risky securities. The fact that we obtain very similar results for CRRA-utility investors in the dynamic setting is not entirely unexpected given that it is well-known that, for the model we consider, these investors’ optimal unconstrained portfolios are instantaneously mean-variance efficient.2 Nonetheless, to date, some of these results, particularly those concerning shortsale constraints, have been missing from the continuous-time literature. Grossman and Vila (1992), using a stochastic dynamic programming approach, study the optimal intertemporal portfolio policies of a borrowing-constrained power-utility investor in the standard Merton (constant-coefficient) setting. Rather than restricting investment in the risky assets to be less than some constant proportion of wealth, as we do here, Grossman and Vila consider the effects of a borrowing limit which is affine in wealth.3 Because their model features only one risky asset, it does not identify how a borrowing

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1 The relevant rate is either the borrowing rate, the lending rate, or a rate at which the optimal investment results in neither borrowing nor lending.

2 See, for example, Merton (1990, pp. 170–171). Here we assume a deterministic investment opportunity set, so there is no hedging demand.

3 This feature leads the investor to alter his optimal portfolio holdings even when the constraint is not binding (see Grossman and Vila, 1992).
restriction affects relative investment in different risky assets. Fleming and Zariphopoulou (1991) arrive at results analogous to those in Brennan (1971) for a power-utility investor facing differing borrowing and lending rates. Their model also features one risky asset. Xu and Shreve (1992a,b) use a martingale approach and duality methods to characterize solutions subject only to shortsale constraints in a model with multiple risky assets. Although Xu and Shreve establish that, in the constant-coefficient setting, a ‘mutual fund’ theorem still holds, they do not explicitly identify this mutual fund. Optimal portfolio policies when portfolio values are more generally constrained to a closed, convex, nonempty subset, $K$, of $\mathbb{R}^n$, are studied by Cvitanic and Karatzas (1992). These results form the departure point for the analysis in this paper. In their paper, Cvitanic and Karatzas treat an example, with borrowing and shortsale constraints, for a log-utility investor with two risky assets having uncorrelated returns of equal volatility. Their approach, which rests on exhaustive enumeration of optimal portfolios given various relationships between different parameters of the model, is not ideally suited to a setting with a much larger number of risky assets, whose returns may be correlated.

In the following section we describe the economy and formulate the unconstrained investor’s problem. Section 2 reviews key results for the constrained problem from Cvitanic and Karatzas (1992). Building on these results, Section 3 examines the optimal policies of the CRRA-utility investor when borrowing and shortsale constraints are imposed, first separately and then concurrently. In the borrowing-constrained-only setting, the investor’s optimal portfolio is identified explicitly. Whenever shortsale constraints are present and binding, it is not known a priori which assets are held in positive amounts in the optimal portfolio. In such instances, further characterization of the optimal constrained portfolios is provided, leading to an algorithm for their calculation. Section 4 concludes.

2. The economic setting and unconstrained problem

We consider a standard model of a dynamically complete financial market defined on the time interval $[0, T]$. The market consists of a bond and $n$ risky securities, whose price processes satisfy

$$dP_0^0 = P_0^0 r_t \, dt, \quad P_0^0 = 1,$$

$$dP_t = P_t (\mu_t \, dt + \sigma_t \, dw_t),$$

respectively, where $w$ is an $n$-dimensional standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$. We let $\mathcal{F}_t$ denote the $P$-augmentation of

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4 Whereby all investors, regardless of preferences, invest in the same portfolio of risky securities.
the filtration generated by \( w \). The functions \( r_t: \Omega \times [0, T] \to \mathbb{R}_+, \mu_t: \Omega \times [0, T] \to \mathbb{R}^n \) and \( \sigma_t: \Omega \times [0, T] \to \mathbb{R}^{n \times n} \) are \( \mathcal{F}_t \)-progressively measurable. We denote the \( i \)th row of \( \sigma_t \) by \( s_{i,t} \). The instantaneous covariance matrix of the risky securities, \( V_t \equiv \sigma_t \sigma_t^T \), is assumed to be non-singular for all \( t \in [0, T] \). The \( \mathbb{R}^n \)-valued market risk premium process \( \theta_t \equiv \sigma_t^{-1}(\mu_t - r_t \cdot 1) \), where \( 1 \) denotes the vector whose every component is one, satisfies the Novikov condition \( E[\int_0^T \exp(\frac{1}{2}||\theta_t||^2) \, dt] < \infty \).

The investor in our model is endowed with initial wealth \( W_0 > 0 \) and at each time \( t \in [0, T] \) chooses the proportions \( \pi_t = (\pi_{1,t}, \pi_{2,t}, \ldots, \pi_{n,t}) \), of wealth to invest in the risky securities. The process \( \pi \) is \( \{\mathcal{F}_t\} \)-progressively measurable, and satisfies \( \int_0^T ||\pi_t|| \, dt < \infty \) a.s. The balance of the investor’s wealth is invested in the bond. The investor’s unconstrained optimization problem is to maximize expected terminal utility of wealth, subject to the standard dynamic budget constraint

\[
\max_{\pi} \quad E[U(W_T)] \\
\text{s.t.} \quad dW_t = (r_t + \pi_t^T(\mu_t - r_t \cdot 1))W_t \, dt + \pi_t^T \sigma_t \, dW_t.
\]

The investor is assumed to have CRRA preferences of the form \( U(W_T) = W_T^{\alpha}/\alpha \), where\(^5\) \( 0 < \alpha < 1 \).

### 3. The constrained problem

When the investor faces portfolio constraints, the market becomes incomplete and there will be more than one set of prices consistent with no arbitrage. Corresponding to each such price system is a complete ‘auxiliary’ financial market in which the riskfree rate and the security returns are adjusted from their original values. The manner in which these parameters can be modified without giving rise to arbitrage opportunities is determined by the nature of the portfolio constraints. In the most general case, considered by Cvitanić and Karatzas (1992, hereafter C&K), in which portfolio values are constrained to a closed, convex, non-empty subset \( K \) of \( \mathbb{R}^n \), the price processes of the riskfree and risky assets in any arbitrage-free auxiliary market satisfy\(^6\)

\[
dP^0_t = P^0_t(r_t + \delta(t)) \, dt, \quad P^0_t = 1, \tag{1}
\]

\[
dP_t = P_t((\mu_t + \delta(t) \cdot 1) \, dt + \sigma_t \, dw_t), \tag{2}
\]

\(^5\) Following Cvitanić and Karatzas (1992). The results extend trivially to the cases \( \alpha > 1 \) and \( \alpha < 0 \). Identical qualitative results can be derived for logarithmic preferences employing a similar line of analysis as for power utility. The logarithmic case is briefly discussed in the following section.

\(^6\) See C&K (p. 776, Eqs. (8.3) and (8.4)).
where \( \delta(x) \) and \( \tilde{K} \), respectively, the support function and the barrier cone of the set \(-K\), are defined by\(^7\)

\[
\delta(x) \equiv \delta(x|K) \equiv \sup_{x \in K} (-x^T) ,
\]

\[
\tilde{K} \equiv \{ x \in \mathbb{R}^n ; \delta(x|K) < \infty \}
\]

and \( v = \{v_t; 0 \leq t \leq T\} \) is a process in the space \( V(\tilde{K}) \) of square-integrable, progressively measurable processes taking values in \( \tilde{K} \).

C&K subsequently establish that the investor’s optimal constrained portfolio is equivalent to the optimal unconstrained portfolio in the auxiliary market corresponding to a particular \( v^* \in V(\tilde{K}) \). Thus, employing a standard martingale approach,\(^8\) the optimal portfolio can be shown to be one which finances the terminal wealth \( W_T^* \), where \( W_T^* \) solves:

\[
\sup_{W_t} E[U(W_T)] \tag{3}
\]

s.t. \( E[W_TH_T^{v*}] \leq W_0, \tag{4} \)

where \( H^{v*} \), the state-price density for this auxiliary market, satisfies

\[
dH_t^{v*} = -(r_t + \delta(v_t^*))H_t^{v*} dt - (\theta_t + \sigma_t^{-1}v_t^*)H_t^{v*} dw_t, \quad H_0^{v*} = 1. \tag{5} \]

C&K identify \( v^* \) as the solution to the dual problem\(^9\)

\[
\min_{y \in V(\tilde{K})} E[\tilde{U}(yH_T^{v})], \quad 0 < y < \infty, \tag{6} \]

where \( H^v \) is defined analogously to \( H^{v*} \) in (5) for all \( v \in \tilde{K}, \)

\[
\tilde{U}(x) \equiv \sup_{z > 0} [U(z) - zx], \quad 0 < x < \infty \]

donates the convex conjugate of \(-U(-z)\), and \( y \) denotes the Lagrange multiplier for the constraint in (4).

For the investor with logarithmic preferences of the form \( U(W_T) = \ln W_T \), C&K show via this dual formulation\(^10\) that \( v^* \) is given by pointwise minimization over \( v \in \tilde{K} \) for all \( t \in [0, T] \) of

\[
\|\theta_t + \sigma_t^{-1}v_t\|^2 + 2\delta(v_t) \tag{7} \]

\(^7\) See C&K (p. 771, Eqs. (4.1) and (4.2)).

\(^8\) See Karatzas et al. (1987), or Cox and Huang (1989).

\(^9\) See Section 12 of C&K. Xu and Shreve (1992a,b) also use a dual approach to characterize optimal portfolio policies when the investor faces shortsale constraints only.

\(^10\) See Section 11 of C&K.
and that the investor’s optimal portfolio is given by
\[ \pi_t = V_t^{-1}(\mu_t - r_t \cdot 1 + \nu_t^*). \] (8)
For other utility specifications and deterministic coefficients, the optimal \( \nu^* \) is characterized as the solution of the ‘dual’ HJB problem defined in (6). That is, \( \nu^* \) solves\(^{11}\)
\[
Q_t - yQ_y r_t + \min_{v \in \bar{K}} \left[ \frac{1}{2} y^2 Q_{yy} ||\theta_t + \sigma^{-1}_t v_t||^2 - y Q_y \delta(v_t) \right] = 0, \quad t \in [0, T) \] (9)
with the boundary condition \( Q(T, y) = \bar{U}(y) \). The investor’s optimal portfolio is identified as\(^{12}\)
\[
\pi_t = - V_t^{-1}(\mu_t - r_t \cdot 1 + \nu_t^*) \frac{U'(W_t)}{W_t U''(W_t)}, \]
(10)
where \( W_t \) denotes the investor’s wealth at the optimal policy. Thus, if \( \delta(v) = 0 \) for all \( v \in \bar{K}, \nu_t^* \) is determined simply by minimizing \( ||\theta_t + \sigma^{-1}_t v_t||^2 \). Otherwise it is generally necessary to first derive the form of \( Q \). C&K perform this exercise for the investor with power utility of the form \( U(W_T) = W_T^\alpha / \alpha \), where \( 0 < \alpha < 1 \), and establish in this case that \( \nu^* \) is independent of \( y \) and that it is given by\(^{13}\)
\[
\nu_t^* = \arg \min_{v \in \bar{K}} \left[ ||\theta_t + \sigma^{-1}_t v_t||^2 + 2(1 - \alpha)\delta(v_t) \right]. \]
(11)
From (10), the optimal portfolio for this investor is
\[
\pi_t = \frac{1}{1 - \alpha} V_t^{-1}(\mu_t - r_t \cdot 1 + \nu_t^*). \]
(12)
In the following section we analyze the optimal borrowing and/or shortsale-constrained policies of the power-utility investor in a setting with deterministic coefficients, by exploiting the first-order conditions for (11). The same analysis, based on the first-order conditions to (7), can be carried out for the case of logarithmic preferences and leads to the same qualitative results. For notational simplicity, we suppress the dependence of all variables on \( t \).

4. The power-utility investor

4.1. Borrowing constraints

We consider a borrowing constraint which places a maximum limit \( a \) on the proportion of wealth that can be invested in the risky assets. Given this

\(^{11}\) See C&K (p. 800, Eqs. (15.1) and (15.2)). Subscripts on \( Q \) in Eq. (9) denote partial derivatives.

\(^{12}\) See C&K (p. 803, Eq. (15.10)).

\(^{13}\) See C&K (p. 802, Eq. (15.7)).
assumption, we have that, independent of investor preferences,

\[ K = \{ \pi \in \mathbb{R}^n; \pi^T 1 \leq a \} \]

and that the support function, \( \delta \), and the barrier cone, \( \bar{K} \), of \( -K \) are given by

\[ \delta(x) = \begin{cases} 
- a\bar{x} & \text{if } x = \bar{x} \cdot 1 \text{ for some scalar } \bar{x} \leq 0, \\
\infty & \text{otherwise,} 
\end{cases} \]

\[ \bar{K} = \{ x \in \mathbb{R}^n; x = \bar{x} \cdot 1 \text{ for some scalar } \bar{x} \leq 0 \}, \]

respectively. The problem in (11) becomes

\[
\min_{\nu \in \mathbb{R}^n} \left[ \| \theta + \sigma^{-1} \nu \|^2 + 2(1 - z)\delta(\nu) \right] \quad \text{s.t.} \quad \nu \leq 0, \quad (13)
\]

where \( \nu \equiv \tilde{\nu} \cdot 1 \), for some non-positive scalar \( \tilde{\nu} \). If we introduce the Lagrange multiplier process \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \) for the constraint in (13), we obtain the first-order condition

\[
1^T V^{-1}(\mu - r \cdot 1) + 1^T V^{-1} \nu^* - a(1 - z) - \lambda = 0, \quad (14)
\]

where \( \nu^* = \tilde{\nu}^* \cdot 1 \). From the complementary slackness conditions, if \( \nu^* = 0 \) (and \( \lambda \geq 0 \)) then we obtain the unconstrained solution,

\[
\pi = \frac{1}{1 - z} V^{-1}(\mu - r \cdot 1). 
\]

Alternatively, if \( \nu^* \leq 0 \) and \( \lambda = 0 \), the optimal portfolio is given by

\[
\pi = \frac{1}{1 - z} V^{-1}(\mu - (r - \tilde{\nu}^*) \cdot 1) \quad \text{where} \quad \tilde{\nu}^* = \frac{a(1 - z) - 1^T V^{-1}(\mu - r \cdot 1)}{1^T V^{-1}1}. 
\]

Thus, when the borrowing constraint is binding, a power-utility investor acts as if unconstrained but facing a higher interest rate (given that \( \tilde{\nu}^* \) is negative).

Using results in later sections,\(^\text{14}\) it is possible to derive an expression for \( \pi_i(a) \), the optimal proportion of wealth invested in any individual risky asset as a function of the borrowing limit. Straightforward differentiation indicates that while \( \pi_i(a) \) is linear in \( a \), this relationship may be either positive or negative. Thus, whereas the imposition of a borrowing constraint in a model with only one risky asset will always result in reduced investment in that asset relative to an unconstrained setting, with multiple risky assets it may be the case that

\(^{14}\) Employing Lemma 1 with \( m = n - 1 \) and \( \Theta = (\mu - (r - v^*) \cdot 1) \) gives

\[
\pi_i(a) = \frac{(\mu_i - (r - v^*)) - s_i \sigma_{i-1}^T V_{n-1}^{-1}(\mu - (r - v^*) \cdot 1)_{n-1}}{s_i (I - \sigma_{i-1}^T V_{n-1}^{-1} \sigma_{i-1}^T)^T},
\]

where \( V_{n-1} = \sigma_{n-1} \sigma_{n-1}^T \) denotes the instantaneous covariance matrix for the \((n - 1)\) assets not including asset \( i \) and \((\mu - (r - v^*) \cdot 1)_{n-1}\) denotes the adjusted excess return vector for these assets.
investment in some of the risky assets actually increases when the investor is borrowing-constrained.\textsuperscript{15} In the simplest two-asset setting, it can be shown that investment in the first risky asset increases whereas investment in the second decreases, relative to the unconstrained setting, if

\[ \sigma_1^2 > \sigma_{12} > \sigma_2^2, \] (15)

that is, if the correlation between the two assets is sufficiently high.\textsuperscript{16}

To understand what is happening, it is easiest to consider how the tangency point on the mean-variance efficient frontier changes with the risk-free rate when there are two risky assets. As the risk-free rate of interest increases (which is how a binding borrowing constraint can be interpreted), then the slope of the tangency line decreases and the proportion of total wealth invested in the asset with the higher variance increases. If the covariance between the two assets is high enough, then this increase outweighs the effect of a total overall decrease in investment due to the borrowing constraint.

4.2. Shortsall constraints

For the case of shortsall constraints only,

\[ K = \mathbb{R}_n^+ \equiv \{ \pi \in \mathbb{R}^n; \pi_i \geq 0, \forall i \}, \]

\[ \delta(x) = 0, \]

\[ \mathcal{K} = K. \]

The problem in (11) becomes

\[ \min_{v \in \mathbb{R}^n} \left[ ||\mu + \sigma^{-1}v||^2 + 2(1 - z)\delta(v) \right] \quad \text{s.t.} \quad -v \leq 0. \] (16)

If we introduce a Lagrange multiplier process \( \psi = (\psi_1, \psi_2, \ldots, \psi_n) \) for the constraint in (16), the first-order condition is

\[ V^{-1}(\mu - r \cdot 1 + v^*) + \psi = 0. \]

\textsuperscript{15}This concurs with Admati (1985) who establishes, in a rational expectations equilibrium model, that results and intuitions based on models with a single risky asset need not carry over to a setting with multiple assets.

\textsuperscript{16}See the appendix. For example, consider an investor whose only risky investments are the S&P500 and Nasdaq indices. The covariance matrix for these two indices, based on weekly return data from January 1994 to January 1999, is given by

\[ V = \begin{bmatrix} 0.0295 & 0.0186 \\ 0.0186 & 0.0179 \end{bmatrix}. \]

In this case, if a borrowing constraint is imposed and binding, investment in the S&P500 index increases, relative to the unconstrained setting, whereas investment in the Nasdaq index falls.
Once again, if $v^* = 0$ (and $\psi \geq 0$) then we obtain the unconstrained solution. In order to consider the more interesting setting in which some of the shortsale constraints are binding, we first establish the following general result concerning ‘tangency’ portfolios of the form $V^{-1}\Theta$, where $\Theta$ is any $n$-dimensional vector with bounded elements.

**Proposition 1.** For any bounded $n$-dimensional vector $\Theta$ and for $1 \leq m < n$,

$$[V^{-1}_n \Theta_n] = \begin{bmatrix} V^{-1}_m \Theta_m - V^{-1}_m \sigma_m \sigma_{n-m}^T \beta^{\Theta}_{n-m} \\
\vdots \\
\beta^{\Theta}_{n-m} \end{bmatrix}$$

where $\sigma_m$ and $\sigma_{n-m}$ denote the first $m$ and last $(n-m)$ rows of $\sigma$, respectively, $V_m = \sigma_m \sigma_m^T$, $\Theta_m$ denotes the vector composed of the first $m$ entries of $\Theta$ and $\beta^{\Theta}_{n-m}$ denotes the vector composed of the last $(n-m)$ entries of $V^{-1}_n \Theta_n$. Remaining subscripts denote dimensions.

**Proof.** See the appendix. 

This proposition simply states that, regardless of how we order the individual entries of $V^{-1}\Theta$, any subset of the entries of $V^{-1}\Theta$ can in fact be written as the sum of the ‘tangency’ portfolio for those assets alone and the ‘hedge’ portfolio of these assets whose return has maximum negative correlation with the return on the portfolio given by the remaining entries of $V^{-1}\Theta$.

Returning to our shortsale-constrained problem, suppose that for the first $k$ assets $v^{*}_i = 0$ and $\psi_i \geq 0$, whereas $v^{*}_i \geq 0$ and $\psi_i = 0$ for the remainder. Using Proposition 1, the first order conditions to (11) can be rewritten as

$$\begin{bmatrix} V^{-1}_k (\mu - r \cdot 1 + v^*)_k - V^{-1}_k \sigma_k \sigma_{n-k}^T \beta^{\mu-r+v^*}_{n-k} \\
\vdots \\
\beta^{\mu-r+v^*}_{n-k} \end{bmatrix} + \begin{bmatrix} \psi_k \\
\cdots \\
0_{n-k} \end{bmatrix} = 0. \quad (17)$$

Therefore $\beta^{\mu-r+v^*}_{n-k} = 0_{n-k}$ and the optimal portfolio is identified (using (12) and Proposition 1) as

$$\pi = \begin{bmatrix} \frac{1}{1-\beta} V^{-1}_k (\mu - r \cdot 1)_k \\
\vdots \\
0_{n-k} \end{bmatrix},$$

which corresponds to the optimal unconstrained portfolio in $k$ dimensions for this investor. In the constant coefficient setting, Xu and Shreve (1992b) originally
Clearly, exhaustive enumeration of all such portfolios is one approach. With \( n \) securities, there are \( (2^n - 1) \) such portfolios to compute. It may be possible to derive a more efficient algorithm.

Note that all shortsale-constrained investors, independent of their preferences, minimize the same function to determine \( v^*_i \) and thus that they all optimally invest in the same portfolio of risky assets. The additional insight furnished by the above analysis is that Xu and Shreve’s ‘mutual fund’ is simply the mean-variance tangency portfolio, or ‘unconstrained’ mutual fund, for a subset of the risky securities. (Note that this is essentially a partial equilibrium result, since someone has to hold the assets which the shortsale-constrained investors choose not to invest in. Nonetheless, it could be consistent with general equilibrium in, say, an international context if investors face different costs of holding domestic versus foreign assets and are therefore affected differently by shortsale constraints, as in Cooper and Kaplanis (1999).)

Our analysis thus far has characterized the optimal portfolio, but has not identified which \( k \) assets are held in the optimal solution. Some progress in this direction can be made by exploiting as yet unused information contained in the complementary slackness conditions.

**Proposition 2.** The condition \( v^*_i \geq 0 \) for all \( i = k + 1 \) to \( n \) implies that if we add any asset which has zero weight in the optimal solution to the set of \( k \) assets which have strictly positive weights and re-solve for the unconstrained portfolio, this asset is either shortsold or not held at all.

**Proof.** See the appendix. □

Given that the portfolio of assets with strictly positive weights is itself equivalent to a shortsale-unconstrained portfolio, this implies that the optimal shortsale-constrained portfolio can theoretically be identified by computing only optimal unconstrained portfolios in different dimensions.\(^{17}\)

Finally, the above analysis is easily extended to allow for the more general constraint set

\[
c_i \leq \pi_i \leq d_i, \quad i = 1 \text{ to } n,
\]

where, for any \( i \), \( c_i \) and \( d_i \) are constants satisfying \(-\infty \leq c_i \leq 0 \leq d_i \leq \infty\). In this case,\(^{18}\) if \( \Phi \) denotes the \((n - k)\)-dimensional vector representing investment in assets for which either the upper- or lower-bound constraint is binding, then the optimal investment in the \( k \) remaining assets can be shown to be equal to the tangency portfolio for those assets alone plus the ‘hedge’ portfolio for \( \Phi \):

\[
\pi_k = V_k^{-1}(\mu - r \cdot 1)_k - V_k^{-1}\sigma_k\sigma_n^{T} \Phi.
\]

\(^{17}\) Clearly, exhaustive enumeration of all such portfolios is one approach. With \( n \) securities, there are \((2^n - 1)\) such portfolios to compute. It may be possible to derive a more efficient algorithm.

\(^{18}\) See Example 14.7 of C&K for characterizations of \( \delta(x) \) and \( \tilde{K} \) in this setting.
4.3. Borrowing and shortsale constraints

For the case with both borrowing and shortsale constraints,

\[ K = \{ \pi \in \mathbb{R}^n_+ ; \pi^T 1 \leq a \}, \]

\[ \delta(x) = a \max_j ( - x_j, 0), \quad j = 1 \text{ to } n, \]

\[ \tilde{K} = \mathbb{R}^n. \]

Here we assume that both the borrowing constraint and some subset of the shortsale constraints are binding, all other possible cases having already been discussed. This corresponds to a setting in which \( \delta(v) \) is non-zero and not all of the entries of \( v^* \) are the same. In particular, suppose that the first \( k \) assets have \( v^*_i = \arg \max_j ( - v_j, 0) \equiv \phi \). The first-order conditions to (11), using Proposition 1, are

\[
1^T V^{-1}_k (\mu - r \cdot 1)_k + 1^T V^{-1}_k 1 \phi - 1^T V^{-1}_k \sigma^T_{n-k} \beta^u_{n-k} = a(1 - \alpha), \tag{18}
\]

\[
\beta^u_{n-k} = 0 \tag{19}
\]

whence, combining (18) and (19), we obtain that

\[ \phi = \frac{a(1 - \alpha) - 1^T V^{-1}_k (\mu - r \cdot 1)_k}{1^T V^{-1}_k 1}. \]

Thus, from (12), the optimal borrowing and shortsale-constrained portfolio is

\[ \pi = \begin{bmatrix}
1^T V^{-1}_k (\mu - (r - \phi) \cdot 1)_k \\
\vdots \\
0_{n-k}
\end{bmatrix}, \]

which is the optimal borrowing-constrained-only portfolio in \( k \) dimensions.

**Proposition 3.** The condition \( v^*_i > \phi \) for \( i = k + 1 \) to \( n \), implies that if we add any asset which has zero weight in the optimal solution to the set of \( k \) assets which have strictly positive weights and re-solve for the borrowing-constrained-only portfolio, this asset is shortsold.

This characterization, and the fact that the optimal constrained portfolio is itself equivalent to an optimal borrowing-constrained-only portfolio, implies that the optimal borrowing-and-shortsale-constrained portfolio can be identified by computing only optimal borrowing-constrained-only portfolios in different dimensions.
5. Concluding remarks

The goal of the above analysis has been to determine the investor’s optimal portfolio policy under various constraints by identifying the vector \( v^* \). Summarizing our results, we have shown that \( v^* \) is characterized as follows:\(^{19}\)

(i) borrowing-constrained only: \( v^* = \bar{v}^* \cdot 1 \) where \( \bar{v}^* \) is a non-positive scalar,
(ii) shortsale-constrained only:
\[
\begin{align*}
    v^*_i \geq 0 & \quad \text{if the shortsale constraint is binding for asset } i, \\
    v^*_i = 0 & \quad \text{otherwise},
\end{align*}
\]
(iii) borrowing and shortsale constrained: \( v^* = \phi \cdot 1 + u^* \), where \( \phi \) is a non-positive scalar and
\[
\begin{align*}
    u^*_i > 0 & \quad \text{if the shortsale constraint is binding for asset } i, \\
    u^*_i = 0 & \quad \text{otherwise}.
\end{align*}
\]

Thus, in the presence of both borrowing and shortsale constraints, \( v^* \) can be broken down into a borrowing-constrained component, \( \phi \cdot 1 \), and a shortsale-constrained component, \( u^* \). This decomposition allows us to link Propositions 2 and 3. While having very similar interpretations, the conditions underlying these two propositions (\( v^*_i \geq 0 \) and \( v^*_i > \phi \), respectively, in each case for any asset \( i \) for which the shortsale constraint is binding) appear somewhat different in nature. However, from (iii) above we see that the condition in Proposition 3 can be restated in terms of the shortsale-constrained component of \( v^* \) as \( u^*_i > 0 \). Thus the condition being interpreted is essentially the same in both propositions.

Appendix. Proofs

Proof of the result in Eq. (15). When there are two risky assets and the investor faces binding borrowing constraints, optimal investment is given by
\[
\begin{align*}
    \pi_1 &= \frac{\sigma_1^2(\mu_1 - r + v^*) - \sigma_{12}(\mu_2 - r + v^*)}{\sigma_1^2 \sigma_2^2 - \sigma_{12}^2}, \\
    \pi_2 &= \frac{\sigma_2^2(\mu_2 - r + v^*) - \sigma_{12}(\mu_1 - r + v^*)}{\sigma_1^2 \sigma_2^2 - \sigma_{12}^2}.
\end{align*}
\]

\(^{19}\) Assuming that in each case the constraints are binding.
Thus, given that $\frac{\partial v^*}{\partial a} < 0$,

$$\frac{\partial \pi_1}{\partial a} = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 - \sigma_{12}} \frac{\partial v^*}{\partial a} = \begin{cases} > 0 & \text{if } \sigma_{12} > \sigma_2^2, \\ < 0 & \text{if } \sigma_{12} < \sigma_2^2, \end{cases}$$

$$\frac{\partial \pi_2}{\partial a} = \frac{\sigma_1^2 - \sigma_{12}}{\sigma_1^2 - \sigma_{12}} \frac{\partial v^*}{\partial a} = \begin{cases} > 0 & \text{if } \sigma_{12} > \sigma_1^2, \\ < 0 & \text{if } \sigma_{12} < \sigma_1^2. \end{cases}$$

Proof of Proposition 1. The proof follows along the lines of the general proof for the inverse of a partitioned matrix (see, for example, pp. 134–136 of Johnston, 1984). Write the inverse of $V_n$ as

$$\begin{bmatrix} A_m : B_{m \cdot (n-m)} \\ \vdots \\ C_{(n-m) \cdot m} : D_{n-m} \end{bmatrix}$$

so that

$$\begin{bmatrix} V_m : \sigma_m \sigma_{n-m}^T \\ \vdots \\ \sigma_{n-m} \sigma_m^T : \sigma_{n-m} \sigma_{n-m}^T \end{bmatrix} \begin{bmatrix} A \vdots B \\ \vdots \\ C \vdots D \end{bmatrix} = \begin{bmatrix} I_m : 0_{(n-m) \cdot m} \\ \vdots \\ 0_{n \cdot (n-m)} : I_{n-m} \end{bmatrix}$$

where $\sigma_m$ and $\sigma_{n-m}$ denote, respectively, the first $m$ and the last $(n - m)$ rows of $\sigma$, $I$ denotes the identity matrix, and where for all other variables subscripts denote dimensions. This yields

$$V_mA + \sigma_m \sigma_{n-m}^T C = I, \quad (A.1)$$

$$\sigma_{n-m} \sigma_m^T A + \sigma_{n-m} \sigma_{n-m}^T C = 0, \quad (A.2)$$

$$V_mB + \sigma_m \sigma_{n-m}^T D = 0^T, \quad (A.3)$$

$$\sigma_{n-m} \sigma_m^T B + \sigma_{n-m} \sigma_{n-m}^T D = I. \quad (A.4)$$

The claim is that

$$\begin{bmatrix} A \vdots B \\ \vdots \\ C \vdots D \end{bmatrix} \begin{bmatrix} \Theta_m \\ \vdots \\ \Theta_{n-m} \end{bmatrix} = \begin{bmatrix} V_m^{-1} \Theta_m - V_m^{-1} \sigma_m \sigma_{n-m} \beta_{n-m}^\Theta \\ \vdots \\ \beta_{n-m}^\Theta \end{bmatrix},$$

where $\Theta_{n-m}$ and $\beta_{n-m}^\Theta$ denote the last $(n - m)$ entries of $\Theta$ and $V_n^{-1} \Theta_n$, respectively. Clearly,

$$\beta_{n-m}^\Theta = C \Theta_m + D \Theta_{n-m}.$$
Substituting $\beta_{n-m}^{\theta}$ into
\[ A\Theta_m + B\Theta_{n-m} = V_m^{-1}\Theta_m - V_m^{-1}\sigma_m\sigma_{n-m}^{\text{T}}\beta_{n-m}^{\theta} \]
and using Eqs. (A.1) and (A.3) veriﬁes the claim. \(\square\)

**Proof of Proposition 2.** Applying Proposition 1 to the ﬁrst-order condition $V^{-1}(\mu - r \cdot 1 + v) - \psi = 0$, we obtain that, for $i = k + 1$ to $n$, 
\[ v_i^* = -s_i\sigma_k^{\text{T}}\psi_k - (\mu_i - r). \] (A.5)

From (17) we have that $\psi_k = -V_k^{-1}(\mu - r \cdot 1)_k$. Substituting into (A.5) yields
\[ v_i^* = s_i\sigma_k^{\text{T}}V_k^{-1}(\mu - r \cdot 1)_k - (\mu_i - r) \quad \text{for} \quad i = k + 1 \text{ to } n. \] (A.6)

By assumption, $v_i^* \geq 0$, for $i = k + 1$ to $n$. To interpret this condition we ﬁrst establish the following lemma. \(\square\)

**Lemma A.1.** Let $\theta_{m+1}$ and $\theta_{m+1}^{\theta}$ denote the $(m+1)$th entries of $\Theta$ and $V_{m+1}^{-1}\Theta_{m+1}$, respectively. Then
\[ \theta_{m+1}^{\theta} = \frac{\theta_{m+1} - s_{m+1}\sigma_m^{\text{T}}V_m^{-1}\Theta_m}{s_{m+1}(I - \sigma_m^{\text{T}}V_m^{-1}\sigma_m)s_{m+1}^{\text{T}}}, \] \[ \text{(A.7)} \]

where
\[ s_{m+1}(I - \sigma_m^{\text{T}}V_m^{-1}\sigma_m)s_{m+1}^{\text{T}} > 0. \]

**Proof of Lemma A.1.** The solution to Eqs. (A.1)–(A.3) for the case of $(n - m) = 1$ yields
\[ C = \frac{\sigma_m^{\text{T}}V_m^{-1}}{s_{m+1}(I - \sigma_m^{\text{T}}V_m^{-1}\sigma_m)s_{m+1}^{\text{T}}}, \]
\[ D = \frac{1}{s_{m+1}(I - \sigma_m^{\text{T}}V_m^{-1}\sigma_m)s_{m+1}^{\text{T}}}. \]

Substituting into
\[ \theta_{m+1}^{\theta} = C\Theta_m + D\theta_{m+1}, \]
yields (A.7). We show that $s_{m+1}(I - \sigma_m^{\text{T}}V_m^{-1}\sigma_m)s_{m+1}^{\text{T}} > 0$ by establishing that this is the residual instantaneous variance of the $(m + 1)$st asset unhedged by the ﬁrst $m$ assets. Let $q$ be portfolio weights on the ﬁrst $m$ assets and consider the problem of minimizing the total instantaneous variance of the return on the $(m + 1)$st asset plus the return on this portfolio. That is, $q^*$ solves
\[ \min_q s_{m+1}s_{m+1}^{\text{T}} + q^{\text{T}}V_mq + 2q^{\text{T}}\sigma_m s_{m+1}^{\text{T}}. \]
The first-order conditions imply that \( q^* = -V_m^{-1}\sigma_m s_{m+1}^T \) and that the minimum value of the objective function is \( s_{m+1}(1 - \sigma_m V_m^{-1}\sigma_m) s_{m+1}^T \), which, by definition, is strictly greater than zero (given that \( V_m \) is non-singular). \( \square \)

If we take \( \Theta = (\mu - r \cdot 1) \) and \( m = k \), the expression for \(-v_i^*\) in (A.6) is equivalent, for \( i = k + 1 \), to the numerator of the expression on the right-hand side of (A.6), whereas (A.6) itself represents the optimal unconstrained investment in the \((k + 1)\)st asset when it is held in combination with the first \( k \) assets. Thus the condition in (A.5) implies that if we add any asset which has zero weight in the optimal solution to the set of \( k \) assets which have strictly positive weights and re-solve for the unconstrained portfolio, this asset is either shortsold or not held at all. \( \square \)

**Proof of Proposition 3.** Using Proposition 1, the restriction \( v_i^* > \phi \) for \( i = k + 1 \) to \( n \) can be rewritten as

\[
 s_i \sigma_k^T V_k^{-1}(\mu - r \cdot 1 + \phi \cdot 1)_k - (\mu_i - r + \phi) > 0 \quad \text{for } i = k + 1 \text{ to } n. \quad (A.8)
\]

This expression does not have any immediate ‘intuitive’ interpretation (that is, the left-hand side of (A.8) cannot be directly interpreted as being proportional to investment in asset \( i \) in a given portfolio, as was the case in the shortsale-constrained-only setting). However, we show that a characterization similar to that established in the shortsale-constrained-only setting also obtains in the presence of borrowing constraints.

Employing Lemma A.1 with \( \Theta = 1 \), the \((m + 1)\)st entry of \( V_{m+1}^{-1}1_{m+1} \) is

\[
 \zeta_{m+1} = \frac{1 - s_{m+1} \sigma_m^T V_{m+1}^{-1}1_m}{s_{m+1}(1 - \sigma_m V_{m+1}^{-1}\sigma_m)s_{m+1}^T} \quad (A.9)
\]

for any \( 1 \leq m < n \). Thus, taking \( m = k \), we can re-write (A.8) in the simpler form

\[
 \zeta_{k+1} \leq -\phi \zeta_{k+1}, \quad (A.10)
\]

where the \((k + 1)\)st asset is any asset \( j = k + 1 \) to \( n \). The following lemma establishes that we can interchange \( \phi \) in (A.10) with the analogous quantity that would be computed as part of the borrowing-constrained-only solution for an opportunity set comprising the first \( k \) assets and asset \( j \) (rather than just the first \( k \) assets).

**Lemma A.2.** Let \( \phi_i = (a(1 - x) - 1_i^TV_i^{-1}(\mu - r \cdot 1)_i)/1_i^TV_i^{-1}1_i \) for any \( i = 1 \) to \( n \). Then

\[
 \zeta_{k+1} \leq -\phi \zeta_{k+1} \quad \text{if and only if} \quad \zeta_{k+1} \leq -\phi \zeta_{k+1}. \]

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Proof of Lemma A.2. Using Proposition 1,
\begin{align*}
1^T V_{m+1}^{-1} 1 &= 1^T V_m^{-1} 1 + \zeta_{m+1}^{-1} (1 - s_{m+1} \sigma_m^T V_m^{-1} 1), \\
1^T V_{m+1}^{-1} (\mu - r \cdot 1)_{m+1} &= 1^T V_m^{-1} (\mu - r \cdot 1)_m + \zeta_{m+1}^{-1} (1 - s_{m+1} \sigma_m^T V_m^{-1} 1) .
\end{align*}
(A.11) (A.12)

For any \( i = 1 \) to \( n \), \( V_i^{-1} \) is a symmetric positive definite matrix, thus \( 1^T V_i^{-1} 1 > 0 \). Using this property to both multiply and divide and the relations in (A.11) and (A.12) for \( m = k \), we have equivalence of the four inequalities
\begin{align*}
\zeta_k^{\mu - r} s_{k+1}^{-1} &\leq - \phi_{k+1} \zeta_k^{1} , \\
\zeta_k^{\mu - r} 1^T V_{k+1}^{-1} 1 &\leq (1^T V_{k+1}^{-1} (\mu - r \cdot 1)_{k+1} - a(1 - \alpha)) \zeta_k^{-1} , \\
\zeta_k^{\mu - r} 1^T V_k^{-1} 1 &\leq (1^T V_k^{-1} (\mu - r \cdot 1)_{k} - a(1 - \alpha)) \zeta_k^{1} , \\
\zeta_k^{\mu - r} &\leq - \phi_k \zeta_k^{1} .
\end{align*}

This proves the result. \( \square \)

Applying Lemma A.2, (A.8) can be written as
\[ (\mu_j - r + \phi_{k+1}) - s_j \sigma_k V_k^{-1} (\mu - r \cdot 1 + \phi_{k+1} \cdot 1)_k < 0. \] (A.13)

Using Lemma A.1, it is now straightforward to interpret (A.13) as implying that if we add any asset which has zero weight in the optimal solution to the set of \( k \) assets with strictly positive weights, and solve the borrowing-constrained-only problem for the new set of \( (k + 1) \) assets, the new asset is shortsold in the new solution.

References