Approximating payoffs and pricing formulas

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Abstract

We use the ideas developed by Madan and Milne (1994. Mathematical Finance 3, 223–245), Lacoste (1996. Mathematical Finance 6, 197–213) to explore the optimality of polynomial approximations in pricing securities. In particular, we look at the approximations for security payoffs as well as the associated pricing formula in a $L^2$ framework. We apply these ideas to two examples, one where the state variable follows an Ornstein–Uhlenbeck process and one based on Brownian motion with reflecting barriers, to illustrate the strengths and weaknesses of the approach. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

It is a common practice in financial institutions to approximate a given payoff by a more simple one. Then, an approximated price is computed as the true price of the approximated payoff. For instance, if the payoff is twice differentiable, one may make a second-order Taylor expansion and consider the price of this second-order expansion. When considering differential swaps or complex FRNs, accurate approximations of prices can be obtained with this approach. This relies on the assumption that the contribution to the price of higher order terms is small.

A similar and related problem consists in getting a simple and tractable approximation to a given price profile of an options portfolio. Even if one knows theoretically the value of an options portfolio and its dependence on the underlying risky asset price (or on state variables’ values), the practical computation of the price profile is not so straightforward: one has to compute the portfolio price for many different prices of the underlying risky asset, each valuation step involving costly numerical procedures. It may then be useful to get a good functional approximation of the price profile. This is related to delta/gamma local analysis. This problem has become of special importance for the computation of value at risk (VaR) for banks managing large option portfolios. Value at risk is often performed by Monte Carlo simulations and each simulation step requires the computation of the portfolio value which may be costly for books where the number of different contracts is large.

Recently, Madan and Milne (1994), Lacoste (1996) and Abken et al. (1996) have proposed approaches where a given option payoff is approximated through $L^2$ projections on orthogonal subspaces. They exhibit some countable basis of non-correlated payoffs, and each derivative payoff can thus be expressed as a linear combination of basis elements. The pricing of a given payoff requires the knowledge of its coordinates (or risk sensitivities) and of the price of the basis elements. One may also think of the basis elements as analogous to factors in asset pricing, but the space of claims considered here includes non-linear transformations of asset returns.

Basis elements may also be viewed as statically completing markets as in John (1981), John (1984) and Amershi (1985). Typically, in the case of a Gaussian measure, Madan and Milne (1994) and Lacoste (1996) provide examples based on the basis of Hermite polynomials. On the other hand, Bansal et al. (1993b) and Chapman (1997) have considered the approximation of the pricing kernel through polynomial approximations. It can be noticed that the space generated by the first $N + 1$ Hermite polynomials is the space of polynomials of order up to $N$.

The purpose of this paper is to investigate whether the use of such a polynomial basis is optimal and what should be an optimal basis. Indeed, there are many countable bases in $L^2$. Moreover, it is not a priori obvious that using the
subspaces generated by polynomials of increasing order is optimal. We propose an integrated framework where the approximation of payoffs is related to the approximation of the pricing operator. After carefully stating the approximation problem, we show that the spectral analysis of the pricing operator is the key tool. An eigenfunction of the pricing operator is simply a payoff whose associated pricing formula is equal, up to a constant, to the payoff. The optimal subspace to approximate payoffs or pricing formulas is the subspace generated by the first $N$ eigenfunctions. Such an analysis is very similar to the principal component analysis of a set of $K$ risky assets when only the first $N$, $N < K$, principal components are considered. Our framework allows us to show that the polynomial approximations of Madan and Milne (1994) are optimal (in a $L^2$ sense) when the eigenfunctions of the pricing operator are polynomials of increasing order.

The paper is organized as follows. In Section 2, we define the market model and introduce an orthogonal basis for the space of payoffs. Section 3 addresses the kernel representation of the pricing models. Section 4 states the main properties regarding the approximation of payoffs, pricing formulas and pricing operators. Section 5 provides two examples: one where the state variable follows a mean-reverting Ornstein–Uhlenbeck process. In that case the polynomial approximation appears to be optimal. We also provide an explicit counterexample based on a Brownian state variable with reflecting barriers, where the optimal approximation is based on Fourier series expansions. Section 6 presents two numerical applications and Section 7 concludes. Proofs are gathered in Appendix A.

2. Market model

We consider a model with $M$ assets, whose price vector at date $t$ is denoted $S_t$, where $t = 1, \ldots, T$. The price processes are defined on a probability space $(\Omega, \mathcal{F}, Q)$, where $Q$ is a risk-neutral probability, and $\mathcal{F}$ a filtration. We assume the existence of a stationary Markovian representation, i.e. there exist $D$ stationary state variables $X$ adapted to $\mathcal{F}$ and with $\sigma(X_t) = \mathcal{F}_t$. Under the risk-neutral probability, the state variables vector is modeled by a homogeneous, $D$-dimensional Markov process.

We consider European-type contingent claims, i.e., whose payoff $\Psi_T$ at time $T$ is a function of the state variables $X$ at time $T$: $\Psi_T = f(X_T)$. The function $f$ is assumed to belong to the space of square integrable functions $L^2(Q^T)$, where $Q^T$ is the marginal distribution of the state variable $X_T$. The price of the contingent claim at time $t$ is equal to the conditional expectation of $f(X_T)$ (for the sake of simplicity, the riskless rate is set to zero)

$$E^Q[f(X_T)/\mathcal{F}_t].$$

(2.1)
Using the Markov property of the state variables \( X_t \) (2.1) can also be expressed as

\[
E^Q[f(X_T)/X_t].
\]  
\[ (2.2) \]

Obviously, this price depends on the value at time \( t \) of the state variables \( X_t \). This leads to the following definitions.

**Definition 2.1.** For any payoff \( f \), the mapping \( V(f) : \mathbb{R}^D \rightarrow \mathbb{R}, \)

\[
x \rightarrow V(f)(x) = E^Q[f(X_T)/X_t = x],
\]
is called a pricing formula. The mapping \( V : L^2(Q^T) \rightarrow L^2(Q'), \)

\[
f \rightarrow V(f),
\]
is called a pricing operator.

Thus, the pricing formula at a date \( t \) of a payoff \( f \), with \( f \in L^2(Q^T) \) is the projection of this payoff on \( L^2(Q') \), where \( Q' \) is the marginal distribution of the state variables \( X_t \). Because of the stationarity assumption made on state variables \( X_t \), it is possible to identify these two functional spaces (we denote \( L^2(Q^T) = L^2(Q') = L^2 \) and thus, the pricing operator \( V \) is a linear contraction of the space \( L^2 \).

As in Madan and Milne (1994), Lacoste (1996), we rely heavily on the use of an orthonormal basis of \( L^2 \). This will allow a simple representation of pricing operators. We recall that every Hilbert space contains an orthonormal basis for itself (see Dunford and Schwartz, 1988, Theorem 12, p. 252). Moreover, we will make the following assumption:

**Assumption 2.1.** The space \( L^2 \) defined above is separable.

Then, there exists a countable orthonormal basis \( w_i, i = 1, \ldots \), such that\(^1 \)

\[
\langle w_i, w_j \rangle = 1 \text{ if } i = j, \text{ and } 0 \text{ otherwise. Hence, given a payoff in } L^2, \text{ we can write its orthonormal decomposition as}
\]

\[
f = \lim_{k \to \infty} \sum_{i=1}^{k} \langle f, w_i \rangle w_i = \sum_{i=1}^{\infty} \langle f, w_i \rangle w_i.
\]  
\[ (2.3) \]

\(^1\) For any payoff \( f, g \in L^2, \langle f, g \rangle = E[f(X)g(X)] \) is the usual inner product in \( L^2 \).
In the same way, if we focus on the approximation of the pricing formula, one can define the orthonormal decomposition of any formula \( V(f) \) of \( L^2 \) by

\[
V(f) = \lim_{k \to \infty} \sum_{i=1}^{k} \langle Vf, w_i \rangle w_i = \sum_{i=1}^{\infty} \langle Vf, w_i \rangle w_i. \tag{2.4}
\]

A pricing formula approximation can be obtained by cutting the previous infinite summation at rank \( N \). However, since the formula \( V(f) \) may be unknown, the computation of the inner products \( \langle Vf, w_i \rangle \) cannot be always done. An alternative approach is to compute the true price on an approximated payoff. Interchanging the limit and conditional expectation operators, Eq. (2.3) allows us to write directly the true price of \( f \) in terms of those of the basis functions:

\[
V(f) = \sum_{i=1}^{\infty} \langle f, w_i \rangle V(w_i). \tag{2.5}
\]

Now, the price approximation is naturally introduced by cutting the previous infinite summation at the rank \( N \). We denote by \( E \) the space generated by \( w_i, i = 1, \ldots, N \). The approximated pricing formula for \( f \) is then given by

\[
V \circ P_E(f) = \sum_{i=1}^{N} \langle f, w_i \rangle V(w_i), \tag{2.6}
\]

where \( P_E \) is the orthogonal projector on \( E \). This setting is analogous to the one used in Madan and Milne (1994) and considers the spaces of payoffs and pricing formulas. The main issues that we will now address concern the validity and the optimality of such an approach:

1. When the dimension of the approximation space goes to infinity, do we have the convergence of the approximated price to the true one, for any payoff \( f \) of \( L^2 \)? In other words, do we have:

\[
\| V - V \circ P_E \| \to 0,
\]

when \( N \to \infty \) and for a convenient choice of the norm \( \| \cdot \| \) ?

2. There are usually many orthonormal bases in \( L^2 \). Can we then characterize the best finite-dimensional approximation space, i.e. the basis functions which minimize the pricing error when we use the approximated payoff instead of the true one?

3. And, finally, is it equivalent to first approximate the payoff and then compute the price of this approximation, or to approximate directly the pricing formula. In other words, do we have:

\[
V \circ P_E = P_E \circ V
\]

for the optimal choice of \( E \)?
3. Pricing operator properties

This section states the two assumptions made on the dynamics of the factors to ensure the validity of the projection approach.

Assumption 3.2. The pricing operator admits an integral representation, i.e. there exist a transition kernel $k$ and a probability measure $dm$ such that

$$E^Q[f(X_{t+1})/X_t = x] = \int k(y, x)f(y) \, dm(y).$$

Remark 3.1. If the measure $dm$ has a density $m(y)$ with respect to Lebesgue measure $dy$, then the transition probability $p(y/x)$ of the process $X$ is computed directly with the kernel $k$ and the invariant probability $m$ by the formula:

$$p(y/x) = k(y, x) \times m(y).$$

Assumption 3.3. The transition kernel $k$ of the pricing operator satisfies the integrability condition:

$$\int \int k^2(y, x) \, dm(x) \, dm(y) < \infty.$$  

Remark 3.2. The last assumption is equivalent to the following condition: $k$ belongs to the Hilbert space $L^2$ defined as the space of square integrable functions on the product space $R^D \times R^D$, for the product measure $Q \otimes Q$. Since $w_i(y)w_j(x)$, $i, j = 1, \ldots$, is an orthonormal basis of $L^2$, we can decompose $k$ as follows:

$$k(y, x) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \alpha_{ij} w_i(y)w_j(x),$$

where $\alpha_{ij} = \int \int k(y, x)w_i(y)w_j(x) \, dm(x) \, dm(y)$ ($= \langle Vw_i, w_j \rangle$ by Fubini’s Theorem) are the coordinates of $k$ in $L^2$.

The two former assumptions on the existence and regularity of the kernel will guarantee that the pricing operators have good properties. In particular, we will be able to work with conditional expectations operators as with infinite dimensional matrices (see Darolles et al., 1998). In the APT approach, the pricing kernel is a linear function of the state variables. Here, we represent the pricing kernel $k$ as a linear combination of possibly non-linear functions of the factors (see Hansen and Jagannathan, 1997; Bansal and Viswanathan, 1993a; Bansal et al. 1993b; Chapman, 1997, for related approaches and discussions). If the coordinates $\alpha_{ij}$ of $k$ are small enough for $i, j > N$, then we can make a finite approximation of the pricing kernel. Unlike the APT, this approximation may not be linear in the factors.
The set of pricing operators satisfying the Assumptions 3.2 and 3.3 is a subspace of the bounded linear mappings in $L^2$, complete for the norm $\| \cdot \|$ defined by

$$\|V\|^2 = \sum_{i=1}^{\infty} |V(w_i)|^2 = \int \int k(y,x) dm(x) dm(y),$$

(3.2)

where the family $w_i, i = 1, \ldots, N$, is an arbitrary orthonormal basis of the space $L^2$, and $|f|^2 = \langle f, f \rangle$ is the usual norm in $L^2$. Assumptions 3.2 and 3.3 made on the pricing operator imply its compactness. Therefore, in the self-adjoint case, this justifies, for example, the decomposition of $V$ with respect to an eigenfunction basis. One can also remark that the norm defined in (3.2) is equal to the $L^2$-norm of the transition kernel $k$.

In the general case, the compactness is not sufficient to easily introduce the spectral decomposition of $V$. Hence, we consider both self-adjoint compact operators $V^*V$ and $VV^*$. Using the spectral decomposition of $V^*V$, we build the eigenfunction basis defined by the relation

$$V^* e_i = \lambda_i e_i,$$

(3.3)

where the sequence $\lambda_i, i = 1, \ldots, N$, decrease to 0, and $\lambda_0 = 1$ is the eigenvalue associated to the constant function. Hence, let us now introduce the space $E$ built with the $N$ first eigenfunctions $e_i, i = 1, \ldots, N$. All the $N$-rank approximations $V_E$ of the pricing operator are then defined by first projecting the payoff:

$$V_E(f) = V \circ P_E(f),$$

(3.4)

where $E$ is any subspace of $L^2$, with dimension $N$. This corresponds to the computation of the pricing formula associated with an approximated payoff as in (2.6). We are now able to prove that the best approximation is obtained using the $E$ subspace introduced above.

4. Main theorem and applications

To introduce the concept of best approximation of an integral operator, we use some results on kernel approximation on the space $L^2$. We suppose in this section that the pricing operator satisfies Assumptions 3.2 and 3.3. One can

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$^2$ Assumptions 3.2 and 3.3 are necessary and sufficient conditions for the pricing operator to be in the class of Hilbert–Schmidt operators (see Riesz and Nagy, 1990).

$^3$ We recall that the adjoint $V^*$ of $V$ is such that for $f, g \in L^2$, $\langle Vf, g \rangle = \langle f, V^*g \rangle$. $V$ is self-adjoint iff $V = V^*$. 

define its kernel \( k \) as an element of the Hilbert space \( L^2 \). \( E \) denotes the subspace generated by \( w_i, \ i = 1, \ldots, N \). Then, the \( N \)-rank approximation (2.6) of the pricing operator \( V \) admits an integral representation:

\[
V \circ P_E(f) = \sum_{i=1}^{N} \langle f, w_i \rangle V(w_i)
= \sum_{i=1}^{N} \int w_i(y)f(y)\,dm(y) \cdot \sum_{j=1}^{\infty} \langle Vw_i, w_j \rangle w_j(x)
= \int \sum_{i=1}^{N} \sum_{j=1}^{\infty} [z_{ij}w_i(y)w_j(x)]f(y)\,dm(y),
\]

where \( z_{ij} = \langle Vw_i, w_j \rangle \) are the coordinates of \( k \) in the basis of \( L^2 \) built using the arbitrary orthonormal basis \( w_i, i = 1, \ldots, \) of \( L^2 \). Hence, the kernel:

\[
k_E(y, x) = \sum_{i=1}^{N} \sum_{j=1}^{\infty} z_{ij}w_i(y)w_j(x)
\]

is associated with the \( N \)-rank approximation \( V \circ P_E \) and is also an approximation in \( L^2 \) of the true kernel \( k \).

**Theorem 4.1.** Let \( E \) be any \( N \)-dimensional subspace of \( L^2 \). Then, we have:

\[
\sup_{f \in L^2} \frac{|V(f) - V \circ P_E(f)|^2}{|f|^2} \to 0,
\]

when \( N \) tends to infinity.

This gives an answer to the first issue we raised in Section 2 and corresponds to the uniform convergence of \( V \circ P_E \) towards \( V \). When the pricing operator has good properties, we can first approximate the payoff, and then compute the approximated price as the price of the approximated payoff. This uniform convergence property will prove to be very useful in practice. It will allow us to provide bounds on the pricing error that are independent of the payoff to price (see numerical applications). The proof of Theorem 4.1 is detailed in Appendix A.

The best operator approximation is defined as the \( N \)-rank operator minimizing the \( L^2 \) error, i.e. \( |k - k_E|^2 \), which arises when we use \( k_E \) instead of \( k \) and is equal to \( ||V - V_E||^2 \), where \( V_E \) denotes \( V \circ P_E \).

**Theorem 4.2.** Define \( \mathcal{E}_N \) as the set of all \( N \)-dimensional subspaces of \( L^2 \). The space \( E^* \) is obtained as the solution to the problem

\[
\min_{E \in \mathcal{E}_N} ||V - V_E||,
\]

\footnote{For any payoff \( f \in L^2 \), \( |f| = \text{E}[f^2(X)]^{1/2} \) is the usual norm in \( L^2 \).}
and is generated by the first $N$ eigenfunctions of the operator $V^*V$, associated with the first $N$ eigenvalues sorted by decreasing order.

We just give a sketch of the proof. The complete one is detailed in Appendix A. We begin the proof by stating a lemma first.

**Lemma 4.1.** Consider a closed subspace $E$ of $L^2$ and $V$, an operator satisfying Assumptions 3.2 and 3.3. We denote by $P_E$ the orthonormal projection on the space $E$, then

$$
\|V\|^2 = \|V - V \circ P_E\|^2 + \|V \circ P_E\|^2.
$$

This triangular equality allows us to rewrite the minimization problem (4.2) in the following form:

**Corollary 4.1.** The minimization problem (4.2) is equivalent to the maximization one:

$$
\text{Max}_{E \in \mathcal{E}} \|V_E\|^2.
$$

The idea is then to solve iteratively the maximization problem after using

$$
\|V_E\|^2 = \sum_{i=1}^{N} \langle V^*Vw_i, w_i \rangle.
$$

(see Appendix A for more details). Theorem 4.2 provides the optimal projection space in terms of the eigenfunctions of $V^*V$. We build the $N$-dimensional optimal projection space by computing the eigenfunctions of the operator $V^*V$ in the general case, or of the operator $V$ in the self-adjoint case.

We now assume that the subspace $E$ is the optimal one. In this optimal setting, we are going to prove that it is equivalent to approximate payoffs, pricing formulas or pricing operators. We first study the case of approximating pricing formulas, i.e. we consider the operator $P_E \circ V$ defined by

$$
P_E \circ V(f) = \sum_{i=1}^{N} \langle Vf, e_i \rangle e_i,
$$

where $E$ denotes the subspace generated by $e_i$, $i = 1, \ldots, N$, the eigenfunctions of $V^*V$. Hence, using the same approach as in the case of $V \circ P_E$, one can prove that $P_E \circ V$ is also an integral operator, with kernel:

$$
K_E(y, x) = \sum_{i=1}^{\infty} \sum_{j=1}^{N} \langle Ve_i, e_j \rangle e_i(y)e_j(x).
$$
Corollary 4.2. Consider $V$, an operator satisfying Assumptions 3.2 and 3.3, and the subspace $E$ of $L^2$ generated by the first $N$ eigenfunctions $e_i$, $i = 1, \ldots, N$ of $V^*V$, then

$$V \circ P_E = P_E \circ V.$$ 

This gives an answer to the third issue we raised in Section 2.

Corollary 4.3. Consider $V$, a self-adjoint operator satisfying Assumptions 3.2 and 3.3, and the subspace $E$ of $L^2$ generated by the first $N$ eigenfunctions $e_i$, $i = 1, \ldots, N$ of $V$, associated with the eigenvalues $\lambda_i$, $i = 1, \ldots, N$, then the pricing formula approximation, associated with the solution of

$$\min_{E \in \mathcal{E}_n} ||V - P_E \circ V||,$$

takes, for any payoff of $L^2$, the following form:

$$P_E \circ V f(x) = \sum_{i=1}^{N} \lambda_i \langle f, e_i \rangle e_i(x).$$

Corollaries 4.2 and 4.3 allow us to write the optimal approximated pricing formula as a finite summation involving the eigenfunctions and eigenvalues of $V$ in the self-adjoint case. Hence, approximating payoffs and pricing formulas are closely related problems. Consider now the pricing kernel approximation problem. As in the case of $V \circ P_E$, one can prove that $P_E \circ V \circ P_E$ is an integral operator, with kernel:

$$\tilde{k}_E(y, x) = \sum_{i=1}^{N} \sum_{j=1}^{N} \langle Ve_i, e_j \rangle e_i(y)e_j(x),$$

which is the $N$-rank approximation of the true pricing kernel $k$ (see Chapman, 1997 for a similar approach). We have:

Corollary 4.4. Consider $V$, an operator satisfying Assumptions 3.2 and 3.3, and the subspace $E$ of $L^2$ generated by the first $N$ eigenfunctions of $V^*V$, then $E$ is a solution of the pricing operator approximation problem

$$\min_{E \in \mathcal{E}_n} ||V - P_E \circ V \circ P_E||.$$ 

The operator $P_E \circ V \circ P_E$ is an approximating pricing operator such that the space used to approximate payoffs, $E$, is the same as the one used to approximate pricing formulas. Let us also notice that for the optimal subspace, we have

$$P_E \circ V \circ P_E = P_E \circ V = V \circ P_E.$$
The next result gives a multi-period framework interpretation of the spectral decomposition of pricing operator. \( V^P \) denotes the pricing operator for payoffs that are paid \( P \) periods (\( P \geq 1 \)) ahead of the current date. Indeed, from the stationarity of \( X \) under \( Q \), and from the law of iterated expectations, we get

\[
E^Q[f(X_{PT})/X_0 = X] = V^P(f),
\]

and we obtain immediately the following optimality results:

**Corollary 4.5.** Consider \( V \), a self-adjoint operator satisfying Assumptions 3.2 and 3.3, and \( P \geq 1 \); the subspace \( E \) of \( L^2 \) generated by the first \( N \) eigenfunctions of \( V \) is the solution of

\[
\min_{E \in E_n} ||V^P - P_{E^o}V^P||,
\]

and the optimal approximation is

\[
P_{E^o}V^P f(x) = \sum_{i=1}^{N} \lambda_i^P \langle f, e_i \rangle w_i(x).
\]

We obtain directly these results after seeing that \( V^p \) fulfills the assumptions of Theorem 4.2 (indeed, \( ||V^p||^2 = \sum_{i=1}^{\infty} \lambda_i^{2p} \leq \sum_{i=1}^{\infty} \lambda_i^2 = ||V||^2 \) since \( 0 \leq \lambda_i \leq 1 \)), and that eigenvalues of \( V^p*V^p \) are the \( P \)-power of the ones of \( V*V \), and that eigenfunctions of \( V^p*V^p \) and \( V*V \) are the same.

From the previous corollary, we are then able to approximate pricing formulas corresponding to payoffs that are paid \( P \) periods ahead of the current date. We notice that all the properties proven for \( V \) apply to \( V^P \). Let us remark that, since the optimal basis remains the same, we have to compute only a single decomposition if we deal with a portfolio of identical payoffs but paid at different dates. Finally, since \( ||V^P|| \leq ||V|| \), the error in pricing formulas

\[
||V^P(f) - P_{E^o}V^P(f)||,
\]

decreases as \( P \) increases.

5. **Link with the infinitesimal generator**

This section deals with the case of a univariate stationary continuous Markovian state variable \( X \). In this particular case, the link between the infinitesimal generator and its associated semi-group allows us to easily compute the eigenelements of the self-adjoint conditional expectation operator \( V \). Such an approach has proved to be fruitful in the econometrics of Markov processes (see, e.g. Hansen and Scheinkman, 1995; Hansen et al., 1998; Florens et al., 1998; Darolles and Gouriéroux, 1997; Darolles et al., 1998).
Assumption 5.4. The Markov process $X_t$ is reversible.

Assumption 5.4 ensures that the pricing operators are self-adjoint. Practically, the calculus of the eigenfunctions of the pricing operator $V$ is done as follows: the semi-group $V_\delta f = E^0[f(X_{t+\delta})/X_t]$, with $\delta > 0$, and its generator $A$ has the same eigenfunctions. Choosing the form of the stochastic differential equation followed by the factor allows us to derive the infinitesimal generator. Hence, we diagonalize the continuous time model to determine the $N$ first eigenfunctions.

If $(\rho_i, e_i)$ are the eigenelements of the conditional expectation operator with horizon $\delta$, then the eigenelements of the associated infinitesimal generator are $(\lambda_i, e_i)$, with $\lambda_i = (1/\delta)\ln(\rho_i)$ (see Davies, 1980, Theorem 2.16). Therefore, if we know how to diagonalize the generator, we obtain the eigenvalues and eigenfunctions that we need to build the optimal approximation of the conditional expectation operator.

5.1. The Vasicek example

Let us assume that the factor $X_t$ is driven by an Ornstein–Uhlenbeck process

$$dX_t = \beta (\alpha - X_t) dt + \sigma dW_t,$$

where $W_t$ is a Brownian motion. In this case, $L^2$ is the space of square integrable functions with respect to a Gaussian measure, with mean $\alpha$ and variance $\sigma^2/2\beta$. Indeed, this space is separable and Assumption 2.1 is fulfilled. The associated infinitesimal generator $A$ is defined by

$$Af = \beta (\alpha - x) \frac{d}{dx} f + \frac{1}{2} \sigma^2 \frac{d^2}{dx^2} f,$$

(5.1)

where $f$ belongs to the domain $D(A)$, defined by

$$D(A) = \{ f \in L^2: Af \text{ exists} \}.$$

The eigenelements $(\lambda_k, e_k)$ of the infinitesimal generator $A$ satisfy the following differential equation:

$$\beta (\alpha - x)e_k' + \frac{1}{2} \sigma^2 e_k'' = \lambda_k e_k,$$

(5.2)

From Assumption 5.4, the infinitesimal generator (5.1) is self-adjoint. Hence, all the eigenvalues $\lambda_k$ are real. Moreover, $-A$ is a positive operator, and then, the differential equation (5.2) has no solution for $\lambda_k > 0$. For negative $\lambda_k$, the

5 For any function $f \in D(A)$, $\langle -Af, f \rangle \geq 0$. 
solutions have discrete eigenvalues\(^6\) and are given by \(\lambda_k = -\beta k\), associated with the Hermite polynomials\(^7\) \(e_k(x) = He_k(x), k = 0, 1, 2, \ldots\).

Let \(V\) be the pricing operator defined by (2.2), \(T\) the maturity date of the payoff \(f\), and \(t\) the current date at which we consider the pricing formula. The norm of \(V\) is equal to \(\sum_{k=0}^{\infty} e^{-2\beta k(T-t)}\), which is finite. Hence, the integrability condition (3.3) is fulfilled and we can apply Theorem 4.2. For any payoff \(f\) of \(L^2\), the optimal approximation of the pricing formula takes the following form:

\[
V_E f(x) = \sum_{k=0}^{N} e^{-\beta k(T-t)} \langle f, He_k \rangle He_k(x).
\]

Since each Hermite polynomial \(He_k\) is a linear combination of \(x^i, i \leq k\), the projection on the space generated by the first \(N+1\) eigenfunctions is equivalent to the best polynomial approximation of order \(N\) of the payoff.

5.2. The Brownian example

In this example, the dynamics of the state variable \(X\) are modeled by a Brownian motion, with variance \(\sigma^2\), and with two reflecting barriers at 0 and \(l\) to ensure the stationarity assumption. Hence, the stochastic differential equation followed by \(X\) is

\[
dX_t = \sigma dW_t,
\]

where \(W_t\) is a Brownian motion. In this case, \(L^2\) is the space of square integrable functions with respect to Lebesgue measure on \([0, l]\). Indeed, this space is separable and Assumption 2.1 is fulfilled. The associated infinitesimal generator \(A\) is

\[
Af = \frac{1}{2} \sigma^2 \frac{d^2}{dx^2} f,
\]

(5.3)

where \(f\) belongs to the domain \(D(A)\), defined by

\[
D(A) = \{ f \in L^2: Af \text{ exists and } f'(0) = f'(l) = 0 \},
\]

because of the reflecting property. The eigenelements \((\lambda_k, e_k)\) of \(A\) satisfy the following differential equation:

\[
\frac{1}{2} \sigma^2 e_k'' = \lambda_k e_k
\]

(5.4)

\(^6\) This would not hold if we would not restrict to \(L^2\), see Holland (1990).

\(^7\) See Wong (1964) for a set of conditions that allow such decomposition. It is also possible to directly check that Assumptions 3.2 and 3.3 are satisfied in the Ornstein–Uhlenbeck case. Thus, the operator \(A\) is compact and then has a discrete spectrum.
with boundary conditions $e_k'(0) = 0$ and $e_k'(l) = 0$. The infinitesimal generator (5.3) is self-adjoint. Hence, all the eigenvalues $\lambda_k$ are real. Moreover, $-A$ is a positive operator, and then, the differential equation (5.4) has no solution for $\lambda_k > 0$. The general form of the solution for negative $\lambda_k$ is the following:

$$e(x) = C_1 \cos\left(\frac{\sqrt{-2\lambda}}{\sigma} x\right) + C_2 \sin\left(\frac{\sqrt{-2\lambda}}{\sigma} x\right).$$

The first boundary condition, $e'(0) = 0$, gives $C_2 = 0$, and the second one, $e'(l) = C_1 \sqrt{-2\lambda}/\sigma \sin((\sqrt{-2\lambda}/\sigma)l) = 0$, allows us to compute solutions of (5.4):

$$\lambda_k = -\frac{1}{2} \left(\frac{k\pi}{l}\right)^2, \quad e_k(x) = \cos\left(\frac{k\pi}{T} x\right), \quad k = 1, \ldots.$$

Let $V$ be the pricing operator defined by (2.2), $T$ the maturity date of the payoff $f$, and $t$ the current date at which we consider the pricing formula. The norm of $V$ is equal to $\sum_{k=1}^{\infty} e^{-(k\pi/l)^2(T-t)}$, which is finite. Hence, the integrability condition 3.3 is fulfilled and, we can then apply Theorem 4.2. For any payoff $f$ of $L^2$, the optimal approximation of the pricing formula takes the following form:

$$V_E f(x) = \sum_{k=1}^{N} e^{-T-t/(2(k\pi/l)^2)} \left\langle f, \cos\left(\frac{k\pi}{T} x\right) \right\rangle \cos\left(\frac{k\pi}{T} x\right).$$

This formula does not correspond to a polynomial approximation of order $N$ as in the previous example. Let us notice that, in this case, the optimal approximation of order $N$ corresponds to a Fourier series expansion of the payoff.

6. Numerical applications

We develop in this section a numerical study in the Vasicek framework introduced in Section 5.1. The factor is the short-term interest rate $r_t$, the solution of the stochastic differential equation:

$$dr_t = \beta(z - r_t) dt + \sigma dW_t,$$

where the parameters values are $z = 0.06$, $\beta = 0.3$, $\sigma = 0.01$ and the initial rate $r_0 = 0.06$.

6.1. Call option on zero-coupon bond example

Vasicek (1977) shows that there exists an explicit solution for the price of the zero-coupons, and Jamshidian (1989) obtained simple closed-form results for
prices of European call options on the $U$-maturity zero-coupon bond with exercise rate $r_k$ and expiration date $T \leq U$. We represent in Figs. 1 and 2 the payoff of a call option of maturity $T = 1$ and strike rate $r_K = 0.06$ on a $(T + 1)$-maturity zero-coupon bond, and its pricing formula at time $t = 0$.

The computation of the approximated price consists first of projecting the payoff onto the basis formed by the Hermite polynomials of order $N$, orthogonal under the marginal law of the factor $r_t$. Then the approximated price corresponds to the price of the approximated payoff. Figs. 3–6 give two examples of this method, for two choices of the dimension of the space on which we project the payoff. We represent the approximated and the true payoffs, and the corresponding pricing formulas.

At this point, we see that, for a call with a long maturity (one year in our example), the price approximation of degree 2 (delta/gamma approximation) has some good properties (see Fig. 3). Due to the smoothing effect of the pricing operator, this ensures a good approximation to the pricing formula (see Fig. 4). As an example, Figs. 5 and 6 show the payoff and the pricing formula approximations when using polynomials of higher order. We see that, for a call payoff, the increase in accuracy is rather small and that a second degree approximation may seem reasonable.

These figures allow us to make the following six comments:

1. The accuracy of the price approximation decreases when the interest rate $r$ becomes different from 6%. This comes from the fact we use a weighting

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8 We will further only subscript $r$ when it is necessary to prevent confusion.
function centered on the mean of $r$ (i.e. the marginal density of $r$) to compute the best approximation.

2. The payoff approximation around $r = 6\%$ is bad. This is due to the irregularity of the payoff at this point, and the difficulty to use a polynomial approximation for non-differentiable functions. This problem does not occur for
pricing formulas because of the regularization properties of the pricing operator.

3. Recall that $L^2$ convergence implies convergence in distribution. So the distributions of the true pricing formula and of the approximated pricing formula are close. This gives us the convergence of value at risk measures because they are computed as quantiles of the portfolio value distribution.
4. The smoothing effect may be analyzed as follows. Since

\[ \| V^p - V^{pE} \|^2 = \sum_{i=N+1}^{\infty} \lambda_i^{2p}, \]

and \(0 \leq \lambda_i \leq 1\), we see that we get better approximations of the pricing operator corresponding to payoffs being paid far-away (\(\| V^p - V^{pE} \|^2 \) decreases in \(P\)).

5. From the proof of Theorem 4.1, we can state that

\[ \sup_{f \in L_2} \frac{|V(f) - V^{pE}(f)|^2}{|f|^2} \leq \| V - V^{pE} \|^2. \]

Thus, it is possible to give an upper bound on the error of the pricing formula that is independent on the payoff.

6. Now, let us compare, for a given payoff, the errors of the pricing formula \( |V(f) - V^{pE}(f)| \) and of the payoff \( |(I - P_E)(f)| \). We can write \( f = \sum_{i=1}^{\infty} \langle f, e_i \rangle e_i \), where \( e_i \) are the eigenfunctions of \( V \). Then

\[ |V(f) - V^{pE}(f)|^2 = \sum_{i=N+1}^{\infty} \lambda_i^2 \langle f, e_i \rangle^2 \leq \lambda_{N+1}^2 |(I - P_E)(f)|^2, \]

since the sequence \( \lambda_i, i = 1, \ldots \), is decreasing. We know that in the case of Vasicek model \( \lambda_i = \exp(-\beta i) \). Thus,

\[ |V(f) - V^{pE}(f)|^2 \leq e^{-2\beta(N+1)} |(I - P_E)(f)|^2. \]
We have then obtained a uniform upper bound for the pricing error in our Vasicek example. There is a sharp decline in the approximation error of the pricing formula.

It is clear that the pricing error decreases when the dimension of the approximation space increases. To have a more exact idea of the validity of the projection approach, we compute the pricing error norm \( |V(f) - V \cdot P_E(f)|/|V(f)| \) for different choices of the dimension of \( E \), where \( f \) is the payoff of the call option on a zero-coupon bond. We obtain results given in Table 1.

### 6.2. Call spread example

Let us now consider a call spread option built from the previous example by buying a European call option with strike rate \( r_k = 0.06 \) and by selling a European call option with strike rate \( r_k = 0.055 \).

As in the previous subsection, the two call options are on the \( U \)-maturity zero-coupon bond and have an expiration date \( T \leq U \). We fix the maturities \( T = 1 \) and \( U = T + 1 \). The true price is computed as the difference of the two call option prices given by the closed-form formula available for European call options on zero coupon bonds.

Figs. 7 and 8 represent the approximated and the true payoffs, and the corresponding pricing formulas. In this example the price approximation of order two does not have good properties, due to the high nonlinearity of the payoff. We need to use a higher order approximation to obtain better results (see Figs. 9 and 10).

This example also shows how the approximation procedure can be used in a portfolio context: if we add a new option to a portfolio, we need only compute its payoff approximation and to add it to the portfolio approximation. So, it is not necessary to compute again the whole portfolio approximation.
7. Conclusion

We have considered the problem of the optimal approximation of payoffs and pricing formulas. Using Hilbert space techniques and the spectral analysis of the
pricing operator we have been able to prove the optimality of the polynomial approximations when the eigenfunctions of the pricing operator are polynomials of increasing order. Two examples based on diffusion processes show that the computation of such eigenfunctions is tractable. These techniques may be relevant when a bank has to compute the value at risk of large options.
portfolios: our approximation ensures that value at risk is well approximated. It is easy to aggregate a new product in a portfolio by simply adding its approximation. We are able to deal with options of several maturities. At least we can provide upper bounds on the approximation error due to the uniform convergence property. Extensions to other cases such as the Cox, Ingersoll and Ross model or multifactor models may also be considered within our framework.

Appendix A. Proofs

A.1. Proof of Theorem 4.1

We consider an Hilbertian basis \( w_i, i = 1, \ldots \), of the space \( L^2 \). This basis allows us to build an orthogonal basis \( w_{ij}, i,j = 1, \ldots \), of \( L^2 \) by setting \( w_{ij} = w_i \otimes w_j \). Let \( E \) be the \( N \)-dimensional subspace of \( L^2 \) generated by the first \( N \) elements of the basis \( w_i, i = 1, \ldots, N \), and \( P_E \) the projection operator on this subspace. For any linear bounded operator \( V \), with kernel \( k \), satisfying Assumptions 3.2 and 3.3, \( V \circ P_E \) is also an integral operator, and we have

\[
\|V - V \circ P_E\|^2 = |k - k_E|^2,
\]

where \( k_E \) is the \( N \)-rank kernel associated with \( V \circ P_E \). From the decomposition (3.1) and (4.1) of kernels \( k \) and \( k_E \), we can write \( k - k_E \) as

\[
k(x, y) - k_E(x, y) = \sum_{i=N+1}^{\infty} \sum_{j=1}^{\infty} \alpha_{ij} w_i(y) w_j(x),
\]

where \( \alpha_{ij} = \langle V w_i, w_j \rangle \) are the coordinates of \( k \) and \( k_E \) in \( L^2 \). Then, we have

\[
\|V - V \circ P_E\|^2 = \sum_{i=N+1}^{\infty} \sum_{j=1}^{\infty} |\alpha_{ij}|^2 = \sum_{i=N+1}^{\infty} |V w_i|^2.
\]

Using property (3.2), we have the convergence to zero of the series \( \sum_{i=N+1}^{\infty} |V w_i|^2 \) when \( N \) goes to infinity. The following step consists of the majorization:

\[
|V(f) - V \circ P_E(f)|^2 = \left| \sum_{i=N+1}^{\infty} \langle f, w_i \rangle V w_i \right|^2 \leq \sum_{i=N+1}^{\infty} |\beta_i|^2 \cdot \sum_{i=N+1}^{\infty} |V w_i|^2,
\]

where \( \sum_{i=N+1}^{\infty} |\beta_i|^2 = \sum_{i=N+1}^{\infty} |\langle f, w_i \rangle|^2 \leq |f|^2 \). This inequality gives a majorization of the supremum norm:

\[
\sup_{f \in L^2} \frac{|V(f) - V \circ P_E(f)|^2}{|f|^2} \leq \|V - V \circ P_E\|^2,
\]

and the convergence to zero of the pricing error when \( N \) goes to infinity.
A.2. Proof of Lemma 4.1

We consider an Hilbertian basis \( w_i, i = 1, \ldots, \) of the space \( L^2 \). This basis allows us to build an orthogonal basis \( w_{ij}, i,j = 1, \ldots, \) of \( L^2 \) by setting \( w_{ij} = w_i \otimes w_j \). Let \( E \) be the \( N \)-dimensional subspace of \( L^2 \) generated by the first \( N \) elements of the basis \( w_i, i = 1, \ldots, N \), and \( P_E \) the projection operator on this subspace. So, for any linear bounded operator \( V \), with kernel \( k \), satisfying Assumptions 3.2 and 3.3, we have

\[
\|V \circ P_E\|^2 = \|k_E\|^2 = \sum_{i=1}^{N} \sum_{j=1}^{\infty} |x_{ij}|^2 = \sum_{i=1}^{N} |V(w_i)|^2,
\]

where \( x_{ij} = \langle Vw_i, w_j \rangle \). Using the same decomposition idea as used in the proof of Theorem 4.1, we have

\[
\|V - V \circ P_E\|^2 = \|k - k_E\|^2 = \sum_{i=N+1}^{\infty} \sum_{j=1}^{\infty} |x_{ij}|^2 = \sum_{i=N+1}^{\infty} |V(w_i)|^2.
\]

The summation of the two previous terms gives us the equality

\[
\|V\|^2 = \|V \circ P_E\|^2 + \|V - V \circ P_E\|^2.
\]

A.3. Proof of Theorem 4.2

We consider an Hilbertian basis \( w_i, i = 1, \ldots, \) of the space \( L^2 \). Looking for the subspace \( E \), the solution of problem (4.2) is equivalent to looking for the subspace solution to the problem (4.3), using Corollary 4.1. We write \( I = \|V\|^2 \) and \( I(E) = \|V \circ P_E\|^2 \). Properties of the norm give us directly \( I = I(E) + I(E^\perp) \).

Rewriting the criterion \( I(E) \) in terms of operator \( V \) gives us

\[
I(E) = \sum_{i=1}^{N} \sum_{j=1}^{\infty} |\langle V w_i, w_j \rangle|^2 = \sum_{i=1}^{N} |V(w_i)|^2
\]

\[
= \sum_{i=1}^{N} \langle V w_i, V w_i \rangle = \sum_{i=1}^{N} \langle V^* V w_i, w_i \rangle,
\]

and the problem to solve is then

\[
\text{Max} \sum_{(w_i)}^{N} \langle V^* V w_i, w_i \rangle.
\]

We are going to build iteratively the \( N \)-dimensional optimal projection space. Suppose an \( N - 1 \)-dimensional optimal projection space is built with the first \( N - 1 \) eigenfunctions of the operator \( V^* V \), then

\[
I(E_{N-1}) = \sum_{i=1}^{N-1} \lambda_i^2.
\]
Let $E$ be any $N$-dimensional subspace of $L^2$. Then $I(E)$ can be decomposed:

$$I(E) = I(P_{E_{N-1}}(E)) + I(P_{E_N}(E)).$$

Since,

$$I(P_{E_{N-1}}(E)) \leq \sum_{i=1}^{p} \lambda_i^2,$$

where $p$ is the dimension of the space $P_E(E_{N-1})$, and

$$I(P_{E_N}(E)) \leq \sum_{i=N}^{N+(N-p)} \lambda_i^2 \leq \sum_{i=p}^{N} \lambda_i^2,$$

because the sequence of eigenvalues is strictly decreasing, we have

$$I(E) \leq \sum_{i=1}^{N} \lambda_i^2,$$

for any $N$-dimensional space $E$. The supremum is reached if we choose the space of the first $N$ eigenfunctions, which is the $N$-dimensional projection space maximizing $I(E)$.

### A.4. Proof of Corollary 4.2

Consider $V$, an operator, satisfying Assumptions 3.2 and 3.3, and the subspace $E$ of $L^2$ generated by the first $N$ elements $w_i, i = 1, \ldots, N$, of an Hilbertian basis, then, following the proof of Theorem 4.2, we have to maximize the criterion $\tilde{I}(E) = \|P_E \circ V\|^2$, which can be written as

$$\tilde{I}(E) = \sum_{i=1}^{\infty} \sum_{j=1}^{N} |\langle V w_i, w_j \rangle|^2 = \sum_{j=1}^{N} |V^* w_j\rangle|^2$$

$$= \sum_{i=1}^{N} \langle V^* w_i, V^* w_i \rangle = \sum_{i=1}^{N} \langle V V^* w_i, w_i \rangle.$$

It is then sufficient to notice that, if $w_i, i = 1, \ldots, N$, are the first $N$ eigenfunctions of $V^* V$ associated with the eigenvalues $\rho_i$, then $\rho_i^{-1/2} V w_i, i = 1, \ldots, N$, are the first $N$ eigenfunctions of $V V^*$ associated with the same eigenvalues. These functions generate the optimal subspace and the optimal pricing formula approximation is, for any $f$ in $L^2$:

$$P_E \circ V(f) = \sum_{i=1}^{N} \langle V f, \rho_i^{-1/2} V w_i \rangle \rho_i^{-1/2} V(w_i) = \sum_{i=1}^{N} \rho_i^{-1} \langle V^* V f, w_i \rangle V(w_i)$$

$$= \sum_{i=1}^{N} \langle f, w_i \rangle V(w_i) = V \circ P_E(f).$$
A.5. Proof of Corollary 4.3

Consider $V$, a self-adjoint operator, satisfying Assumptions 3.2 and 3.3, and the subspace $E$ of $L^2$ generated by the first $N$ eigenfunctions $w_i$, $i = 1, \ldots, N$, of $V$, then, following Theorem 4.2 and Corollary 4.2, we have directly, for any $f$ in $L^2$

$$V_E(f) = \sum_{i=1}^{N} \langle f, w_i \rangle V(w_i),$$

because operators $V$ and $V^*V$ have the same eigenfunctions in the self-adjoint case. Noting that the eigenfunction $w_i$ has eigenvalue $\lambda_i$, we can write

$$V_E(f) = \sum_{i=1}^{N} \lambda_i \langle f, w_i \rangle w_i.$$

A.6. Proof of Corollary 4.4

Consider $V$, an operator, satisfying Assumptions 3.2 and 3.3, and the subspace $E$ of $L^2$ generated by the first $N$ elements $w_i$, $i = 1, \ldots, N$, of an Hilbertian basis. $P_E \circ V \circ P_E$ is the integral operator with kernel:

$$\tilde{k}_E(y, x) = \sum_{i=1}^{N} \sum_{j=1}^{N} \langle Vw_i, w_j \rangle w_i(y)w_j(x).$$

Following the proof of Theorem 4.2, we have to maximize the criterion $	ilde{I}(E) = \|P_E \circ V \circ P_E\|^2$, which can be written as

$$\tilde{I}(E) = \sum_{i=1}^{N} \sum_{j=1}^{N} |\langle Vw_i, w_j \rangle|^2.$$

The solution of this problem is the space generated by the $N$ first eigenfunctions of $V^*V$ (see Riesz and Nagy, 1990, Chapter 6, p. 243).

References


