On dynamic investment strategies

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Abstract

This paper presents a new approach for analyzing dynamic investment strategies. Previous studies have obtained explicit results by restricting utility functions to a few specific forms; not surprisingly, the resultant dynamic strategies have exhibited a very limited range of behavior. In contrast, we examine what might be called the inverse problem: given any specific dynamic strategy, can we characterize the results of following it through time? More precisely, can we determine whether it is self-financing, yields path-independent returns, and is consistent with optimal behavior for some expected utility maximizing investor? We provide necessary and sufficient conditions for a dynamic strategy to satisfy each of these properties. \copyright\ 2000 Published by Elsevier Science B.V. All rights reserved.

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Editor's note: Although presented in the privately circulated Proceedings of the Seminar on the Analysis of Security Prices, Center for Research in Security Prices, University of Chicago, this paper has remained unpublished for a variety of reasons. For almost 20 years it has been frequently referred to and used by many financial economists, in areas as diverse as portfolio insurance and risk premium analysis. I am certain that its publication will ensure its wider availability and further research into fundamental issues.
1. Introduction

This paper presents a new approach for analyzing dynamic investment strategies. Previous studies have obtained explicit results by restricting utility functions to a few specific forms; not surprisingly, the resultant dynamic strategies have exhibited a very limited range of behavior. In contrast, we examine what might be called the inverse problem: given any specific dynamic strategy, can we characterize the results of following it through time? More precisely, can we determine whether it is self-financing, yields path-independent returns, and is consistent with optimal behavior for some expected utility maximizing investor? We provide necessary and sufficient conditions for a dynamic strategy to satisfy each of these properties.

Our results permit assessment of a wide range of commonly used dynamic investment strategies, including 'rebalancing', 'constant equity exposure', 'portfolio insurance', 'stop loss', and 'dollar averaging' policies.

Indeed, any dynamic strategy that specifies the amount of risky investment or cash held as a function of the level of investor wealth, or of the risky asset price, can be analyzed with our techniques.

We obtain explicit results for general dynamic strategies by assuming a specific description of uncertainty. We consider a world with one risky asset (a stock) and one safe asset (a bond). We assume that the bond price grows deterministically at a constant interest rate and that the stock price follows a multiplicative random walk that includes geometric Brownian motion as a limiting special case. This limiting case, which implies that the price of the risky asset has a lognormal distribution, has been widely used in financial economics. The restriction to a single risky asset involves no significant loss of generality, since it can be taken to be a mutual fund. Furthermore, our basic approach can be applied to other kinds of price movements. We assume that the risky asset pays no dividends. We explain later why this too involves no real loss of generality. We allow both borrowing and short sales with full use of the proceeds. We further assume that all markets are frictionless and competitive.

We wish to make full use of the tractibility that continuous time provides for characterizing optimal policies. However, we appreciate the view that the economic content of a continuous-time model is clearer when it is obtained as the limit of a discrete-time model. Accordingly, we always establish our results in a setting in which trading takes place at discrete times and the stock price

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1 For example, see Hakansson (1970), Levrari and Srinivasan (1969), Merton (1971), Phelps (1962), and Samuelson (1969).

2 For example, see Merton (1969) and Merton (1971). See also Black and Scholes (1973) and much of the extensive literature based on their article.

3 See Merton (1971) and Ross (1978).
follows a multiplicative random walk. In the statement of our propositions and in our examples, we emphasize the limiting form of our results as the trading interval approaches zero. It is well known that the (logarithms of the) approximating random walks so obtained provide a constructive definition of Brownian motion. Alternatively, our basic approach can be formulated directly in continuous time.

To address the issues, we shall need the following definitions. An investment strategy specifies for the current period and each future period:

(a) the amount to be invested in the risky asset,
(b) the amount to be invested in the safe asset, and
(c) the amount to be withdrawn from the portfolio.

In general, the amounts specified for any given period can depend on all of the information that will be available at that time. A feasible investment strategy is one that satisfies the following two requirements. The first is the self-financing condition: the value of the portfolio at the end of each period must always be exactly equal to the value of the investments and withdrawals required at the beginning of the following period. The second is the nonnegativity condition: the value of the portfolio must always be greater than or equal to zero. Feasible investment strategies are thus the only economically meaningful ones that can be followed at all times and in all states without being supplemented or collateralized by outside funds. A path-independent investment strategy is one for which the controls (amounts (a)–(c)) can be written as functions only of time and the price of the risky asset. Hence, the value of the portfolio at any future time will depend on the stock (and bond) price at that time, but it will not depend on the path followed by the stock in reaching that price. We shall see that path independence is necessary for expected utility maximization. Furthermore, portfolio managers may find path independence to be very desirable even when they are not acting as expected utility maximizers. For example, without a path-independent strategy, a portfolio manager could hold a long position throughout a rising market yet still lose money because of the particular price fluctuations that happened to occur along the way.

In Section 2, we establish some preliminary results that will be needed in later sections. We consider the case where the controls of an investment strategy are

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4 For an example of the use of a discrete random walk in an investment problem, see Griffeth and Snell (1974). Our subsequent arguments draw on an application of the multiplicative random walk initiated by Sharpe (1981) and developed further in an expository article by Cox et al. (1979).

5 See Knight (1962). In particular, consider a sequence of random walks and a limit Brownian motion defined on the same probability space. It can be shown that the sample paths of the random walks converge uniformly with probability one to the corresponding paths of the Brownian motion. The random walks thus provide a constructive definition of the path functions, field of measurable sets, and probability measure of Brownian motion.
given as functions of time and the value of the risky asset. This situation often arises in the valuation of contingent claims, so our results will also be of some use there. We find necessary and sufficient conditions for the investment strategy to be feasible. These conditions take the form of a set of linear partial differential equations that must be satisfied by the amounts invested. In other words, if these equations are not satisfied, then the strategy cannot always be maintained; if they are satisfied, then it can be. By construction, the policy in this case is path independent.

In the third section, we consider the case where the controls depend on time and the value of the portfolio (wealth). Here we find that feasibility places no substantive restrictions on the investment strategy, but path independence does. We show that a given investment strategy will be path independent if and only if the amounts satisfy a particular nonlinear partial differential equation.

Finally, in Section 4 we develop necessary and sufficient conditions for a given investment strategy to be consistent with expected utility maximization for some nondecreasing concave utility function. It turns out that these conditions are closely related to the results of Section 3.6

We shall use the following notation:

- \( S(t) \) stock price at time \( t \)
- \( u \) one plus the rate of return from an upward move
- \( d \) one plus the rate of return from a downward move
- \( r \) the one period interest rate (in the limiting cases, \( r \) stands for the continuous interest rate)
- \( q \) probability of an upward move
- \( p = \frac{(1 + r) - d}{u - d} \)
- \( m \) local mean of the limiting lognormal process
- \( \sigma^2 \) local variance of the limiting lognormal process
- \( G(S(t), t) \) portfolio control specifying the number of dollars invested in stock at time \( t \) as a function of the stock price at time \( t \)
- \( H(S(t), t) \) portfolio control specifying the number of dollars invested in bonds at time \( t \) as a function of the stock price at time \( t \)
- \( K(S(t), t) \) portfolio control specifying the number of dollars withdrawn from the portfolio at time \( t \) as a function of the stock price at time \( t \)
- \( W(t) \) wealth at time \( t \)
- \( A(W(t), t) \) portfolio control specifying the number of dollars invested in stock at time \( t \) as a function of wealth at time \( t \)

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6 In a related paper, Dybvig (1982) shows that our path-independence condition has important implications for the recoverability of individual preferences from observed actions.
$B(W(t), t)$ portfolio control specifying the number of dollars invested in bonds at time $t$ as a function of wealth at time $t$

$C(W(t), t)$ portfolio control specifying the number of dollars withdrawn from the portfolio at time $t$ as a function of wealth at time $t$

Subscripts on $A$, $B$, $C$, $G$, $H$, and $K$ indicate partial derivatives. We shall say that the functions $A$, $B$, and $C$ are differentiable if the partial derivatives $A_t$, $A_W$, $A_{WW}$, $B_t$, $B_W$, $B_{WW}$, and $C_W$ are continuous on $[t, T) \times (z, \infty)$ for a specified $z$ and $T$. A similar definition applied for $G$, $H$, and $K$.

With this notation, the multiplicative random walk can be specified in the following way: for each time $t$, the stock price at time $t+1$ conditional on the stock price at time $t$ will be either $uS(t)$ with probability $q$ or $dS(t)$ with probability $1 - q$. To rule out degenerate or pathological cases, we require that $1 > q > 0$ and $u > 1 + r > d$.

2. Controls that are functions of the value of the risky asset and time

In this section, we establish some preliminary results that will be needed later. Consider the case where the controls are specified as functions of the value of the risky asset (the stock price) and time. Such a strategy is obviously path independent. Furthermore, the nonnegativity requirement can be verified directly from inspection of the functions $G(S(t), t)$ and $H(S(t), t)$: $G + H$ must be greater than or equal to zero for all $s(t)$ and $t$. The self-financing requirement is discussed below.

Proposition 1. Necessary and sufficient conditions for the differentiable functions $G(S(t), t)$, $H(S(t), t)$ and $K(S(t), t)$ to be the controls of a self-financing investment strategy are that $G$, $H$, and $K$ satisfy

\[
\frac{1}{2} \sigma^2 S^2 G_{SS} + r S G_S + G_t - r G + SK_S = 0
\]

(1)

\[
\frac{1}{2} \sigma^2 S^2 H_{SS} + r S H_S + H_t - r H + K - SK_S = 0
\]

(2)

\[
S(G_S + H_S) = G
\]

(3)

for all $S(t)$ and $t$.

Proof. We first show that necessary and sufficient conditions for self-financing in the discrete-time model are

\[
pG(uS(t), t + 1) + (1 - p)G(dS(t), t + 1) - (1 + r)G(S(t), t)
\]

\[+ [(1 + r)/(u - d)] [K(uS(t), t + 1) - K(dS(t), t + 1)] = 0,
\]

(4)
\[ \begin{align*}
  pH(uS(t), t + 1) + (1 - p)H(dS(t), t + 1) - (1 + r)H(S(t), t) \\
  + pK(uS(t), t + 1) + (1 - p)K(dS(t), t + 1) \\
  - [(1 + r)/(u - d)] [K(uS(t), t + 1) - K(dS(t), t + 1)] = 0, \\
  \end{align*} \]

(5)

\[ \begin{align*}
  [G(uS(t), t + 1) - G(dS(t), t + 1)] + [H(uS(t), t + 1) - H(dS(t), t + 1)] \\
  + [K(uS(t), t + 1) - K(dS(t), t + 1)] = (u - d)G(S(t), t). \\
  \end{align*} \]

(6)

By definition, an investment strategy is self-financing if and only if

\[ \begin{align*}
  uG(S(t), t) + (1 + r)H(S(t), t) &= G(uS(t), t + 1) + H(uS(t), t + 1) \\
  + K(uS(t), t + 1), \\
  \end{align*} \]

(7)

\[ \begin{align*}
  dG(S(t), t) + (1 + r)H(S(t), t) &= G(dS(t), t + 1) + H(dS(t), t + 1) \\
  + K(dS(t), t + 1). \\
  \end{align*} \]

(8)

The equivalence of (4)–(6) and (7)–(8) follows directly from a series of straightforward operations.

**Sufficiency:** Eqs. (4)–(6) imply self-financing. Let \( F(S(t), t) \) denote \( G(S(t), t) + H(S(t), t) \). Eq. (6) can then be written as

\[ \begin{align*}
  F(uS(t), t + 1) - F(dS(t), t + 1) = (u - d)G(S(t), t). \\
  \end{align*} \]

(9)

Eqs. (4) and (5) imply that

\[ \begin{align*}
  pF(uS(t), t + 1) + (1 - p)F(dS(t), t + 1) \\
  = (1 + r)[G(S(t), t) + H(S(t), t)]. \\
  \end{align*} \]

(10)

Solving (9) and (10) for \( F(uS(t), t + 1) \) and \( F(dS(t), t + 1) \) in terms of \( G(S(t), t) \) and \( H(S(t), t) \) gives (7) and (8).

**Necessity:** Self-financing implies (4)–(6). Subtracting each side of (8) from the corresponding side of (7) gives

\[ \begin{align*}
  G(S(t), t) = [F(uS(t), t + 1) - F(dS(t), t + 1)]/(u - d), \\
  \end{align*} \]

(11)

which gives (6). To see that self-financing implies (4) and (5), add \( p \) times (7) to (1 - \( p \)) times (8) and rearrange to get

\[ \begin{align*}
  pF(uS(t), t + 1) + (1 - p)F(dS(t), t + 1) \\
  - (1 + r)F(S(t), t) + (1 + r)K(S(t), t) = 0. \\
  \end{align*} \]

(12)

Eq. (11) implies that

\[ \begin{align*}
  G(uS(t), t + 1) - G(dS(t), t + 1) \\
  = [F(u^2S(t), t + 2) - 2F(udS(t), t + 2) + F(d^2S(t), t + 2)]/(u - d). \\
  \end{align*} \]

(13)
By using (12) to express the right-hand side of (13) in terms of \(F(uS(t), t + 1)\) and \(F(dS(t), t + 1)\) and rearranging, we obtain (4). Eq. (5) follows from using the definition of \(F\) to express (4) in terms of \(H\) and \(K\) and then applying (12).

We conclude the proof by outlining the argument showing that (4)–(6) converge to (1)–(3) as the trading interval approaches zero. Consider dividing each trading period into \(n\) periods, each of length \(d\). To obtain convergence to geometric Brownian motion, let \(u\), \(d\) and \(q\) depend on \(d\) in the following way:

\[
u = 1 + \sigma \sqrt{d}, \quad d = 1 - \sigma \sqrt{d}, \quad q = \frac{1}{2} \left[ 1 + \frac{m}{\sigma \sqrt{d}} \right]. \tag{14}\]

To maintain the appropriate interest and withdrawal flow, let the interest rate and withdrawal amount for each new period be \(d\) times the corresponding amounts for the original period (of unit length). Now note that (4) (and analogously (5) and (6)) can be rewritten using (14) as

\[
\frac{1}{2} \sigma^2 S^2 \delta \left[ \frac{G(S + \eta, t) - 2G(S, t) + G(S - \eta, t)}{\eta^2} \right] + rS\delta \left[ \frac{G(S + \eta, t) - G(S - \eta, t)}{2\eta} \right] + \delta \left[ \frac{G(S, t) - G(S, t - \delta)}{\delta} \right] - r\delta G(S, t - \delta)
\]

\[
= - \left[ \delta + r\delta^2 \right] S \left[ \frac{K(S + \eta, t) - (K(S - \eta, t))}{2\eta} \right]. \tag{15}\]

where \(\eta = \sigma S \sqrt{\delta}\). The limits of the terms in the square brackets as \(\delta\) approaches zero are the corresponding derivatives in (1).

In the presentation of the continuous-time case in Merton (1969), it is implicit that necessary and sufficient conditions for self-financing are

\[
\frac{1}{2} \sigma^2 S^2 F_{SS} + rSF_S + F_t - rF + K = 0, \tag{16}\]

\[
G = S(G_S + H_S). \tag{17}\]

Our analysis thus far develops this conclusion in a simple and explicit way and gives separate necessary conditions on each component of the portfolio.

Example 1. Consider the control \(G(S(t), t) = S(t)[1 - \exp(-\lambda(t)S(t))]\), where \(\lambda\) is a function of time. This is an apparently reasonable plan that increases the number of shares held to one as the stock price rises and decreases the number of shares held to zero as the stock price falls. However, it can be verified by direct substitution that with \(K = 0\) this control does not satisfy the necessary
condition on $G$ and hence there is no accompanying $H(S(t), t)$ for which the overall strategy will be feasible.

A few other comments are worth mentioning. When there are no withdrawals, both the amount invested in stock and the amount invested in bonds satisfies the same equation as the total value of the portfolio. Consequently, Eqs. (1) and (2) can be thought of as value conservation relations on the components of the portfolio. Even though the portfolio composition is changing dynamically over time, there can be no anticipated shifts in value between the stock and bond components. The current value of the random amount that will be held in stock on any future date must be the amount held in stock today and the same is true for bonds.

We conclude this section by citing a result that will be needed frequently in Section 4. Consider a hypothetical stock price process having the same sample paths as the true process but with a different probability measure defined on these paths. Let this probability measure be chosen so that the expected rate of return on the hypothetical process is in each period equal to the interest rate. In our model, this is accomplished by letting $q = p$. Let $E$ denote expectation with respect to this measure conditional on the current true stock price.

Lemma 1. Consider a claim paying the amount $\Phi(S(T))$ at time $T$, where $\Phi$ is a known function. Then

(i) the payoff to this claim can be replicated by a controlled self-financing portfolio of stock and bonds
(ii) the value of the claim at time $t$ is the value of the replicating portfolio and is given by

$$F(S(t), t) = G(S(t), t) + H(S(t), t) = e^{-r(T-t)}E\Phi(S(T)).$$


3. Controls that are functions of the value of the portfolio and time

Now consider the case where the controls are specified as functions of the value of the portfolio (wealth). Here the necessary and sufficient condition for self-financing is obvious: at each time $t$, the sum of the amount invested in stock, the amount invested in bonds, and the amount withdrawn from the portfolio must equal the value of the portfolio, $W(t)$. As the trading interval approaches zero, the withdrawal flow $C$ becomes negligible relative to the stocks $A$ and $B$, and the self-financing condition reduces to $A(W(t), t) + B(W(t), t) = W(t)$.

The conditions for nonnegativity are given below in Lemma 3. The conclusions of this lemma are readily apparent and widely known; our purpose is to
summarize them in our terms. We then derive the main result of this section, the condition for path independence.

**Lemma 2.** Necessary and sufficient conditions for the functions $A(W(t), t)$, $B(W(t), t) = W - A(W(t), t)$, and $C(W(t), t)$ to be the controls of a nonnegative investment strategy for all nonnegative initial values of wealth are

$$A(0, t) = 0, \quad C(0, t) = 0$$

for all $t$.

**Proof.** In the discrete case, a sufficient condition for wealth to be nonnegative is

$$[d - (1 + r)]A(W(t), t) + (1 + r)[W(t) - C(W(t), t)] \geq 0$$

for $A > 0$ and

$$[u - (1 + r)]A(W(t), t) + (1 + r)[W(t) - C(W(t), t)] \geq 0$$

for $A \leq 0$. This can be written as

$$A(W(t), t) = 0 \quad \text{and} \quad C(W(t), t) = 0 \quad \text{for} \quad W(t) = 0$$

and

$$\left[\frac{(1 + r)[W(t) - C(W(t), t)]}{(1 + r) - d}\right] \geq A(W(t), t)$$

for $W(t) > 0$. \hfill (20)

$$\left[\frac{-(1 + r)[W(t) - C(W(t), t)]}{u - (1 + r)}\right] \geq A(W(t), t)$$

for $W(t) > 0$. \hfill (21)

In the limiting continuous case described in the proof of Lemma 1, the left and right sides of inequality (21) go to $+\infty$ and $-\infty$, respectively, leaving (20) as the effective condition. This condition is obviously necessary as well if non-negativity is to hold for all possible nonnegative initial values. \hfill \square

For some strategies, (19) will not be necessary if initial wealth is restricted to lie in a specified interval. For example, if there is a function $g(t) = 0$ such that $A(g(t), t) = 0$ and $(1 + r)g(t) - C(g(t), t) = g'(t)$ and initial wealth $W(t)$ is greater than or equal to $g(t)$, then $W(t)$ will be greater than or equal to $g(t)$ for all $t$.

**Example 2.** Consider the fixed strategy in which the initial portfolio consists of the amount $L$ in bonds and the difference, $W(t) - L$, in stock. Subsequently, there are no changes in or withdrawals from the portfolio. At time $t > \tau$, $W(t) = [(W(\tau) - L)/S(\tau)]S(t) + Le^{(\tau - \tau)}$, so $A(W(t), t) = W(t) - Le^{(\tau - \tau)}$. If $W(t) \geq L \geq 0$, then the value of the portfolio can never become negative, since $S(t) \geq 0$. In this case, which corresponds to a simple buy-and-hold strategy, $g(t) = Le^{(\tau - \tau)}$. However, $A(W(t), t)$ does not satisfy (19), so this policy cannot be consistent with $W(t) \geq 0$ for all $W(t) \geq 0$. If $L > W(t) > 0$ or if $L < 0$, the value of the portfolio can become negative for sufficiently low stock prices.
In the first section, the restrictions imposed by feasibility are substantive, while those imposed by path independence are trivial. Now just the opposite is true: the restrictions imposed by feasibility are relatively trivial, but those imposed by path independence are substantive.

Proposition 2. A necessary and sufficient condition for the differentiable functions $A(W(t), t), B(W(t), t) = W - A(W(t), t)$, and $C(W(t), t)$ to be the controls of a path-independent investment strategy is that $A$ and $C$ satisfy

$$\frac{1}{2}\sigma^2 A_{ww} + (rW - C)A_w + A_t - (r - C)A = 0$$

for all $W(t)$ and $t$.

Proof. We first establish that a necessary and sufficient condition for path independence in the discrete-time model is that $A$ and $C$ satisfy

$$pA(W(t + 1)|u, t + 1) + (1 - p)A(W(t + 1)|d, t + 1) - (1 + r)A(W(t), t)$$

$$+ [(1 + r)/(u - d)] [C(W(t + 1)|u, t + 1) - C(W(t + 1)|d, t + 1)] = 0,$$

where $W(t + 1)|u(W(t + 1)|d)$ denotes wealth at time $t + 1$ conditional on the stock price moving up (down).

Sufficiency: Eq. (23) implies

$$A([u - (1 + r)]A(W(t), t) + (1 + r)W(t) - (1 + r)C(W(t), t), t + 1)$$

$$\times [d - (1 + r)] + (1 + r)[[u - (1 + r)]A(W(t), t) + (1 + r)W(t)$$

$$- (1 + r)C(W(t), t)] - (1 + r)C([u - (1 + r)]A(W(t), t)$$

$$+ (1 + r)W(t) - (1 + r)C(W(t), t), t + 1)$$

$$= A([d - (1 + r)]A(W(t), t) - (1 + r)W(t)$$

$$- (1 + r)C(W(t), t), t + 1)[u - (1 + r)]$$

$$+ (1 + r)[[d - (1 + r)]A(W(t), t) + (1 + r)W(t) - (1 + r)C(W(t), t)]$$

$$- (1 + r)C([d - (1 + r)]A(W(t), t) + (1 + r)W(t)$$

$$- (1 + r)C(W(t), t), t + 1)).$$

This follows from multiplying (23) by $(d - u)$, adding $(1 + r)[(1 + r)W(t) - (1 + r)C(W(t), t) - (1 + r)A(W(t), t)]$ to each side, and then rearranging to get (24). The left (right) hand side of (24) is the value of the portfolio at time $t + 2$ given that wealth at time $t$ is $W(t)$ and that the stock goes up (down) in the first period and down (up) in the second period. Consequently, (23) implies that the
stock price sequence $S \rightarrow uS \rightarrow udS$ leads to the same wealth at time $t + 2$ as the sequence $S \rightarrow dS \rightarrow udS$. This means that over an arbitrary number of periods any two paths that differ only in the interchanging of an adjoining $u$ and $d$ result in the same wealth. Furthermore, all paths leading to a given final stock price can be obtained from any single path by repeated interchanging of adjoining $u$’s and $d$’s. Hence, (23) implies that there is a single level of wealth associated with each final stock price.

**Necessity:** If $A(W(t), t), B(W(t), t)$, and $C(W(t), t)$ are the controls of a self-financing path-independent investment strategy, then there exists a function $F(S(t), t)$ satisfying

$$A(F(S(t), t), t)[u - (1 + r)] + (1 + r)F(S(t), t) - (1 + r)C(F(S(t), t), t)$$

$$= F(uS(t), t + 1),$$

$$A(F(S(t), t), t)[d - (1 + r)] + (1 + r)F(S(t), t) - (1 + r)C(F(S(t), t), t)$$

$$= F(dS(t), t + 1),$$

for all $S(t)$ and $t$. Eqs. (25) and (26) imply the following two equations:

$$pF(uS(t), t + 1) + (1 - p)F(dS(t), t + 1) - (1 + r)F(S(t), t)$$

$$+ (1 + r)C(F(S(t), t), t) = 0,$$

$$A(F(S(t), t), t) = S(t) \left[ \frac{F(uS(t), t + 1) - F(dS(t), t + 1)}{(u - d)S(t)} \right]$$

$$= G(S(t), t).$$

By using Lemma 1, we obtain

$$pA(F(uS(t), t + 1), t + 1) + (1 - p)A(F(dS(t), t + 1), t + 1)$$

$$- (1 + r)A(F(S(t), t), t)$$

$$= pG(uS(t), t + 1)$$

$$+ (1 - p)G(dS(t), t + 1) - (1 + r)G(S(t), t)$$

$$= - [(1 + r)/(u - d)] [C(F(uS(t), t + 1), t + 1)$$

$$- C(F(dS(t), t + 1), t + 1)],$$

which can be rewritten as (23),

$$pA(W(t + 1) | u, t + 1) + (1 - p)A(W(t + 1) | d, t + 1) - (1 + r)A(W(t), t)$$

$$+ [(1 + r)/(u - d)] [C(W(t + 1) | u, t + 1) - C(W(t + 1) | d, t + 1)] = 0.$$
To verify that (23) converges to (22), consider first the case \( C = 0 \). With \( u \) and \( d \) specified as in (14), (25) can be written as

\[
\frac{1}{2} \sigma^2 A^2(W(t), t) \delta \left[ \frac{A(\tilde{W}(t) + \eta, t) - 2A(\tilde{W}(t), t) + A(\tilde{W}(t) - \eta, t)}{\eta^2} \right]
+ rA(W(t), t) \delta \left[ \frac{A(\tilde{W}(t) + \eta, t) - A(\tilde{W}(t) - \eta, t)}{2\eta} \right]
+ \frac{\epsilon}{\delta} \frac{A(\tilde{W}(t), t) - A(\tilde{W}(t) - \epsilon, t - \delta)}{A(\tilde{W}(t) - \epsilon, t - \delta)} = r\delta A(\tilde{W}(t) - \epsilon, t - \delta),
\]

(30)

where \( \eta = \sigma A(W(t), t) \sqrt{\delta} \). \( \epsilon = r[W(t) - A(W(t), t)] \delta \). and \( \tilde{W}(t) = W(t) + \epsilon \). The limits of the terms in square brackets as \( \delta \) approaches zero are the corresponding derivatives in (22). The case \( C \neq 0 \) follows in the same way.

**Example 3.** Consider the controls \( A(W(t), t) = \lambda(t - \tau) \), \( B(W(t), t) = W(t) - \lambda(t - \tau) \), and \( C(W(t), t) = 0 \), where \( \lambda \) is a constant. This is a type of ‘dollar averaging’ strategy in which the amount invested in stock is being increased by a constant dollar flow. Trying this \( A \) as a solution to (22) shows that the equation is not satisfied. Consequently, \( A(W(t), t) = \lambda(t - \tau) \) cannot be the control of a path-independent portfolio. However, \( A(W(t), t) = \lambda e^{\rho_1(t - \tau)} \) does satisfy the equation, so path independence will hold if the dollar amount invested in the stock grows geometrically at the riskless interest rate rather than linearly. Hence it is possible to represent the value of the portfolio as a function of the stock price. It can be verified by direct substitution into the continuous-time limits of (27) and (28) that for this new choice of \( A \) the value of the portfolio at time \( t \), given an initial investment of \( W(\tau) \) at time \( \tau \), is

\[
F(S(t), t) = e^{\rho_1(t - \tau)}[W(\tau) + \lambda \log(S(t)/S(\tau)) + \lambda(1/2)\sigma^2 - (t - \tau)].
\]

(31)

Although \( A(W(t), t) = \lambda e^{\rho_1(t - \tau)} \) satisfies condition (22), it does not satisfy the nonnegativity condition (19). The consequences are obvious from (31): for sufficiently low stock prices, the portfolio can take on arbitrarily large negative values. Hence, this policy is path independent but not feasible.

**Example 4.** Now consider the investment strategy given by \( A(W(t), t) = \lambda W(t) \), \( B(W(t), t) = (1 - \lambda)W(t) \), and \( C(W(t), t) = \kappa W(t) \), where \( \lambda \) and \( \kappa \) are constants. This ‘rebalancing’ strategy is widely used in institutional portfolios. Unlike the previous example, this strategy is completely feasible, and since \( A \) and \( C \) satisfy (22), it is also path independent. By proceeding as before, we can verify that the value of the portfolio at time \( t \), given an initial investment of \( W(\tau) \) at time \( \tau \), is

\[
F(S(t), t) = W(\tau)(S(t)/S(\tau))^\lambda \exp[(1 - \lambda)(r + (1/2)\sigma^2\lambda) - \kappa(t - \tau)].
\]

(32)
An examination of (32) shows that $F(S(t), t)$ is an increasing convex function of $S(t)$ if $\lambda > 1$, an increasing concave function if $0 \leq \lambda \leq 1$, and a decreasing convex function if $\lambda < 0$.

One further case is of some practical interest. This is the situation where the withdrawal flow and one control are specified as functions of the stock price and time, with the other control passively absorbing the remainder of wealth. We discuss this case here in Section 3 because path independence rather than self-financing is the substantive issue, but it is the results of Section 2 that are of immediate relevance. For suppose the controls specified as functions of $S(t)$ and $t$ fail to meet condition (1) or (2). Then from Proposition 1 we know that no self-financing strategy can express the other control as a function of $S(t)$ and $t$ alone. Thus, any self-financing strategy cannot be path independent. Consequently, condition (1) or (2) is necessary for path independence in this case. An argument very similar to that used to establish Proposition 2 can be used to show that it is also sufficient. That is, if the withdrawal flow and one control satisfy (1) or (2), with the other control absorbing the remainder of wealth, then the resultant strategy will be path independent.

Any investment strategy constructed in this way is obviously self-financing, but nonnegativity must be checked indirectly by expressing the value of the portfolio, $W(t)$, as a function of the stock price, $G(S(t), t) + H(S(t), t)$. Since the resultant strategy is path independent, for any specified $G(S(t), t)$ or $H(S(t), t)$ such a function always exists and can be found by using (1)–(3) and the condition $G(S(\tau), \tau) + H(S(\tau), \tau) = W(\tau)$, where $\tau$ is the time at which the portfolio is initiated and $W(\tau)$ is initial wealth. Having obtained $G + H$, one can then verify immediately whether or not it is nonnegative for all times and stock prices, as shown in Example 5.

**Example 5.** Suppose that the amount to be invested in stock is specified as

$$G(S(t), t) = e^{\sigma(t - \tau)}[\log S(t) + (\frac{1}{2} \sigma^2 - r)(t - \tau)],$$

the withdrawal flow is specified as zero, and the remainder of the portfolio is to be invested in bonds. This choice of $G$ satisfies (1), so the strategy is path independent. By using (1)–(3) and the condition $G(S(\tau), \tau) + H(S(\tau), \tau) = W(\tau)$, we find that

$$W(t) = G(S(t), t) + H(S(t), t)$$

$$= \frac{1}{2} e^{\sigma(t - \tau)}[(\log S(t) + (\frac{1}{2} \sigma^2 - r)(t - \tau))^2 - \sigma^2(t - \tau)$$

$$- (\log S(\tau))^2 + 2W(\tau)],$$

which can clearly be negative under some conditions. We conclude that this investment strategy is self-financing and path independent but not nonnegative.
4. A characterization of optimal consumption and portfolio strategies

In this section, we examine the optimal behavior of a single individual in the market environment described in Section 1. In Proposition 3, we consider the case of an individual maximizing his expected utility of wealth at a terminal time \( T \). In Proposition 4, we extend our results to include maximization of the expected utility of lifetime consumption as well as terminal wealth.

Since we are taking the market environment as given, we could consider several alternative specifications of the opportunities available to an individual with zero wealth. For example, we could assign utility values to negative levels of terminal wealth and then allow an individual with zero wealth to borrow and continue investing. However, there is little point in considering situations that allow an individual to get something for nothing. Consequently, we do not allow an individual to incur negative wealth. An individual with zero wealth is thus precluded from any subsequent investment or consumption. Similarly, we rule out negative consumption, which has no economic meaning.

Our arguments in this section draw heavily on the correspondence (in the absence of arbitrage opportunities) between the optimal behavior of an expected utility maximizer in a world with complete markets and in a world in which markets can be completed by sequential trading (such as the setting we consider here).\(^7\) We also make frequent use of Lemma 1 and Proposition 2 and their proofs. The results of Section 3 are relevant here because the distribution of the rate of return on the stock, and hence the individual’s opportunity set, does not depend on the stock price. As a result, the optimal controls for an expected utility maximizer will depend on wealth but not on the stock price, which is the situation discussed in Section 3.

For future reference, we cite several properties of concave functions. A real-valued nondecreasing concave function \( \mathcal{V} \) defined on \( I = [0, \infty) \) is lower semicontinuous. At every interior point of \( I \), \( \mathcal{V} \) is continuous and has both a right-hand derivative \( \mathcal{V}'_+ \) and a left-hand derivative \( \mathcal{V}'_- \). Furthermore, if \( x > y \) are two points interior to \( I \), then \( \mathcal{V}'_+(x) \leq \mathcal{V}'_-(x) < \mathcal{V}'_+(y) \). Conversely, if \( \mathcal{V} \) is a lower semicontinuous function defined on \( I \), is continuous in the interior of \( I \), has a right-hand derivative \( \mathcal{V}'_+ \) at every point in the interior of \( I \), and \( \mathcal{V}'_+ \) is nonnegative and nonincreasing, then \( \mathcal{V} \) is nondecreasing and concave.

When we subsequently refer to a nondecreasing concave utility function, it is to be understood that we exclude the trivial case of a constant function.

Proposition 3. Necessary and sufficient conditions for the differentiable functions \( A(W(t), t) \) and \( B(W(t), t) = W(t) - A(W(t), t) \) to be optimal controls for some

nondecreasing concave utility of terminal wealth function are that \( A \) satisfy
\[
\frac{1}{2} \sigma^2 A^2 W W + r W A + A_r - r A = 0,  \tag{33}
\]
\[A(0, t) = 0, \tag{34}\]
\[A(W(t), t) \geq 0 (A(W(t), t) \leq 0) \text{ for all } W(t) > 0 \]
and \( A(W(t), t) > 0 (A(W(t), t) < 0) \text{ for some } W(t) > 0 \)
\[\text{when } m > r (m < r), \tag{35}\]
\[\text{for all } W(t) > 0 \text{ and all } t < T.\]

\textbf{Proof. Sufficiency:} We begin with some definitions. Let the states \( i \) denote the possible paths followed by the stock price from the current time \( t \) to the terminal time \( T \). Let \( \Pi_i(t) \) denote the current state price for state \( i \) and let \( q_i(t) \) denote the current probability of state \( i \). Let \( q \) be the probability of an upward move in any period and, as before, \( p = ((1 + r) - d)/(u - d) \). Let \( W_i \) be the terminal wealth in state \( i \) resulting from having initial wealth \( W(t) \) and following a policy satisfying the discrete versions of Eqs. (33)–(35) (i.e., (20), (21), (23), and (35) with \( C = 0 \)).

We need to show that (20), (21), (23) and (35), together with our other assumptions, imply that there exists a nondecreasing concave function \( \varphi^- \) and a nonnegative constant \( \lambda \) such that
\[\varphi^-_+(W_i) \leq \lambda \frac{\Pi_i(t)}{q_i(t)} \text{ for all } W_i = 0 \tag{36}\]
\[\varphi^-_+(W_i) \leq \lambda \frac{\Pi_i(t)}{q_i(t)} \leq \varphi^-_-(W_i) \text{ for all } W_i > 0 \tag{37}\]
\[W_i \geq 0 \text{ for all } W_i \tag{38}\]
\[\sum_i \Pi_i(t)W_i = W(t), \tag{39}\]
where we follow the convention of replacing \( \varphi^-_+(0) \) with \( +\infty \) for \( \varphi^- \) discontinuous at the origin. Conditions (20) and (21) imply that \( W_i \geq 0 \) for all \( W_i \) and the absence of arbitrage opportunities implies that \( \sum_i \Pi_i(t)W_i = W(t) \). Thus, if there is a nondecreasing concave function \( \varphi^- \) satisfying (36) and (37), then the allocation given by \( A \) is optimal for \( \varphi^- \).

The final stock price will be completely determined by the number of upward moves (and the complementary number of downward moves) occurring in the (say) \( n \) periods form \( t \) to \( T \). Consequently, all paths leading to the same final stock price will have the same probability and thus the same state price. This follows from Lemma 1, which implies that the state prices are the discounted probabilities for \( q = p \). Furthermore, (23) implies that all such paths will lead to
the same final wealth. Group the states $i$ according to the final stock price, and let State $J$ denote the occurrence of $j$ upward moves and $n - j$ downward moves. The probability of State $J$ is then $q_J = (t_J^n)q_j(1 - q)^{n-j}$ and the state price of State $J$ is then $P_J = (t_J^n)p_j(1 - p)^{n-j}/(1 + r)^n$. Let $W_J$ be the (unique) level of wealth in State $J$.

Conditions (23) and (35) imply that $W_J$ is a nondecreasing (nonincreasing) function of $j$ when $q > p$ ($q < p$).

Furthermore,  

$$
\frac{\Pi_J(t)}{q_J(t)} = \left(\frac{p(1 - q)}{q(1 - p)}\right)^n \left(\frac{1 - p}{1 - q(1 + r)}\right)^n
$$

is a decreasing (increasing) function of $j$ if $q > p$ ($q < p$). (If $q = p$, $\Pi_J/q_J$ is a constant and optimality requires the same wealth in all states; i.e., no investment in the risky asset.) Thus, for $q \neq p$, as $\Pi_J/q_J$ decreases (increases), $W_J$ increases or remains constant (decreases or remains constant). Consequently, there exists a nondecreasing concave function $J'$, defined on all levels of final wealth accessible from $W(t)$ using $A$, and a nonnegative constant $\lambda$ (depending on $W(t)$) satisfying (36) and (37) for initial wealth $W(t)$. That this remains true in the limiting case follows from the fact that $\Pi_J/q_J$ is then a continuous function of $S(T)$ and $W(S(T))$ has only isolated discontinuities.

We still must show that this $J'$ continues to satisfy (36) and (37) for all the allocations produced by $A$ from other levels of initial wealth. Consider the level of initial wealth $W(t)$ such that a downward move by the stock in the first period will lead to the same level of wealth at the end of the period as did an upward move from initial wealth $W(t)$. Path-independence implies that when there are $j$ upward moves from $t$ to $T$, the initial wealth $W(t)$ produces the same level of final wealth as the initial wealth $W(t)$ did with $j + 1$ upward moves (i.e., $W_i = W_{i+1}$). Consequently, we want to show that there is a nonnegative constant $\lambda$ such that

$$
J'_{i+1}(W_i) = J'_{i+1}(W_{i+1}) \leq \lambda \frac{\Pi_{i+1}(t)}{q_{i+1}(t)} = \lambda \frac{\Pi_i(t)}{q_i(t)} \quad \text{for all } \hat{W}_i = W_{i+1} = 0
$$

(41)

$$
J'_{i+1}(W_i) = J'_{i+1}(W_{i+1}) \leq \lambda \frac{\Pi_{i+1}(t)}{q_{i+1}(t)} = \lambda \frac{\Pi_i(t)}{q_i(t)} \leq J'_{i+1}(\hat{W}_i) = J'_{i+1}(W_{i+1})
$$

for all $\hat{W}_i = W_{i+1} > 0$.

(42)

Since

$$
\frac{\Pi_{i+1}(t)}{q_{i+1}(t)} = \left(\frac{p(1 - q)}{q(1 - p)}\right)\frac{\Pi_i(t)}{q_i(t)}.
$$

(43)
there is such a $\hat{\lambda}$ and it is equal to $[p(1 - q)/q(1 - p)]\hat{\lambda}$. Our earlier results imply that the extension of $\nu$ to the (single) level of wealth that is accessible from $W(t)$ but not $\hat{W}(t)$ is nondecreasing and concave. For our purposes, it is sufficient to consider successively higher (or lower) wealth levels in the same way.

**Necessity**: We want to show that if the differentiable functions $A$ and $B = W - A$ are the optimal controls for some nondecreasing concave utility of terminal wealth function, then $A$ satisfies (20), (21), (23), and (35). For any nondecreasing concave utility of wealth function $\nu$ an optimal portfolio allocation, if it exists, satisfies (36)–(39). We thus need to show that if $A$ is differentiable and leads to an allocation satisfying (36)–(39), then $A$ satisfies (20), (21), (23), and (35).

Conditions (36) and (37) imply that any two states having the same ratio of $\Pi_t$ to $q_t$ must have the same $W_t$. All paths leading to the same final stock price will have the same $\Pi_t$ and $q_t$, hence they must have the same $W_t$. Thus, utility maximization implies path independence, which in turn implies (23). Condition (38) requires that $W_t \geq 0$ for all $i$, which means that the dynamically revised portfolio duplicating this allocation must have a nonnegative value at time $T$. The absence of arbitrage opportunities then implies that the value of the portfolio must be nonnegative for all $t \leq \tau \leq T$, which gives (20) and (21). Recall that $\Pi_{t}/q_{t}$ is a decreasing (increasing) function of $j$ when $q > p$ ($q < p$).

Since $\nu_{+}$ is a nonincreasing function satisfying (36)–(37), $W_{t}$ is a nondecreasing (nonincreasing) function of $J$, which in turn implies (35). □

For a given twice differentiable strictly concave utility of terminal wealth function $V$, there is a simple relation between the terminal condition for (33) and the absolute risk tolerance function. In that case, for $A$ to be optimal it is necessary and sufficient that $A$ satisfies (33)–(35) and, for $W(T) > 0$, the terminal condition

$$A(W(T), T) = \left(\frac{m - r}{\sigma^2}\right) \left(\frac{V'(W(T))}{-V''(W(T))}\right).$$

To see that (44), together with (33)–(35), implies that $\nu = V$, note that since $\nu$ satisfies (41) for all $J$ such the $W_{t} > 0$, then

$$\frac{\nu''(W_{t+1}) - \nu''(W_{t})}{W_{t+1} - W_{t}} = \frac{(\Pi_{t+1}/q_{t+1} - \Pi_{t}/q_{t})\hat{\lambda}}{W_{t+1} - W_{t}} = \frac{(p(1 - q) - q(1 - p))\nu''(W_{t})}{A(W_{t}(T - 1), T - 1)(u - d)}$$

which in the limit becomes

$$\nu''(W(T)) = -\left(\frac{m - r}{\sigma^2}\right)\left(\frac{1}{A(W(T), T)}\right)\nu''(W(T)).$$

(45)
However, (44) says that

\[ V''(W(T)) = - \left( \frac{m - r}{\sigma^2} \right) \left( \frac{1}{A(W(T), T)} \right) V'(W(T)) \]

for all \( W(T) > 0 \), so (to a linear transformation) \( \psi = V \). By a similar argument, if \( A \) is optimal for a given twice differentiable strictly concave \( \psi \), then (44) must hold.

Now consider the situation where the individual also receives utility from consumption during his lifetime. Let the utility function be of the additive form

\[ \int_t^T U(C(s), s) \, ds + V(W(T)); \]

in the discrete version, the integral is replaced with the sum of the one-period utility of consumption functions. We assume that \( U \) is a piecewise continuous function of \( s \). This case leads to the following generalization of Proposition 3.

**Proposition 4.** Necessary and sufficient conditions for the differentiable functions \( A(W(t), t), B(W(t), t) = W(t) - A(W(t), t), \) and \( C(W(t), t) \) to be optimal controls for some nondecreasing concave utility of consumption and terminal wealth functions are that \( A \) and \( C \) satisfy

\[ \frac{1}{2} \sigma^2 A^2 A_{ww} + (rW - C) A_w + A_t - (r - C_w) A = 0, \]

(47)

\[ A(0, t) = 0 \quad \text{and} \quad C(0, t) = 0, \]

(48)

\[ A(W(t), t) \geq 0 \quad (A(W(t), t) \leq 0) \quad \text{for all} \quad W(t) > 0 \quad \text{and} \]

\[ A(W(t), t) > 0 \quad (A(W(t), t) < 0) \quad \text{for some} \quad W(t) > 0 \]

when \( m > r \) (\( m < r \)),

(49)

\[ C_w(W(t), t) \geq 0 \quad \text{for all} \quad W(t) > 0 \quad \text{and} \quad C_w(W(t), t) > 0 \]

for some \( W(t) > 0 \) and all \( t < T \).

(50)

**Proof.** The proof is very similar to that of Proposition 3, so we shall give only a summary that stresses the important differences.

**Sufficiency:** Consider the consumption and final wealth allocation resulting from following a given feasible investment policy satisfying the discrete versions of (47)–(50) (i.e., (20), (21), (23), (49), and (50)). We need to show that there exist nondecreasing concave functions \( \psi(C_{is}, s) \) and \( \psi(W_t) \) for each \( s \) from \( t \) to \( T \) such that this consumption and wealth allocation satisfies the optimality conditions (36)–(38) supplemented with

\[ \psi_+(C_{is}, s) \leq \lambda \frac{P_{is}(t)}{q_{is}(t)} \quad \text{for} \quad C_{is} = 0, \]

(51)
\[ \mathcal{U}_+'(C_{is}, s) \leq \lambda \frac{\Pi_{is}(t)}{q_{is}(t)} \leq \mathcal{U}_-(C_{is}, s) \quad \text{for } C_{is} > 0, \tag{52} \]

\[ C_{is} \geq 0 \quad \text{for all } C_{is} \tag{53} \]

and with (39) replaced by

\[ \sum_{i,s} (\Pi_{is}(t)C_{is} + \Pi_i(t)W_i) = W(t), \tag{54} \]

where \( C_{is}(t), \Pi_{is}(t), \) and \( q_{is}(t) \) are the consumption, state price, and probability at time \( t \) for state \( i \) at time \( s \) and \( \mathcal{U}_+(C_{is}, s) \) is the right-hand derivative of \( \mathcal{U}(C_{is}, s) \) with respect to \( C_{is} \).

Conditions (20), (21), (49) and (50) imply that \( W_i \geq 0 \) for all \( i \) and that \( C_{is} \geq 0 \) for all \( i \) and \( s \). Eq. (54) follows from the absence of arbitrage opportunities. Condition (23) implies path independence, which in turn implies that all paths with the same \( \Pi_{is}(t)/q_{is}(t) \) have the same wealth. Since \( C \) is a function of wealth, they must also have the same consumption. As before, (23) and (49) imply that \( W_j \) is a nondecreasing (nonincreasing) function of \( j \) and \( \Pi_j(t)/q_j(t) \) is a decreasing (increasing) function of \( j \) when \( q > p \) (\( q < p \)). Condition (50) implies that the same is true for consumption for all \( s \). All of this together implies that there exist nondecreasing, concave functions \( \mathcal{U}(s) \) and \( \mathcal{V}(s) \) satisfying (36)–(38) and (51)–(52) for all \( s \) for initial wealth level \( W(t) \). The extension to other wealth levels follows along the same lines as before.

**Necessity:** Let \( \mathcal{U}(s) \) and \( \mathcal{V}(s) \) be nondecreasing concave utility functions for which the differentiable controls \( A, B, \) and \( C \) are optimal and hence satisfy (36)–(38) and (51)–(54). We need to show that these controls satisfy (20), (21), (23), (49), and (50).

Clearly, the optimality conditions again imply path independence, which in turn implies (23). In the same way as before, the nonnegativity conditions (38) and (53) imply (20) and (21). Conditions (37) and (52) imply that for \( q > p \) (\( q < p \)) consumption at time \( \tau \) is a nondecreasing (nonincreasing) function of \( S(\tau) \) and that \( W(T) \) is a nondecreasing (nonincreasing) function of \( S(T) \). Wealth at time \( \tau \), which is the value at time \( \tau \) of future consumption and terminal wealth, is thus a nondecreasing (nonincreasing) function of \( S(\tau) \). Hence, we have conditions (49) and (50). □

Furthermore, suppose that \( C(W(t), t) > 0 \) for all \( W(t) > 0 \). Then a necessary and sufficient condition for \( A \) and \( C \) to be optimal for a given twice differentiable strictly concave utility of terminal wealth function \( V(W(T)) \) and some twice differentiable strictly concave utility of consumption having a given risk tolerance function

\[ R(C(W(t), t), t) = - U'(C(W(t), t), t)/U''(C(W(t), t), t) \]
is that $A$ and $C$ satisfy (48)–(50) and
\[
\frac{1}{2} \sigma^2 A^2 A_{WW} + (rW - C) A_W + A_t - rA + \left( \frac{m - r}{\sigma^2} \right) \left( \frac{U'}{U''} \right) = 0
\]
(55)
with, for $W(T) > 0$, the terminal condition
\[
A(W(T), t) = \left( \frac{m - r}{\sigma^2} \right) \left( \frac{V'(W(T))}{V''(W(T))} \right).
\]
(56)

To see this, note that (52) implies that for each $s$ and for all $J$ such that $C_{J+1} > 0$,
\[
\frac{U'(C_{J+1}) - U'(C_J)}{C_{J+1} - C_J} = \frac{\left( \frac{\Pi_{J+1}}{q_{J+1}} - \frac{\Pi_J}{q_J} \right) \lambda}{\left( \frac{C_{J+1}}{W_{J+1}} - \frac{C_J}{W_J} \right) (W_{J+1} - W_J)}
\]
\[= \frac{p(1 - q)}{q(1 - p)} - 1 \right) U'(C_J)
\]
\[= \left( \frac{C_{J+1} - C_J}{W_{J+1} - W_J} \right) (W_{J+1} - W_J)
\]
(57)
which in the limit becomes
\[
C_W(W(t), t) A(W(t), t) = \left( \frac{m - r}{\sigma^2} \right) \left( \frac{U'(C(W(t), t), t)}{U''(C(W(t), t), t)} \right)
\]
(58)
Combining (58) with (47) gives (55). If $C(W(t), t) = 0$ for some $W(t) > 0$, then the utility function must be restricted to those for which risk tolerance goes to zero as consumption goes to zero.

Note that (55) and (56) give a much weaker result than was obtained with (33) and (44) for the terminal wealth problem. There are two reasons for this. The first is that $U$ is a function of two variables, so it determines but is not determined by its risk tolerance function. (Consider, for example, utility functions of the separable form $U(C, t) = f(t)U(C(t))$). Consequently, (55) and (56) are necessary but not sufficient for $A$, $B = W - A$, and $C$ to be optimal for a given utility function $U(C(t), t)$. The restriction that $R(0, t)$ must equal zero gives a hint of the second reason. This is a result of the fact that differentiability of $C$ forces $C_W(W(t), t)$ to be equal to zero for all positive $W(t)$ such that $C(W(t), t)$ equals zero. This suggests that utility functions that do not satisfy the condition $R(0, t) = 0$ will have optimal consumption strategies that are not differentiable at the largest level of wealth for which consumption equals zero. In some related work, we verify this and completely characterize the class of continuous utility functions for which an optimal strategy is differentiable. The results of Proposition 4 do not apply to these cases.
5. Concluding remarks

In this paper, we have examined several issues in intertemporal portfolio theory. In a specific setting, we have provided complete answers to the following questions: Can a given investment strategy be maintained under all possible conditions, or are there instead some circumstances in which it will have to be modified or abandoned? How is the portfolio value resulting from following a given investment strategy until any future date related to the prices of the underlying securities on that date? Is a given investment strategy consistent with expected utility maximization?

The setting that we have chosen is the most widely used specific model of asset price movements. However, it does have two important features that should not be overlooked. The multiplicative structure of the geometric random walk or Brownian motion greatly simplifies our results. Although our general approach can be applied to many other descriptions of uncertainty, the results will inevitably be more complicated. In addition, our model provides a setting in which contingent claims can be valued by arbitrage methods, and this property plays a crucial role in some of our arguments. Many other possible descriptions of uncertainty also have this property, but our procedures will not directly apply to those that do not.

One specialization of our model is, however, essentially trivial. We assumed that the risky asset pays no dividends. If dividends are allowed, the geometric random walk or Brownian motion would apply to the value of an investment in the risky asset with reinvestment of dividends. It is easy to verify that this would leave Propositions 2–4 completely unchanged. Proposition 1 would have to be modified slightly for dividends, but it seems reasonable that most investors would in fact not want their controls to be affected by price changes caused solely by dividends. Instead, they would like to condition their controls on the returns performance of the stock. In that case, Proposition 1 would remain valid when the stock price is replaced by the value that an investment in one share of stock would have if all dividends were reinvested.

References