Optimal accumulation of pollution: Existence of limit cycles for the social optimum and the competitive equilibrium

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Abstract

This paper extends the recent investigation of Tahvonen and Withagen (1996, Journal of Economic Dynamics and Control 20, 1775–1795) for costly and thus sluggish instead of instantaneous reductions in emissions. In addition to the social optimum, the paper investigates the competitive equilibrium. This plausible extension allows to derive limit cycles as the outcome for both, the social optimum and a competitive equilibrium given rational expectations of the firms. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction and summary of the results

The paper Tahvonen and Withagen (1996) modifies the implausible assumption of depreciation (better of indigenous abatement) proportional to the stock of pollution. However, that paper overlooks the implication of this modification that more complex results are possible if in addition a similarly implausible assumption – emissions or effluents can be changed instantaneously – is dropped. This possibility for the social optimum, the case investigated in Tahvonen...
and Withagen (1996), is an implication of theoretical findings in Feichtinger et al. (1994) and Wirl (1996). These theoretical results have important implications for environmental economics, but tend to be overlooked despite the application in Wirl (1995), may be, because that paper was restricted to the well-known fishery model of Clark, Clarke and Munro (1979). However, this note will not only rephrase this finding for the model of Tahvonen and Withagen (1996) (Section 2), but will consider in addition to the social optimum (Section 3) an outcome under laissez faire (with or without some internalisation, Section 4); the corresponding proofs are relegated to appendices.

The major findings are summarised in Fig. 1 considering an abatement function \( \alpha(z) \) of the shape shown in Fig. 1 and suggested in Tahvonen and Withagen (1996); in fact, the entire notation of this paper follows Tahvonen and Withagen (1996): \( \delta \) denotes the discount rate, \( z \) the stock of pollution and \( \alpha(z) \) describes the 'depreciation', i.e., the indigenous cleaning capabilities of the environment, which are restricted to the interval \([0, \bar{z}]\); \( \bar{z} \) denotes the point where depreciation becomes zero (thus pollution beyond \( \bar{z} \) is irreversible, even if all emissions stop) and \( z^* \) (see Fig. 1) defines the maximum of depreciation. While Tahvonen and Withagen (1996) focus on the difficulties associated with irreversible pollution (non-concavity across \( z = \bar{z} \)), the present paper is restricted to the case of reversible pollution, \( z < \bar{z} \). Now the major results are the following. If the preferences (accounting for the external costs associated with the stock of pollution) are such that low levels of pollution result, more precisely \( z < z^* \), then the optimal policy will always be stable (but may involve, nevertheless, damped oscillations). However, if the parameters are such that the long-run

![Fig. 1. Stability properties of pollution accumulation for non-monotonic depreciation.](image-url)
outcome of pollution exceeds \( z^* \), then complex evolutions including instabilities and limit cycles are possible. Although this characterisation holds for both, the competitive equilibrium and the social optimum, the latter reduces the domain and the probabilities of such complexities. In fact, the reduction is twofold. First, reducing the domain over pollution as highlighted in Fig. 1 and second, at least in the numerical examples, larger cost convexities are necessary for the social optimum in order to yield a Hopf bifurcation. This tends to confirm the intuition that limit cycles, although possibly optimal, are less likely if everything is internalised.

2. Framework

I consider the following extension of the Tahvonen and Withagen (1996) framework:

\[
\max_{(x(t))} \int_0^{\infty} \exp(-\delta t)\left[ (u(z))y(t) - vz(t) - C(x(t)) \right] dt, \tag{1}
\]

\[
y'(t) = -x(t), \quad y(0) = y_0, \tag{2}
\]

\[
z'(t) = y(t) - \alpha(z(t)), \quad z(0) = z_0. \tag{3}
\]

That is the firms have the gross surplus \( U(z, y) = u(z)y \) from discharging \( y \) (emissions or effluents) into the environment; \( u' \leq 0 \) reflects that the surplus per unit emission may diminish as the pollution \( z \) increases. This extension (the inclusion of \( z \) in \( U \) is not present in the separable set up of Tahvonen and Withagen, 1996) is introduced for the investigation of the competitive equilibrium while the study of the social optimum is restricted to the separable case investigated in Tahvonen and Withagen (1996). A practical example of this framework (1)–(3) are factories located on a lake that discharge waste water into the lake in order to save on waste water treatment, i.e., a higher level of effluents \( y \) lowers the firms’ variable costs. On the other hand, a polluted lake reduces this benefit \( u \) either directly or indirectly through higher effluent charges. Furthermore, the owners of the firms may have a direct interest in a clean lake, i.e., they penalise pollution at \( D(z) = vz \). However, this pollution is a public good for each firm and thus will not affect the individual decisions unless properly internalised via an effluent tax. The firms can lower their individual emissions at the rate \( x \) but at a cost, \( C(x) \). The investment \( x \) is everlasting (in order to simplify) and the associated expenditures \( C(x) \) are increasing and convex. Hence, these investments are reversible but without recovering all the expenditures. The stock of pollution increases with the effluents (or emissions) but is reduced through indigenous abatement by the environment (‘depreciation’). Tahvonen and
Withagen (1996) consider an abatement function \( \alpha(z) \) that increases for low levels of pollution and decreases for high levels of pollution (as shown in Fig. 1) instead of the common linear depreciation that has the implausible implication that one can always increase abatement by raising the stock of pollution.

In order to simplify, I introduce the following specifications: linear-quadratic investment expenditures \( C(x) \), surplus per unit emission, \( u(z) \), is either constant or linear in \( z \) and the logistic function for \( \alpha(z) \), which has exactly the shape displayed in Fig. 1 with \( \tilde{z} = 1 \).

\[
C(x) = x + 1/2\gamma x^2, \quad (4)
\]
\[
u(z) = p \text{ for the social optimum and}
\]
\[
u(z) = p - az \text{ for a competitive equilibrium,} \quad (5)
\]
\[
\alpha(z) = z(1 - z). \quad (6)
\]

None of the above specifications (4)–(6) is essential; in fact, considering utilities separable in \( y \) and \( z \) for the social optimum hardens the detection of limit cycles first due to imposing concavity (an assumption retained for all optimisation problems in this paper) and second due to eliminating state interactions that are helpful for limit cycles, compare Wirl (1996). The major advantage of this separable (already in Tahvonen–Withagen) and partially linear set up is that the corresponding Jacobi-matrix is sufficiently simple (because mixed and second-order derivatives do not appear) to allow for a manageable analytical analysis; the same reason (to calculate the steady-state analytically) and the need for a minimum of non-linearity explains the choice of the logistic growth for \( \alpha \). In particular, the same stability results including a similar bifurcation analysis can be carried out for non-linear utilities, costs and other depreciation functions. However, increasing the concavity with respect to the states works against complexities such as limit cycles, while increasing the concavity with respect to the control fosters limit cycles.

### 3. Social optimum

In line with Tahvonen and Withagen (1996) it is assumed that the external costs due to pollution are separable from the benefits of emissions, thus \( u = p \) and \( u' = 0 \). The calculation of the social optimum requires to solve the dynamic optimisation problem (1) subject to Eqs. (2) and (3). Using simplification (4) to (6) and \( \lambda_i, i = 1 \) and 2 to denote the costates, the associated Hamiltonian \( H \) is defined as follows:

\[
H = py - vz - x - 1/2\gamma x^2 - \lambda_1 x + \lambda_2[y - \alpha(z)]. \quad (7)
\]
Therefore, the following first-order optimality conditions result:

\[ H_x = -1 - \gamma x - \lambda_1 = 0 \Rightarrow x^* = -(1 + \lambda_1)/\gamma, \]  
\[ \dot{\lambda}_1 = \delta \lambda_1 - p - \lambda_2, \]  
\[ \dot{\lambda}_2 = (\delta + \alpha'(z))\lambda_2 - v. \]  

Since the Hamiltonian (7) is concave in the control and the states, the first-order conditions are sufficient if a limiting transversality condition is satisfied. This transversality condition is definitely satisfied if the paths converge either to a finite steady state or to a limit cycle so that the conditions (8)–(10) are then sufficient too.

**Proposition 1.** Considering the accumulation of pollution, associated externalities, non-trivial depreciation of the stock of pollution and costly reductions in emissions, stable limit cycles can be socially optimal. The existence of limit cycles requires high stationary levels of pollution, more precisely, \( z_\infty > z^* \). Moreover each steady state in this domain \( (z_\infty > z^*) \) allows for limit cycles (leaving aside the question about the stability of the cycle) if the investments costs \( C \) are sufficiently convex. However, high external costs of pollution so that the long-run optimum is characterised by low levels of pollution, more precisely \( z_\infty \leq z^* \) thus \( \alpha'(z_\infty) \geq 0 \), are sufficient for saddlepoint stability.

Fig. 1 highlights that limit cycles are only possible for \( -\delta < \alpha' < 0 \), which seems to impose a restriction that is not addressed explicitly in Proposition 1. Yet no additional restriction is necessary, because any interior and stable steady state must be to the left of \( \alpha'(z) = -\delta \). The reason is that \( \alpha'(z) = -\delta \) characterises the stationary level of pollution of no external costs \( (D(z), i.e., v = 0, see \ Eq. (10)) and obviously, v > 0 reduces the stationary pollution. Although lowering \( \delta \) reduces the set of potentially complex optimal policies (cf. Fig. 1), arbitrary small discount rates allow for a corresponding bifurcation (by choosing a sufficiently large \( \gamma \)); for details see Appendix A.

**4. Competitive equilibrium**

The calculation of the social optimum in Section 3 accounts explicitly for the impact of the firms’ emissions on the stock of pollution. However, competitive agents consider the evolution of pollution \( z(t) \) as exogenous data, because the state \( z \) is negligibly affected by the (representative) agent’s actions. This implies that \( z \) becomes a public good so that the existence of stationary pollution below \( \bar{z} \) requires that the marginal product of emissions, \( u(z) \), is sufficiently reduced as \( z \) increases. Two reasons for this feedback are conceivable. First, pollution diminishes the surplus \( u \), e.g. if a lake is simultaneously used to discharge
Small letters are used in order to differentiate from the notation in Section 3; this extends to the corresponding Appendix B.

These are sufficient if the transversality condition is satisfied, because $h$ is (jointly) concave in $y$ and $x$.

Although $z(t)$ is exogenous data for each agent, these agents do not treat $z$ as constant but instead foresee the evolution $\{z(t), t \in [0, \infty)\}$ perfectly (rational expectations in this deterministic set up). In short, each agent chooses a trajectory $\{x(t), t \in [0, \infty)\}$ such that the present value aggregate of the individual profits (1) is maximised subject to the individual constraint (2). The assumption of rational expectations implies that the evolution of pollution is determined simultaneously with the (representative) firm’s intertemporal decisions but without adjoining the differential equation of the externality $z$, Eq. (3), to the Hamiltonian. Hence, no costate variable is associated with the external state $z$ and the resulting motions are contained in $\mathbb{R}^3$ instead of the $\mathbb{R}^4$ as in Section 3; however, in both cases the solutions will lie in a two-dimensional manifold, the minimum requirement to allow for cycles and related complexities.

The optimisation problem of the competitive firms – maximising Eq. (1) subject to Eq. (2) and given the externality (3) – leads to the Hamiltonian

$$h = (p - az)y - vz - x - 1/2\gamma x^2 - \mu x,$$

using $\mu$ to denote the costate (of $y$) and to the following first-order conditions

$$h_x = -1 - \gamma x - \mu = 0 \Rightarrow x^* = -(1 + \mu)/\gamma,$$

$$\dot{\mu} = r\mu - p + az.$$  

Therefore, a solution of the following system of differential equations satisfying the transversality condition determines a competitive, rational expectation equilibrium:

$$\dot{y} = (1 + \mu)/\gamma,$$

$$\dot{\mu} = r\mu - p + az,$$

$$\dot{z} = y - z(1 - z).$$  

**Proposition 2.** Stable limit cycles can describe the competitive equilibrium given rational expectations. Moreover, any equilibrium with ‘high’ levels of stationary pollution, i.e., $x'(z_\infty) < 0$, ensures the existence of a pair of purely imaginary
eigenvalues and of limit cycles (the stability of the cycle aside) if investments costs are sufficiently convex. However, an equilibrium with low levels of pollution, more precisely, \( \dot{x}(z_\infty) \geq 0 \), will always be stable.

The difference between Propositions 2 and 1 is that the entire domain \( \dot{x} < 0 \) is feasible for a stationary solution of the competitive equilibrium and this entire set provides candidates for cyclical solutions if \( C'' = \gamma \) is sufficiently large; this allows in particular for limit cycles for arbitrary small discount rates.

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Appendix A.

The existence of limit cycles is proven applying the Hopf-bifurcation theorem. This theorem gives sufficient conditions for generic and stable limit cycles of a family of dynamic systems, if a pair of purely imaginary eigenvalues (of the Jacobian) exists, if the linear dynamics change suddenly from a sink into a source and if the coefficient of the quadratic term of the so called normal form is negative; for details see Guckenheimer and Holmes (1983). The analytical argument is confined to establish the existence of a pair of purely imaginary eigenvalues, while the remaining two conditions are numerically checked. The procedure is similar to the one outlined in Wirl (1995, 1996).

A.1. Proof of Proposition 1

1. Deriving the canonical equations from the necessary optimality conditions yields:

\[
\begin{align*}
\dot{x} &= (1 + \lambda_1)/\gamma, \\
\dot{z} &= y - z(1 - z), \\
\dot{\lambda}_1 &= \delta \lambda_1 - p - \lambda_2, \\
\dot{\lambda}_2 &= (\delta + \dot{\alpha}(y))\lambda_2 + v.
\end{align*}
\]

(A.1a) (A.1b) (A.1c) (A.1d)

2. Calculation of the (unique) steady states of Eqs. (A.1a)–(A.1d):

\[
\begin{align*}
z_\infty &= 1/2[(p + \delta)(1 + \delta) - v]/(p + \delta), \\
y_\infty &= z_\infty(1 - z_\infty), \lambda_{1\infty} = -1, \lambda_{2\infty} = -(p + \delta).
\end{align*}
\]

(A.2)
3. Calculation of the Jacobian (evaluated at the steady state (A.2)):

\[
J = \begin{pmatrix}
0 & 0 & 1/\gamma & 0 \\
1 & -\alpha' & 0 & 0 \\
0 & 0 & \delta & -1 \\
0 & -2\lambda_2 & 0 & \delta + \alpha'
\end{pmatrix}
\]  
(A.3)

4. Calculation of the eigenvalues \(E_i\) using the formula of Dockner (1985),

\[
E_i = \frac{r}{2 \sqrt{\left(\frac{r}{2}\right)^2 - \frac{K}{2} \pm \frac{1}{2} \sqrt{K^2 - 4 \det(J)}}}, \quad i = 1-4,
\]
which requires to calculate

\[
\det(J) = -2\lambda_2/\gamma = 2(p + \delta)/\gamma,
\]
which is positive, and the sum of the principal minors of \(J\) of dimension 2 (denoted as in Dockner, 1985 by \(K\)):

\[
K = -(\delta + \alpha')\alpha'.
\]

(A.5)

According to Dockner and Feichtinger (1991) the inequality \(K > 0\) is necessary to allow for a Hopf bifurcation, which in turn implies due to Eq. (A.5) the growth condition of Wirl (1996), \(\delta > \partial \hat{z}/\partial z = -\alpha' > 0\).

5. The existence of a pair of purely imaginary eigenvalues requires \(K > 0\) and

\[
\det(J) - (K^2/4 + \delta^2 K/2) = [2(p + \delta)/\gamma] - \left[1/4(\delta + \alpha')^2\alpha'^2 - 1/2\delta^2\alpha'(\delta + \alpha')\right] = 0.
\]

(A.6)

This Eq. (A.6) allows for an explicit calculation of the critical value of \(\gamma\), because this parameter does not affect the stationary solution (A.2):

\[
\gamma_{\text{crit}} = 2(p + \delta)\left[1/4(\delta + \alpha')^2\alpha'^2 - 1/2\delta^2\alpha'(\delta + \alpha')\right].
\]

(A.7)

Therefore, if \(\delta > -\alpha' > 0\), then \(K > 0\) (\(\det(J) > 0\) anyway) and a pair of purely imaginary eigenvalues exists, because \(\gamma_{\text{crit}} > 0\) due to Eq. (A.7).

6. The pair of purely imaginary eigenvalues in the domain \(\delta > -\alpha' > 0\) for \(\gamma = \gamma_{\text{crit}}\) is crucial but insufficient for the existence of stable limit cycles, which according to the Hopf-bifurcation theorem requires in addition that the real parts change their sign at \(\gamma = \gamma_{\text{crit}}\) and a stability condition (a negative Lyapunov number) holds. Since the check of these two conditions is very tedious the proof of existence uses the following numerical example, \(\delta = 1/2\), \(p = 1\), \(v = 0.45\), that implies the steady states: \(z_\infty = 0.6\) thus \(y_{\infty} = 0.24\), \(\lambda_{1\infty} = -1\) and \(\lambda_{2\infty} = -1.5\). The stationary pollution is from the critical domain so that \(K > 0\)
and $\gamma_{\text{crit}} = 357.143$. Numerical calculations using LOCBIF (see Khibnik et al., 1992) confirm that a Hopf-bifurcation results at this critical level of $\gamma$ (the corresponding Lyapunov number is $-0.0169$). Therefore, stable limit cycles characterise the optimal strategies for values of $\gamma > \gamma_{\text{crit}}$ (at least locally); yet the system remains stable for $\gamma < \gamma_{\text{crit}}$.

7. It remains to prove the claim of saddlepoint stability for $\lambda^* > 0$. According to Eq. (A.5) $\lambda^* > 0$ is sufficient for $K < 0$. Yet $K < 0$ and $\det(J) > 0$ are sufficient for saddlepoint stability according to Dockner (1985).

Appendix B. Proof of Proposition 2

The proof is similar to the one in Appendix A. (1) The canonical equations system is already given by Eq. (14). (2) The steady states, which are again independent of the parameter $\gamma$, are: $y_\infty = (p + \delta)(a - \delta - p)/a^2$, $\mu_\infty = -1$, $z_\infty = (p + \delta)/a$. (3) The Jacobian (denoted $j$):

$$j = \begin{pmatrix} 0 & 1 & 0 \\ \gamma & 0 & a \\ 1 & 0 & -\lambda^* \end{pmatrix}. \quad (B.1)$$

(4) The eigenvalues of $j$ are the roots of the characteristic polynomial:

$$p(e) = e^3 - \text{tr}(j)e^2 + ke - \det(j), \text{ where } \text{tr}(j) = \delta - \lambda^*,
\quad k = -\delta\lambda^*, \det(j) = a/\gamma. \quad (B.2)$$

Since $\det(j) > 0$, at least one eigenvalue, say $e_3$, must be positive. Hence,

$$p(e) = (e - e_3)(e^2 + Ae + B), \quad A = e_3 - \text{tr}(j),
\quad B = \det(j)/e_3 = k + e_3(e_3 - \text{tr}(j)). \quad (B.3)$$

and the remaining roots that determine the stable manifold are:

$$e_{12} = 1/2(-A \pm \sqrt{A^2 - 4B}). \quad (B.4)$$

(5) A pair of purely imaginary eigenvalues requires due to Eq. (B.4) $A = 0$, thus $\det(j) - \text{tr}(j)k = a/\gamma - \lambda^*\delta - \lambda^* = 0$ and in particular $k > 0$. Therefore, $\lambda^* < 0$ (equivalent to $k > 0$) ensures a pair of purely imaginary eigenvalues (using a subscript to denote the critical value):

$$\gamma = \gamma_{\text{crit}} = a/[\lambda^*\delta(\lambda^* - \delta)]. \quad (B.5)$$

(6) The existence of stable limit cycles is again proven numerically using the parameters of the social optimum $\delta = 1/2$, $p = 1$. Introducing a tax on
emissions, \( \tau = az \), we set \( a = 5/2 \) so that the same stationary pollution, \( z_\infty = 0.6 \), as in Appendix A results; i.e., this tax ensures at least in the long run the social optimum. This leads to a Hopf-bifurcation at \( \gamma_{\text{crit}} = 35.7143 \) (at exactly a tenth for the social optimum), because the Lyapunov number is again negative (\(-0.0377\) calculated with LOCBIF).

(7) The claim concerning stability for \( \alpha' > 0 \) follows because \( k < 0 \) combined with \( \det(j) > 0 \) ensures the eigenvalues \( e_{1,2} \) in Eq. (B.4) are negative or have negative real parts.

References


