Uncertainty aversion and rationality in games of perfect information

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Abstract

This paper shows how uncertainty aversion can resolve ties in extensive games of perfect information, and thereby refine subgame-perfection in such games. This is done by assuming that (a) players are rational with multi-prior expected utility functions of Gilboa and Schmeidler (1989), (Journal of Economic Theory 48, 221–237); and (b) a player’s plan of the game corresponds to set-valued acts at each of his decision nodes, the so-called strategy system, in contrast to single-valued act as specified in the traditional notion of strategy profile. The effects of pre-communication channels among players are studied, and are reflected in each of the three proposed solutions. Finally, existence of equilibria are established for each proposed solution concept as well as comparisons with other existing solutions. © 2000 Elsevier Science B.V. All rights reserved.

1. Introduction

Modern game theory typically adopts strong assumptions about agents’ beliefs. For example, in the notion of Nash equilibrium each agent’s beliefs about the likelihoods of other players’ play are represented by a subjective probability measure in conformity with the Savage (1954) axioms. In such an equilibrium, distinction between risk, where probabilities are available to guide choice,¹ and uncertainty, where information is too imprecise to be summarized

¹ This occurs when a player uses a random device to guide his decision in situations where he has to make a choice among a set of mutually indifferent acts and when all players know other players’ random devices.
adequately by probabilities, is not allowed (see Knight, 1921). In this paper we argue that uncertainty is more common in multi-agent interaction situations and that such uncertainty should be explicitly modelled in the analysis of game situations and be reflected in game-theoretical solution concepts.

This paper presents three solution concepts for extensive form games with perfect information. The main objective is to show how uncertainty aversion can resolve ties in such strategic situations. Precisely, the proposed solution concepts reflect the presumptions that (a) all players are rational and uncertainty averse with beliefs summarized by set-valued probability measures over paths, 2 (b) players’ decisions are summarized by a so-called ‘strategy system’ which corresponds to a plan that involves a set of feasible acts at each decision node, any act in this set is planned to be played when such a decision node is reached. This is in contrast to the traditional literature where player’s decision is summarized by a ‘strategy’ that corresponds to a single act at each decision node. All solution concepts proposed in this paper constitute refinements to the subgame perfect equilibrium (and hence Nash equilibrium).

The difficulties in the existing game-theoretical solution concepts of dealing with ties in strategic interaction situations were first pointed out by Young (1975, pp. 28–29):

Logically speaking, there is an infinite variety of rules of thumb that could be used in assigning subjective probabilities, and game theory offers no persuasive reason to select any one of these rules over the others.

The debates between Kadane-Larkey (1982) and Harsanyi (1982) in the early 1980s on how to approach the strategic interaction in game situations in general were also partially motivated by the failure of existing solution concepts in dealing with situations that involve ties. Kadane and Larkey pointed out correctly that the existence of uncertainty, such as ties, makes it problematic to adopt Savage’s single prior subjective probability in modeling agent’s belief as assumed in many classical game-theoretic solution concepts such as Nash (1952), Harsanyi (1967-1968) and Selton (1972). While Kadane and Larkey failed to provide a positive theory in resolving ties, their arguments suggest clearly that a suitable solution concept should deal with set-valued action profiles in forming players’ plans and beliefs. 3 This paper assumes set-valued plans of

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2 The notion of common knowledge of rationality can be formally formulated following Aumann (1995) by paying special attention to the notion of set-valued strategy system and beliefs.

3 While I entirely agree with professor Harsanyi’s (1982) argument that the common knowledge of rationality is an important constraint in forming player’s belief and in developing normative solution concepts, the condition of common knowledge of rationality in general cannot eliminate uncertainty, although it surely helps to reduce the uncertainty within some limited range.
action and set-valued beliefs which distinguishes it from the existing literature. We argue that when a player makes a plan on future selection of acts at the decision node $v$ in a game tree, one may not be willing to pre-commit to any specific single act at $v$ (e.g., there exists a set of acts that are equally the best), but instead they may choose a set of acts at that node so that all acts in this set will be candidate selections at $v$ once $v$ is reached.

The set-valued approach makes a difference only when there exist ties in a game situation. Otherwise, if payoffs associated with the sets of strategy profiles are distinct from each other and are strictly ordered for all players in a game of perfect information, then backward induction implies the existence of a unique optimal strategy for each player of the game, which constitutes the unique subgame perfect equilibrium. In this case, we say that the game has an obvious play, and uncertainty is not an issue. Indeed, all three solutions proposed in this paper coincide with the unique subgame perfect equilibrium when there are no ties. This, nevertheless, does not make our solution concepts less interesting or less important because a game becomes interesting only when there is no obvious play associated with it. In other words, it is the class of games associated with no obvious plays that deserves serious study. We can also argue that in real life situations, it is the absence of obvious play that makes a game challenging and interesting (see, for example, Kreps (1990, pp. 55–56) discussion on the Stackelberg deter-entry game where the entrant is indifferent to ‘enter’ and ‘stay-out’, while the monopoly incumbent cares very much about the entrant’s decision).

Since set-valued plans of action can generate (Knightian) uncertainty, players’ attitude toward uncertainty must be discussed. In particular, we wonder whether attitude toward uncertainty matters to the solution of a game and how that solution may differ from other classical solution concepts such as Nash equilibrium and its subgame perfection refinement. One way to model player’s attitudes toward uncertainty is through Savage’s (1954) subjective expected utility model by assuming uncertainty-neutrality. This approach of modelling player’s attitude toward uncertainty, however, has been criticized in the decision theory literature. First, Savage’s model of subjective utility is inconsistent with experimental evidence as suggested by the Ellsberg’s paradox (Ellsberg, 1961). The experimental evidence suggests that players exhibit some degree of aversion toward uncertainty. Second, it seems inappropriate and unconvincing to model a player’s belief by specifying a single prior probability in the presence of Knightian uncertainty (see Gilboa, 1987; Gilboa and Schmeidler, 1989; Machina and Schmeidler, 1991; Epstein and Le Breton, 1993; Dow and Werlang, 1994; Kadane and Larkey, 1982; Young, 1975). Given the above criticisms of Savage’s subjective expected utility theory, we adopt Gilboa and Schmeidler’s (1989) multi-prior expected utility model which is also assumed in Dow and Werlang (1994), Epstein (1995), Epstein and Wang (1994), Klibanoff (1993) and Lo (1994) in their study of normal form games.
In contrast to the static decision theory, where uncertainty is exogenously fixed and is thus called ‘exogenous uncertainty’, uncertainty in game situations, which is associated with players’ actions, is endogenously created by the players of a game. The common knowledge of rationality will play an important role in reducing the uncertainty to some limited ranges for any given game. In addition, preplay communication, together with the notion of rationality, can be served as an avenue through which a player can convey information to the others regarding its plan of play aiming to strategically manipulate the belief-formations of the others. The strategic aspect of ‘endogenous uncertainty’ in game situation is formally discussed in this paper and is reflected in our proposed solution concepts:

The first solution concept, called $\sigma^*$-equilibrium, is with respect to the situation where communication among players is strictly prohibited. The $\sigma^*$-equilibrium is shown to be well-defined and unique, and can be axiomatically justified. This equilibrium concept is closely related to the concept ‘stable belief system’ proposed and studied by Luo and Ma (1998). Luo and Ma showed that the $\sigma^*$-equilibrium can actually be interpreted as a unique stable belief system and is stable in a generalized sense of von Neumann and Morgenstern (1947). Though it refines the subgame perfect equilibrium, similar to von Neumann and Morgenstern’s minimax solution for perfect conflict situations (the so-called zero sum games), the $\sigma^*$-equilibrium may rule out some economically appealing outcomes due to the uncertainty averse behavior assumption. Our second solution, called $\Omega^{**}$-equilibrium, is aimed to model the situation where there exists a social planner who may recommend a specific path to all players of a game before the actual play takes place. While players have the option whether or not to follow the recommended path, the $\Omega^{**}$-equilibrium is defined to contain all ‘stable’ paths that, once being recommended, will be followed. It is shown that $\Omega^{**}$-equilibrium contains all $\sigma^*$-equilibrium paths while it still refines the subgame perfect equilibrium. Therefore, $\Omega^{**}$-equilibrium may contain those economically appealing outcomes that are ruled out by the $\sigma^*$-equilibrium. In the third solution concept, called $\Xi^*$-equilibrium, players are allowed to announce their plans of action before the game starts. That is, a player has the option to keep silent, to announce a plan of action openly and honestly, or to tell a lie – to announce a plan which he is not willing to commit at all. In contrast to games with a publicly recommended path, games with pre-announced plans of action allow players to play an active role in deciding what to announce and how to act once announcements are made. For example, player may use announcements to create a credible threat or promise to the

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4The notion of public recommendation in game theory, is first introduced by Greenberg (1989), and is applied to extensive form games by Greenberg (1994). His approach in modeling players’ preference and belief-formation is different from ours. In addition, his solution contains subgame perfect equilibrium paths as a subset (see also Fudenberg and Levine, 1993).
A stable plan could be irrational in the sense that there is no reason for rational players to announce such a plan in the first place. This observation is of particular interest because it suggests that a simple adoption of stability as a necessary and sufficient criterion for rational solution concept could be problematic.

An announcement is called stable if it will be followed once announced. A stable announcement is rational if no body can benefit for sure by announcing another plan given other’s announced plans. The $\Xi^*$-equilibrium is the set of all stable and rational announcements.\(^5\) If we interpret an equilibrium plan as a ‘theory’ or a ‘social norm’, then the theory defined by the $\Xi^*$-equilibrium is ‘good’ since it has the following two attractive features: (a) it will be optimally carried out if it is announced (or proposed); and (b) there is no incentive for any individual player to announce something inconsistent with the theory knowing that the other players all believe in the theory.

The interpretation of the $\Xi^*$-equilibrium offered here is obviously on the line to the development of ‘cheap talk’ games, or signalling games, as a branch of game theory. Given the original contributions made by Crawford and Sobel (1982), van Damme (1987), Farrell and Gibbons (1989), Seidmann (1990, 1992) and Manelli (1996) with many other extended references in that topic, ‘cheap talks’ in this paper are modelled as (announcements of) set-valued plans. Therefore, a cheap talk (in this paper) can be ‘vague’ or ‘explicit’ depending on whether the announced plan is set-valued or single-valued. Similar to Bennett and van Damme (1991) and Farrell and Gibbons (1989), examples (see Section 5) can be constructed to show that an announced plan can serve as a credible threat or a credible promise. Since $\Xi^*$-equilibrium is defined to consist of all credible and rational announcements, it is therefore also called ‘cheap talk equilibrium’.

This paper is organized as follows: In Section 2, we present a simple example to demonstrate the existence of Knightian uncertainty in extensive form games of perfect information and to argue that the universally used Nash equilibrium concept and its refinement might be inappropriate in describing the actual play of the game. In this section, we also provide a brief description of the recently-developed theory of modeling beliefs and preferences in the presence of Knightian uncertainty. Our first solution concept, the $\sigma^*$-equilibrium, is presented in Section 3. This section compares $\sigma^*$-equilibrium with the subgame perfect equilibrium, and shows that the former is a refinement of the latter. In addition, an axiomatic justification of $\sigma^*$-equilibrium is provided. Section 4 studies the situation in which public recommendation is allowed, and our second solution concept, the $\Omega^*$-equilibrium, is introduced and is shown to be a refinement of the subgame perfect equilibrium and to contain $\sigma^*$-equilibrium paths as a subset. Games with pre-announced plans of action are studied in Section 5. The $\Xi^*$-equilibrium is proposed and studied, and is also proved to be

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\(^5\)A stable plan could be irrational in the sense that there is no reason for rational players to announce such a plan in the first place. This observation is of particular interest because it suggests that a simple adoption of stability as a necessary and sufficient criterion for rational solution concept could be problematic.
a refinement of the subgame perfect equilibrium. Section 6 contains some concluding remarks.

2. Knightian uncertainty in extensive form games

This section contains three subsections. Section 2.1 presents a simple example to demonstrate the existence of Knightian uncertainty in extensive form games of perfect information. Section 2.2 introduces Gilboa and Schmeidler’s (1989) theory of decision making under Knightian uncertainty in a static setting. Section 2.3 formalizes the decision problem in extensive form games of perfect information.

2.1. Existence of Knightian uncertainty: An example

In the game described in Fig. 1, we first assume that communication between players 1 and 2 is not allowed. Since player 2 is indifferent to playing L or R, player 1’s belief regarding to player 2’s choice is totally vague in the sense that he has no idea what player 2 will do if he leads to player 2’s decision node by playing r. In other words, no single probability measure over \{L, R\} can qualify as a suitable and rational description of player 1’s belief of player 2’s play. We say that player 1 in this game is facing a Knightian uncertainty.

We can argue that, even if communication is allowed, Knightian uncertainty may still exist. For the same game as above, suppose each player must announce a plan of action to the other. Will player 1’s vague belief regarding player 2’s choice disappear? Not necessarily. Player 2 can still create Knightian uncertainty by announcing that ‘either L or R will be selected’ without mentioning if he will use any particular random device to make a choice between this two acts. In that case, we say that player 2 can control player 1’s vagueness regarding his actual play. Indeed, in many situations, players will use announcements strategically aiming to create favorable outcomes for themselves (see Section 5 of this paper for elaboration).

How does the presence of Knightian uncertainty affect the actual play of the game? Player 1 must evaluate consequences of his playing l or r. If he chooses to play l, he will get payoff of 1 for sure; otherwise, his payoff will be either 0 or 2 depending on player 2’s act. Therefore, in order to answer the above question, we need to explicitly specify a player’s attitude toward uncertainty, preference determination and his formation of beliefs in the presence of

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6 For discussion of uncertainty aversion in normal form games readers are referred to Klibanoff (1993) and Lo (1996).
We consider pure acts only. Generalization to mixed acts can be done in the same fashion (see Gilboa and Schmeidler, 1989).

Knightian uncertainty. Moreover, since existing equilibrium notions of games such as Nash equilibrium and subgame perfect equilibrium are defined by assuming that players are subjective expected utility maximizers with single probability prior, deviations from the Savage model to accommodate aversion to uncertainty make it necessary to redefine equilibrium concepts.

2.2. Decision theory under Knightian uncertainty

This section provides a generalization of recently developed theory of decision making under Knightian uncertainty. Let $X$ be a finite outcome space, $\Omega$ a finite state space, and $M(\Omega)$ the set of probability measures over $\Omega$. Let $F$ denote the act space that consists of all mappings from $\Omega$ to $X$.\(^7\) That is, $f \in F$ if, for each $\omega \in \Omega$, $f$ maps $\omega$ into $X$, and such a map is denoted by $\langle f, \omega \rangle \in X$, $\forall f \in F$, $\omega \in \Omega$.

\(^7\)We consider pure acts only. Generalization to mixed acts can be done in the same fashion (see Gilboa and Schmeidler, 1989).
The preference ordering $\succeq$ is a binary relation defined on the act space $F$. Gilboa and Schmeidler (1989) provide a set of axioms on $\succeq$ so that the preference ordering $\succeq$ admits the following type of representation: there exists an affine function $u: X \rightarrow R$, and a unique, nonempty, closed and convex correspondence $\Lambda: F \rightarrow M(\Omega)$, for all $f, g \in F$,

$$f \succeq g \iff \min_{\omega \in \Lambda(f)} \int_{\Omega} u(\langle f, \omega \rangle) \, dp(\omega) \geq \min_{\omega \in \Lambda(g)} \int_{\Omega} u(\langle g, \omega \rangle) \, dp(\omega).$$  (1)

The set $\Lambda(f) \subseteq M(\Omega)$ is interpreted as the beliefs induced by the preference $\succeq$ and by act $f$. For example, the special case of complete vagueness of a decision maker with respect to the possible realization of the state of nature can be modeled by specifying the belief set to consist of all probability measures on $\Omega$, i.e., $\Lambda = M(\Omega)$. In that case, the preference relation (1) reduces to

$$f \succeq g \iff \min_{\omega \in \Omega} u(\langle f, \omega \rangle) \geq \min_{\omega \in \Omega} u(\langle g, \omega \rangle).$$  (2)

A preference relation $\succeq$ satisfying relation (1) is referred to as a multi-prior expected utility preference. As a special case, when $\Lambda$ is a singleton set, it reduces to Savage’s subjective expected utility.

This formulation of multi-prior expected utility preference described by Eq. (1) has many attractive features in comparison with the traditional Savage single prior model. First, it models situations in which a decision maker’s belief cannot be suitably represented by a single subjective probability. Second, predictions generated by this class of utility functions are consistent with Ellsberg’s paradox (Ellsberg, 1961) in contrast to the single prior model. Finally, the behavioral assumption of uncertainty aversion is also modeled by the above multi-prior expected utility preference: for any given acts $f$ and $g$, the induced vague belief sets are respectively given by $\Lambda(f)$ and $\Lambda(g)$, then according to Eq. (1), act $f$ is preferred to act $g$ if and only if the least well-off (in terms of expected utility) over $\Lambda(f)$ exceeds that over $\Lambda(g)$.

2.3. Extensive form games of perfect information

This section defines $n$-person extensive form games of perfect information in which players’ preferences in the presence of Knightian uncertainty are explicitly

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8 Precisely, this formulation is a generalization of Gilboa and Schmeidler (1989). Here, we do not require that $\Lambda(f) = \Lambda(g)$ for all acts $f, g \in F$. This is because different acts may lead to different contingent state spaces, thus different belief sets. Our formulation has some special advantages in dealing with problems in sequential situations. For example, in the game described in Fig. 1, there is no vagueness regarding the payoff associated with player 1’s playing $l$, while his playing $r$ will result in complete ignorance of all possible outcomes.
formulated. We restrict our attention to bounded games with finite players and finite act spaces. Roughly speaking, an extensive form game is a detailed description of the dynamic structure of a sequence of decisions encountered by a set of players in a strategic situation. A perfect information game describes the situation in which each player, when making decisions, is perfectly informed of the structure of the game tree, and can observe all the events that have previously occurred including the acts of those players who have had chances to act (note: simultaneous decisions are not allowed).

2.3.1. Extensive game form with perfect information

The following definition of extensive game form with perfect information is standard (see, for example, Osborne and Rubinstein, 1994).

**Definition 1.** An extensive game form with perfect information is defined as $G = (N, T, P,(u_i)_{i \in N})$ where

- $N$ is a finite set of players;
- $T = (V, E)$ is an acyclic digraph, called a game tree. $V$ is a finite set of vertices, or nodes, with a typical element $v \in V$. The game tree $T$ is rooted from a unique vertex $v^* \in V$. The set $E$ consists of all directed edges of the game tree $T$. For each $v \in V$, let $E(v) \subseteq V$ be the set of immediate successors of $v$. It is assumed that $E(v)$ admits a one-to-one correspondence with the set of all edges from vertex $v$. The set of terminal vertices $Z \subseteq V$ consists of all vertices $v \in V$ at which $E(v) = \emptyset$. Vertex $v \in V$ is said to be connected to vertex $v' \in V$ through a directed path $\omega = (v, v_1, \ldots, v_n = v', \ldots)$ if, $v_1 \in E(v)$, and $v_{i+1} \in E(v_i)$, for all $i \geq 1$.
- $P = (P_i)_{i \in N}$ is the players partition, that is $\bigcup_{i \in N} P_i = E/Z$ and $\bigcap_{i \in N} P_i = \emptyset$. Here, $P_i$ is the collection of all vertices at which player $i$ is assigned to make a play. The set of all feasible acts for the player at vertex $v \in P_i$ is, of course, given by $E(v)$. It is assumed that whenever $v \in P_i$ is reached, player $i$ knows that he is at vertex $v$.
- At each terminal node $z \in Z$, a unique payoff vector $(u_i(z) \in R)_{i \in N}$ is assigned to all players of the game. Payoffs are measured in unit of utility.

To formulate a player’s decision problem in game form $G$, we need to specify the (endogenous) state space, players’ ‘act’ space – the so-called space of strategy systems defined below, and preference ordering defined on the space of strategy systems. Belief formation with respect to the possible realization of ‘state’ of nature needs also to be discussed.

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*The set $E(v)$ is understood as the set of all feasible acts at vertex $v$ for the player who is supposed to make a move at $v$. 
2.3.2. State space

For each \( v \in V \), the set of all directed paths through which vertex \( v \) is connected to some terminal vertex \( z \in Z \) is denoted by \( \Omega_v \). For \( z \in Z \), \( \Omega_z = \{z\} \); and for \( v \in V \setminus Z \), the set \( \Omega_v \) has the following mathematical formulation:

\[
\Omega_v = \{(e, \omega) : e \in E(v), \omega \in \Omega_v \}.
\]

Let \( \Omega = \bigcup_{e \in E} \Omega_e \). For any \( \omega \in \Omega \), and \( v \in V \), ‘\( v \) is on the path \( \omega \)’ is denoted by \( v \prec \omega \). For \( v \prec \omega \), let \( \omega_v \in \Omega_e \) denote the truncated path starts from \( v \), and \( \omega(v) \in E(v) \), the immediate successor of \( v \) (on the path \( \omega \)). The set \( \Omega \) is referred to as the endogenous state space, or simply the state space of the game form \( G \). Therefore, the state space \( \Omega \) consists of all (terminated) paths rooted from \( v^* \), the origin of the game tree \( T \). The state space summarizes all of the possible plays, or realizations, in the game form \( G \). For each \( v \), \( \Omega_v \) can be embedded as a subset of \( \Omega \), and it summarizes all of possible paths that may be followed in the subgame form \( G_v \equiv (N, T_v, P(v), (u_i)_{i \in N}) \). The state space \( \Omega \) is common to all players of the game. For each \( v \), let \( \mathcal{F}_v = 2^{\Omega_v} \). Then, \( \{\mathcal{F}_v\}_{v \in V} \) is referred to as the information filtration of the state space \( \Omega \). The information filtration summarizes the information structure at each decision node of the game tree. For example, at \( v \), the set of all paths following \( v \) is \( \Omega_v \), and any particular subset of paths in \( \Omega_v \) is an event in \( \mathcal{F}_v \).

2.3.3. Strategy system

Given a game form \( G \) as above, for each \( v \in V \), let \( P(v) \equiv (P_i(v))_{i \in N} \) be the players partition for the subgame \( G_v \), and let

\[
\Sigma_i(v) = \prod_{v' \in P_i(v)} E(v'), \quad i \in N \quad \text{and} \quad \Sigma(v) = \prod_{i \in N} \Sigma_i(v)
\]

be respectively the set of all feasible (pure) strategies for player \( i \) and the set of all feasible (pure) strategy profiles for the subgame \( G_v \) rooted from \( v \). Let \( \Sigma = \{\Sigma(v)\}_{v \in V} \).

Definition 2. A strategy system \( \sigma \) in game form \( G \) is a \( \Sigma \)-valued correspondence such that, (a) for all \( v \in V \), \( \sigma(v) \subseteq \Sigma(v) \); (b) for all \( v \in P_i \), \( v' \in P_i(v) \), projection of \( \sigma_i(v) \) into \( \Sigma_i(v') \), denoted by \( \sigma_i(v, v') \), coincides with \( \sigma_i(v') \), i.e., \( \sigma_i(v, v') = \sigma_i(v') \).

Let \( \Xi \) denote the set of all strategy systems for the game form \( G \). A strategy system \( \sigma \in \Xi \) assigns a set of acts for each player of the game at each of his decision nodes. Therefore, a strategy system is also referred to as a plan. As

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10 Note that each vertex \( v \) connects to the root \( v^* \) of the game tree through a unique directed path (or history), and the player at vertex \( v \) is only concerned with the future plays of the game. Therefore, his uncertainty at vertex \( v \) is concerned with the possible realizations of paths within the set \( \Omega_v \).
mentioned in the introduction, our assumption of set-valued acts is based on the belief that, when a player at vertex \( v \) makes a long-run plan of action, say for vertex \( v' \in P_i(v) \), he may or may not be willing to pre-commit to any specific single act at that vertex \( v' \) (e.g., there exists a set of acts, each of which is equally good to him), but instead he may choose a subset of acts at that vertex so that all acts in this subset will be candidate selections at \( v' \) once \( v' \) is reached. When \( \sigma(v) \) is a singleton, it reduces to a strategy profile in subgame \( G_v \). Condition (b) is the so-called dynamic consistency restriction on the strategy system: player \( i \)'s plan of action, made at vertex \( v \), for the subgames starting at vertex \( v' \in P_i(v) \), coincides with his (ex-post) plan of action at vertex \( v' \). Write \( \sigma(v) = (\sigma_i(v))_{i \in N} \), \( v \in V \), then \( \sigma_i(v) \) is referred to as player \( i \)'s strategy system in subgame \( G_v \). Given the consistency condition, a strategy system \( \sigma \in \Xi \), can be expressed as \( \sigma(v) = (E(\sigma, v'))_{v' \in P(v)} \), where \( E(\sigma, v') \subseteq E(v') \) is the set of local choices at \( v' \) under \( \sigma \).

2.3.4. Belief system

There are two alternative ways to define a player \( i \)'s belief at a vertex \( v \in P_i(v) \); (1) beliefs are about the possible paths to be followed after he makes a plan of action at \( v \) and (2) beliefs are about other players' plans of action in the subgame started at vertex \( v \). These two definitions are actually equivalent. This is because every strategy system reduces to a unique set of paths, and because a collection of paths for each subgame of the game also corresponds to a unique strategy system. Therefore, uncertainties about other players' strategy systems, given one's own strategy system, reduce to uncertainties about the possible realizations of paths for each of the subgames.

Definition 3. Given a game form \( G \), a belief system \( \Delta_i = \{ \Delta_i(v) \}_{v \in P_i(v)} \), for player \( i \) is a collection of \( M(\Omega) \)-valued correspondences: for all \( v \in V \), \( \Delta_i(v) = \{ \Delta_i(\cdot; v, v') \}: \Xi_i \to M(\Omega_i(v)) \}_{v \in P_i(v)} \) is such that, for all \( v' \in P(v) \), and for all \( \sigma_i \in \Xi_i \), \( \Delta_i(\sigma_i; v, v') \subseteq M(\Omega_i(v)) \) is a closed and convex set.

In the definition of belief system, \( \Delta_i(\sigma_i; v, v') \) is interpreted as player \( i \)'s belief at \( v \) regarding the possible paths to be followed after vertex \( v' \in P(v) \). Different plans of action may lead to different sets of beliefs. Let \( A_i \) and \( A \), respectively, denote the set of all belief systems for player \( i \) and that of all players of the game \( G \).

Definition 4. A belief system \( A_i \) is said to be consistent with its plan of action if, for all \( v, v' \in P_i(v) \), and \( \sigma_i(v) = (E(\sigma, v'))_{v' \in P_i(v)} \subseteq \Sigma_i(v) \), where \( E(\sigma, v') \subseteq \Sigma_i(v) \), and for all \( p \in A_i(\sigma_i, v, v') \),

\[
p \{ \omega \in \Omega_v : \omega(v') \in E(\sigma_i, v') \} = 1,
\]

or in words, the probability of player \( i \) playing (planned) acts \( e \in E(\sigma, v') \) at vertex \( v' \) conditional on such a decision node \( v' \) being reached, is one.
A consistent belief system can be interpreted as a system that satisfies the following property: given a player’s plan of action, \( E(\sigma_i, v') \), for decision node \( v' \), beliefs (as probability functions) on the possible acts to be selected at \( v' \) have their supports in \( E(\sigma_i, v') \).

### 2.3.5. Preference system and its representation

For any given game form \( G \), the payoff vector \( (u_i(z) \in R)_{i \in N} \) of \( G \) induces a unique preference ordering for all certainty paths in \( \Omega \) and for each player of the game. For all \( \omega \in \Omega \), let \( z(\omega) \in Z \) be the unique terminal node of \( \omega \). Then, player \( i \)'s utility function \( u_i: \Omega \to R \) is defined by \( u_i(\omega) = u_i(z(\omega)) \), \( \forall \omega \in \Omega \). In general, players’ decision problem in such a sequential situation can be summarized by a so-called preference system:\(^{11}\)

\[
\succeq \equiv \{\succeq_{i(v,v')}: i \in N, v \in P_i, v' \in P_i(v)\},
\]

where \( \succeq_{i(v,v')} \) is a binary relationship defined on \( \mathcal{Z} \) for subgame \( G_{v'} \), and it represents player \( i \)'s preference ordering over all his possible plans for subgame \( G_{v'} \), when player \( i \) is actually at his decision node \( v \). In other words, given his information \( \mathcal{F}_v \) at \( v \), his preference over different plans of action for subgame \( G_{v'} \) is summarized by \( \succeq_{i(v,v')} \). A preference system \( \succeq \) is dynamically consistent if, for all \( i, \sigma_i \) and \( \sigma'_i \),

\[
\sigma_i \succeq_{i(v,v')} \sigma_i' \iff \sigma_i \succeq_{i(v,v')} \sigma'_i, \quad \text{for all } v \in P_i \text{ and } v' \in P_i(v).
\]

Consistent with the multi-prior model of decision making in a static environment described in the previous section, we consider the following specification of preference system for the sequential decision problem in the extensive form game situation. Given a belief system \( \Delta_i \) and other players’ strategy system \( \sigma_{-i} \), player \( i \)'s preference \( \succeq_{i(v,v')} \) is represented by a utility function \( u_i(\tau; \sigma_i, v') \), which is defined to be such that, for all \( v \in P_i \), \( v' \in P_i(v) \) and \( \sigma_k(v) = (E(\sigma_i, v'))_{v' \in P_i(v)} \subseteq \Delta_i(v) \)

\[
u_i(v; \sigma_i, v') \equiv \max_{e \in E(\sigma_i, v')} \min_{p \in \Delta_i(\sigma_i; v, e)} \int_{\Omega} u_i(o) \, dp(o), \quad \forall v' \in P_i(v),
\]

where \( \min_{p \in \Delta_i(\sigma_i; v, e)} \int_{\Omega} u_i(o) \, dp(o) \) is the most conservative payoff that player \( i \) can get by taking action \( e \) at \( v' \) based on his beliefs \( \Delta_i(\sigma_i; v, e) \). In Eq. (6), we assume that player \( i \)'s plan of action at his other decision nodes of the game affecting his utility at \( (v, v') \) is reflected only through his belief system. That is, for each plan \( \sigma_i \) and an immediate act \( e \) within the planned acts at \( v' \), player \( i \) forms a belief \( \Delta_i(\sigma_i; v, e) \) regarding the possible paths to be followed after \( e \). Such a belief is formed by taking into consideration his planned actions in future decision nodes. Therefore, we can say that the right-hand side of Eq. (6) is the maximum

---

\(^{11}\) This formulation of preference system is easy to interpret. It applies to all sequential decision situations.
least well-off that player $i$ can achieve at $v'$ through the plan $\sigma_i$. For terminal node $z \in Z$, the utility $u_i(v; \sigma_i, z)$ coincides with $u_i(z)$. Moreover, when belief system $A_i(\sigma_i; v, e)$ is invariant to $v$, utility function defined in Eq. (6) corresponds to a dynamic consistent preference system.

Remark 1. When $E(\sigma_i, v') = \{e\}$ is a singleton for all $v'$, Eq. (6) reduces to the Gilboa-Schmeidler’s formulation of multi-prior expected utility in a static setting except that the uncertainty is about the other players’ plays at their decision nodes. Actually, we can regard the static decision problem studied by Gilboa and Schmeidler as a special case of ours, since the uncertainty associated with the possible realizations of state of nature in their framework can be treated as uncertainty regarding the possible selection of ‘actions’ made by a ‘dummy player’ – the nature. Axiomatic justification of our preference representation Eq. (6) is beyond the scope of this paper. This will involve some special treatments with respect to the presence of set-valued acts and the nature of sequential decision making and we leave this for a future study.

Remark 2. For the special case of single-player games, say $N = \{i\}$, with a suitable specification of belief system, the corresponding representation Eq. (6) of preference system is also meaningfully well-defined. In that case, the uncertainty at each decision node $v$ of player $i$ is about his own possible plays at his future decision nodes (including node $v$). For any given plan $\sigma_i$ and $v' \in P(v)$, let $\Omega_{\sigma_i}(v')$ denote the set of paths in the subgame rooted from $v'$ that are generated by the plan $\sigma_i(v')$. For each act $e \in E(\sigma_i, v')$ in the set of planned actions at $v'$, let $\tilde{\Omega}_{\sigma_i}(e) \subseteq \Omega_{\sigma_i}(e)$ denote the set of paths following $e$ that maximizes player $i$’s utility among all of the paths in $\Omega_{\sigma_i}(e)$. The corresponding belief set $A_i(\sigma_i; v, e)$ appeared on the right-hand side of Eq. (6) can be thus defined as the set of all probability measures over $\tilde{\Omega}_{\sigma_i}(e)$, i.e., $A_i(\sigma_i; v, e) = M(\tilde{\Omega}_{\sigma_i}(e))$. In this case, player $i$’s utility (6) is well-defined, and becomes

$$u_i(\sigma_i, v') = \max_{e \in \tilde{\Omega}_{\sigma_i}(v')} \max_{v' \in \Omega_{\sigma_i}(v')} u_i(v').$$

The right-hand side is the maximum utility that player $i$ can obtain through plan $\sigma_i$ within the subgame rooted from $v'$. Of course, different plans may lead to different utility levels. An optimal plan is one that gives player $i$ the highest utility. Therefore, our formulation of utility function (6) for a player in a multi-agent extensive form game is consistent with utility function specification for a player in a single-player sequential situation.

\[12\] Obviously, individual rationality is taken into consideration in this formulation of player $i$’s belief system.
Given the above definitions, an extensive form game for a game form \( G \) is summarized by \( \Gamma \equiv (G, \Omega, \Xi, \Lambda) \), with subgames \( \{I_v\}_{v \in V} \).

### 3. Rationality and \( \sigma^\ast \)-equilibrium

This section presents our first solution concept for extensive form games. This solution concept is formulated under the following assumptions: (a) no preplay communication is allowed; (b) it is common knowledge that all players are rational and uncertainty averse. Assumption (b) implies that players’ (optimal) decisions are based on ‘rational’ beliefs, on the possible evolution of the game, that are consistent with players’ utility maximizing behavior.

A strategy system corresponds to a plan of action at each decision node that players might actually choose, so if every player \( i \) is expected to form a plan \( \sigma_i \), then every player \( j \) should understand this fact and accordingly forms his own belief \( A_i \) that consists of all probability distributions on paths that are generated by \( \sigma \). Since nobody is allowed to announce its plan to the rest, and no public recommendation is permitted regarding how the game should be played, a player must use his own reasoning to form beliefs regarding other people’s play and to form a plan for themselves on how to play the game. As a consequence, if at vertex \( v \) player \( i \)’s belief system is \( A_i(\sigma_i; v', v) \), where \( v' \in P_i(v) \), then at vertex \( v \), his beliefs for \( v' \) should be the same, i.e., \( A_i(\sigma_i; v, v') = A_i(\sigma_i; v', v) \) (What can change his beliefs?). This results in a dynamic consistent preference system defined in Eq. (6).

Here, we define \( \sigma^\ast \)-equilibrium as the unique ‘rational’ solution in extensive form games of perfect information. More precisely,

**Definition 5.** A \( \sigma^\ast \)-equilibrium for game \( \Gamma \) is a dynamic consistent strategy system \( \sigma^\ast \in \Xi \) such that

(a) There exists a belief system \( A^\ast \in \Lambda \) that is consistent with \( \sigma^\ast \): For all \( i, v \in V \), and \( v' \in P(v), A_i^\ast(\sigma_i^\ast; v, v') = M(\Omega_i^\ast) \), where \( \Omega_i^\ast \) consists of all paths that are generated by \( \sigma^\ast \) within the subgame \( G_v \).

(b) Let \( \sigma_i^\ast = (E(\sigma_i^\ast; v))_{v \in P_i} \), \( \forall i \in N \). For all \( i, v \in P_i \), and for the belief system \( A^\ast \) given in (a),

\[
E(\sigma_i^\ast; v) = \arg \max_{\omega \in E(v)} \min_{\omega \in \Omega_i^\ast} u_i(\omega). \tag{7}
\]

In the above definition, condition (a) is a consistency and rationality restriction on players’ belief system. Every player knows that all players commit on the strategy system \( \sigma^\ast \), but is uncertain (vague) about the realization of any particular path within the set of paths induced by system \( \sigma^\ast \). Condition (b) is an optimality and rationality restriction on agents’ formation of strategy...
system: First, every act \( e \in E(\sigma^*, v) \) in the planned actions set must be optimal in the sense that it solves the maximization problem on the right-hand side of Eq. (7). Second, any act that maximizes player \( i \)'s utility at \( v \), should be included in his 'plan' of actions at that node. This is because all optimal acts deliver the same level of utility to player \( i \), and eliminate any particular set of acts from the optimal set in forming a plan of action cannot be justified, and is irrational.\(^{13}\)

The rational solution concept present here is different than that of rationalizable solutions proposed by Bernheim (1984) and Pearce (1984) and further extended by Battigalli (1993), Epstein (1995) and Luo (1995). First, in this paper, we assume it is common knowledge that all players are uncertainty averse. Second, an equilibrium strategy system, corresponding to a set-valued acts at each decision node, must be optimal instead of 'rationalizable', in the sense that it maximizes each player's uncertainty averse utility given her (rational) beliefs. Third, in this paper, the sources of uncertainty come from the nature of set-valued plans of action since each act in the planned action set at a node might turn out the actual play at that node; while, in Bernheim and Pearce, uncertainty results from the presumption that players are uncertain about their preferences. More precisely, they are uncertain about players' 'subjective probability' in forming the Savage's single prior expected utilities (which are presumably unknown to players' of the game).\(^{14}\)

3.1. \( \sigma^* \)-equilibrium as a refinement to subgame perfect equilibrium

Here is the main result of this section:

**Theorem 1.** For any given game \( \Gamma \), there exists a unique non-empty \( \sigma^* \)-equilibrium. In particular, this \( \sigma^* \)-equilibrium is a refinement to the subgame perfect equilibrium: for each subgame \( \Gamma_v \), the set of \( \sigma^* \)-equilibrium paths \( \Omega^*_v \) is non-empty, and constitutes a subset of \( \Omega_{0,v} \) – the set of subgame perfect equilibrium paths for subgame \( \Gamma_v \).

\(^{13}\)This restriction can be dropped when communication is allowed: if an announced plan, or a recommended act, is included in the set of optimal acts, then it is natural to include only these announced acts or the recommended act in forming a player’s belief. See Sections 5 for details.

\(^{14}\)The primary emphasis of rationalizability is also on the formation of set-valued 'rational beliefs'. That is, every player knows that all players conform to some Savage’s subjective expected utility functions with possibly unknown (yet to be determined through rational reasoning) set of subjective probabilities. In general, in such circumstances a player cannot say if an arbitrary strategy is optimal or not; but they may be able to rule out those strategies that cannot be rationalized by any (rationalizable) beliefs (such as those strictly dominated strategies). Therefore, strategies that have survived the interacted elimination of un-rationalizable strategies are called ‘rationalizable’ strategies.
To prove this theorem, we first need to introduce the following two propositions:

**Proposition 1.** For any given game \( \Gamma \), the set of subgame perfect equilibrium paths \( \{ \Omega_{0,v} \}_{v \in V} \) solves uniquely the following recursive system: for \( v \in \mathbb{Z} \), \( \Omega_{0,v} = \{ v \} \); and for \( v \in P_i/\mathbb{Z} \), \( i \in N \),

\[
\Omega_{0,v} = \left\{ (e, \omega) \in \Omega_v : e \in E(v), \omega \in \Omega_{0,e}, \text{s.t. } u_i(\omega) \geq \max_{\omega' \in \Omega_{0,e}} \min_{e \in E(v)} u_i(\omega') \right\}.
\]  

(8)

**Proof.** For any \( v \in P_i \), according to the definition of subgame perfect equilibrium (see Greenberg, 1989, Theorem 8.2.6), an act \( e \in E(v) \) at \( v \), together with any subgame perfect equilibrium path \( \omega \in \Omega_{0,e} \) for subgame \( \Gamma_e \), forms a subgame perfect equilibrium path \( (e, \omega) \) for subgame \( \Gamma_v \) if, and only if, there does not exist an act \( e' \in E(v) \), such that for all \( \omega' \in \Omega_{0,e'} \), \( u_i(\omega') \geq u_i(\omega) \); or, equivalently, there does not exist an \( e' \in E(v) \), such that \( u_i(\omega) \leq \min_{\omega' \in \Omega_{0,e'}} u_i(\omega') \). The latter statement is equivalent to the condition that, \( u_i(\omega) \geq \max_{e \in E(v)} \min_{\omega' \in \Omega_{0,e'}} u_i(\omega') \). This ends the proof. \( \square \)

According to Proposition 1, to find the subgame perfect equilibrium paths, we need to solve a sequence of recursive minimization problems: For each \( v \in V \) and \( v \in P_i \), given the knowledge of all subgame perfect paths, \( \Omega_{0,v} \) for subgame \( \Gamma_v \) that is rooted from \( e \in E(v) \), player \( i \) computes the minimum utility \( \min_{\omega \in \Omega_{0,e}} u_i(\omega) \) among all paths in \( \Omega_{0,e} \) for each \( e \in E(v) \); therefore, the least well-off that player \( i \) can get at \( v \) is \( \max_{e \in E(v)} \min_{\omega \in \Omega_{0,e}} u_i(\omega) \). The subgame perfect equilibrium paths for the subgame started from \( v \) consists of all paths \( (e, \omega) \in \Omega_v \) such that \( \omega \in \Omega_{0,e} \) is a subgame perfect equilibrium path for the subgame rooted from \( e \in E(v) \), and that player \( i \)'s utility at \( \omega \) is no less than \( \max_{e \in E(v)} \min_{\omega \in \Omega_{0,e}} u_i(\omega') \). Note that, a path \( (e, \omega) \in \Omega_{0,v} \), which is a subgame perfect equilibrium path, could not be located in those subgames \( \Gamma_e, e \in E^*(v) \), that give player \( i \) the maximum least well-off. Therefore, some subgame perfect equilibrium paths cannot be rationalized by assuming that all players have uncertainty averse utility functions described in the previous section, i.e., subgame perfect equilibrium contains paths that are unlikely to be followed by uncertainty averse players. In particular, the ‘beliefs’ that all players follow subgame perfect equilibrium paths in all of its subgames, as is implicitly assumed in the subgame perfect equilibrium solution concept, is also inconsistent with the uncertainty averse behavior assumption.

**Proposition 2.** For any given game \( \Gamma \), the set of \( \sigma^* \)-equilibrium paths \( \{ \Omega^*_v \}_{v \in V} \) uniquely solves the following recursive system: for \( v \in \mathbb{Z} \), \( \Omega^*_v = \{ v \} \), and for \( v \in P_i/\mathbb{Z} \), \( i \in N \),

\[
\Omega^*_v = \left\{ (e, \omega) \in \Omega_v : \omega \in \Omega^*_e \quad \text{and} \quad e \in \arg \max_{e \in E(v)} \min_{\omega \in \Omega^*_e} u_i(\omega) \right\}.
\]  

(9)
Proof. Follows obviously from the definition of $\sigma^*$-equilibrium. The details are thus omitted. □

In contrast to the subgame perfect equilibrium, for each subgame $\Gamma_v, v \in P$, the corresponding set of $\sigma^*$-equilibrium paths consists of all $(e, \omega) \in \Omega_e$, such that, $\omega \in \Omega^*_e$ and $e \in E^*(v)$, where $E^*(v) \subseteq E(v)$ is the set of vertices $e \in E(v)$ such that player $i$’s least well-off among all $\sigma^*$-equilibrium paths in subgame $\Gamma_e$ is $\max_{e \in E(v)} \min_{\omega \in \Omega^*_e} u_i(\omega)$. Of course, for any selection $\omega \in \Omega^*_v$, player $i$’s utility at $\omega$ is no less than the maximum least well-off, i.e., $u_i(\omega) \geq \max_{e \in E(v)} \min_{\omega \in \Omega^*_e} u_i(\omega)$.

Proof of Theorem 1. The existence and uniqueness of $\sigma^*$-equilibrium follows immediately from Proposition 2. The collection of paths $\{\Omega^*_v\}_{v \in V}$, which is unique, defines, uniquely, a set of acts for each vertex $v \in V$, which thus forms, uniquely, a strategy system $\sigma^*$. This strategy system $\sigma^*$, together with the belief system $\Delta^i_\sigma(\sigma^*_i; v, v') = M(\Omega^*_v)$, $\forall v \in V, v' \in P(v), \sigma \in \Sigma$, obviously satisfies Definition 5.

We use the backward induction argument to prove the second part of the theorem. Let $l$ denote the length of a game tree $T$, which is defined as the maximum number of nodes that forms a path in $T$. For $v \in Z$, we have $\Omega^*_v = \Omega_{0, v} = \{v\}$. Suppose that $\Omega^*_v \subseteq \Omega_{0, v'}$ for all subgames $\Gamma_{v'}$ that have a length less or equal to $l$. Now consider the subgame $\Gamma_v$ that has a length of $l + 1$. We have, for any $\omega \in \Omega^*_v$,

$$u_i(\omega) \geq \max_{e \in E(v)} \min_{\omega' \in \Omega^*_e} u_i(\omega') \geq \max_{e \in E(v)} \min_{\omega' \in \Omega_{0, v}} u_i(\omega'),$$

where the first inequality is by Proposition 2 and the second inequality is due to the fact that the length of game $\Gamma_v$ is less than or equal to $l$ and that $\Omega^*_v \subseteq \Omega_{0, v}$ by assumption. Therefore, by Propositions 1 and 2, we have $\omega \in \Omega_{0, v}$ since there exists an $e \in E(v)$ such that $\omega \in \Omega^*_e \subseteq \Omega_{0, v}$. □

The following example shows that $\sigma^*$-equilibrium can be a strict-refinement to the subgame perfect equilibrium.

Example 1. The game described in Fig. 1a has a unique $\sigma^*$-equilibrium path $l$, while it has two subgame perfect equilibrium paths: $l$ and $(r, L)$. Since communication is not allowed in this game, player 1, who is uncertainty averse, will take action $l$ in fear of player 2’s playing $R$. (Note that player 2 is indifferent between $L$ and $R$.)

Under what condition does the set of (pure strategy) subgame perfect equilibrium paths coincide with the set of $\sigma^*$-equilibrium paths? Obviously, these two
sets coincide whenever the set of subgame perfect equilibrium paths is a singleton. A well-known sufficient condition for that is when each player has a vector of payoffs (over all the terminal nodes) that are strictly ordered, i.e. when there do not exist two terminal nodes that deliver the same payoff to a same player.

3.2. An axiomatic justification of $\sigma^*$-equilibrium

For any given game $\Gamma$, a value function $u^*$ for $\Gamma$ is a map $u^*: N \times V \rightarrow R$ such that, for all $i \in N$, and vertex $v \in V$, $u^*(i, v)$ is the value assigned to subgame $G_v$ for player $i$. Values of a game (or a subgame) are, therefore, interpreted as the fair prices (in unit of utility) that players are respectively willing to pay in order to play the game and each of the subgames. The following axioms, for the value function $u^*(\cdot, \cdot)$, are consistent with the uncertainty-averse utility maximizing behavior specified in the previous section:

Axiom (U1). For $v \in P_t$, $u^*(i, v) = \text{max}_{e \in E(v)} u^*(i, e)$.

Axiom (U2). For $v \in P_t$, and $j \neq i$, $u^*(j, v) = \text{min}_{e \in E^*(v)} u^*(j, e)$, where $E^*(v) \equiv \text{arg max}_{e \in E(v)} u^*(i, e)$.

Axiom (U3). For $v \in Z$, $u^*(i, v) = u_i(v)$, for all $i \in N$.

Axiom (U1) says that player $i$'s value at his decision node $v$ is the maximum value among all subgames to which he can lead. According to Axiom (U2), player $j$'s value $u^*(j, v)$ at vertex $v$ is the minimum value for player $j$ over subgames $\{G_v\}_{e \in E^*(v)}$ that are likely to be led by player $i$ from $v$. Axiom (U3) is a 'boundary condition' for the value functions $u^*$ imposed on all terminal nodes $v \in Z$.

Theorem 2. For any given extensive form game $\Gamma$, there exists a unique value function $u^*: N \times V \rightarrow R$ satisfying Axioms (U1)–(U3). Moreover, the strategy system $(E^*(v'))_{v' \in P(v)}$, $v \in V$, coincides with the unique $\sigma^*$-equilibrium of $\Gamma$, and $u^*(i, v) = \text{min}_{e \in \Omega^*(v)} u_i(\omega)$, for all $v \in V$, and $i \in N$.

Proof. The existence and uniqueness of the value function follows by construction through the above listed axioms. We prove the second part of the theorem by induction. The statement holds for all subgames of length 1. Suppose, it holds for all subgames whose length are no longer than $l$ ($\geq 1$). Consider a subgame $\Gamma_v$ has a length of $l + 1$, and is rooted from $v \in V$. Without loss of generality, we assume $v \in P_t$. By assumption, for all $e \in E(v)$, in subgame $\Gamma_e$ we have $u^*(i, e) = \text{min}_{e \in \Omega^*(\omega)} u_i(\omega)$. Therefore, $e \in E^*(v)$ if and only if $e \in \text{arg max}_{e \in E(v)} \text{min}_{e \in \Omega^*(\omega)} u_i(\omega) \equiv E(\sigma^*, v)$. We conclude that $(E^*(v'))_{v' \in P(v)}$ coincides with $\sigma^*(v) \equiv (E(\sigma^*, v'))_{v' \in P(v)}$. Moreover, by Axiom (U1), for player $i$ we have

$$u^*(i, v) \equiv \text{max}_{e \in E(v)} u^*(i, e) = \text{max}_{e \in E(v)} \text{min}_{e \in \Omega^*(\omega)} u_i(\omega) = \text{min}_{e \in \Omega^*(\omega)} u_i(\omega),$$
for all $e \in E(\sigma^*, v)$, or equivalently, $u^*(i, v) = \min_{e \in \Omega^*(v)} u_i(\omega)$; similarly, for $j \neq i$, by Axiom (U2), we have

$$u^*(j, v) = \min_{e \in E^*(v)} u^*(j, e) = \min_{e \in E(\sigma^*, v)} \min_{\omega \in \Omega^*(v)} u_j(\omega) = \min_{e \in \Omega^*(v)} u_j(\omega).$$

This ends the proof. $\square$

If we interpret the values as prices that players are willing to pay in order to play the game and each of its subgames, then the theorem says that the unique $\sigma^*$-equilibrium will consist of all possible plays that can actually attain these values. This observation provides a straightforward approach for one to find the $\sigma^*$-equilibrium for a given game. At each decision node $v \in P_i$, player $i$ computes, first, his value at each of the subgames that he can lead to directly from $v$ (this can be done by backward induction). The corresponding set of $\sigma^*$-equilibrium actions at $v$ is then the set of all actions that maximize player $i$'s value at $v$.

4. $\sigma^*$-equilibrium with a publicly recommended path

Given all ‘nice’ properties of $\sigma^*$-equilibrium (without communication), similar to many other solution concepts proposed in the literature, the $\sigma^*$-equilibrium may rules out some intuitively attractive outcomes.\footnote{The assumption of the existence of public recommendation can indeed capture the complexity that one may encounter in many social situations. Readers are referred to Greenberg (1989) for further demonstration of this.} To illustrate this, let us consider the following example:

Example 2. For the game described in Fig. 1a, the unique $\sigma^*$-equilibrium path is given by $l$ with payoff vector $(1,1)$. This equilibrium outcome is obviously Pareto-dominated by the outcome associated with playing $(r, L)$, which gives a payoff vector $(2,2)$.

In this example, suppose that the following public recommendation is offered to both players before they actually start to play the game: player 1 is advised to take action $r$, while player 2 is advised to take action $L$. Will they follow such an advice? The answer is obviously ‘yes’: player 2 cannot increase his utility by deviating from $L$ to $R$, so he will follow the advice. Player 1 knows that player 2 is indifferent between acts $L$ and $R$ and he also knows that player 2 will not deviate from the advice once he himself follows the recommendation to take action $r$. Therefore, given the above public advice, player 1’s vagueness with respect to player 2’s selecting of acts (between $L$ and $R$) disappears; in consequence, both players follow the advice and reach a Pareto-optimal outcome.
4.1. \( \sigma^*_x \)-equilibrium under recommended path \( x \)

This section introduces our second solution concept, \( \sigma^*_x \)-equilibrium with a publicly recommended path, which is referred to as \( \Omega^*_x \)-equilibrium. By public recommendation, we mean advice offered to all players before the start of the game on a specific path along which the players are advised to follow. Given a recommended path, it is assumed that each player has the option to follow the recommended path or to deviate from it. If any player deviates, then the game reduces to a subgame where players act as if they were playing a game described in the previous section.\(^{16}\)

The following definition of \( \sigma^*_x \)-equilibrium is similar to the \( \sigma^* \)-equilibrium except in the definition of the \( \sigma^*_x \)-equilibrium we take into account the fact that the existence of recommended path \( x \) will affect players’ belief formation and optimal choices, thus the outcomes of the game.

**Definition 6.** For any given \( \Gamma \) and \( x \in \Omega \), a recommended path, the \( \sigma^*_x \)-equilibrium of \( \Gamma \) under recommendation \( x \in \Omega \) is a dynamic consistent strategy system \( \sigma^*_x \in \Xi \) such that:

(a) There exists a belief system \( \Delta^*_x \in \Delta \) that is consistent with \( \sigma^*_x \): for all \( i, v \in V \), and \( v' \in P(v) \), \( \Delta^*_x(v^*_x, v, v') = M(\Omega^*_x) \), where \( \Omega^*_x \) consists of all paths that are generated by \( \sigma^*_x \) within the subgame \( \Gamma_{v'} \).

(b) \( \sigma^*_{x,i} = (E(\sigma^*_{x,i}, v))_{v \in P_i}, \forall i \in N \), is such that, for all \( v \in P \), given belief system \( \Delta^*_{x,i} \) in (a),

\[
\begin{align*}
E(\sigma^*_{x,i}, v) &= \{ x(v) \} \quad \text{if } x(v) \in E^*_x(\sigma^*_{x,i}, v), v \propto x, \\
E(\sigma^*_{x,i}, v) &= E^*_x(\sigma^*_{x,i}, v) \quad \text{otherwise},
\end{align*}
\]

where

\[
E^*_x(\sigma^*_{x,i}, v) = \arg \max_{e \in E(v)} \min_{o \in \Omega^*_x} u_i(o).
\]

Condition (a) is a consistency and rationality restriction on players’ belief systems: every player knows that all players commit on the strategy system \( \sigma^*_x \), but is uncertain about the realization of any particular path within the set of paths induced by system \( \sigma^*_x \) except when the path \( x \) is followed by all players. Condition (b) is an optimality and rationality restriction on agent’s formation of strategy system: First, every act \( e \in E(\sigma^*_x, v) \) in the planned actions set must be optimal in the sense that it maximizes the right-hand side of Eq. (11). Second,

\(^{16}\) This specification of players’ belief-formation concerning choices that would be taken in vertices off the recommended path is different to those of Fudenberg and Levine (1993) and Greenberg (1994).
if \( v \in P_1 \) is on the recommended path \( x \), and if the successor \( x(v) \) of \( v \) on path \( x \) belongs to the set of maximizers at \( v \), i.e., \( x(v) \in E^*_x(\sigma^*_x(v), v) \), then player \( i \) will not deviate from the recommendation of playing \( x(v) \) at vertex \( v \). In that case, his plan of action at \( v \) is \( x(v) \); similarly, if either \( v \) is not on the recommended path or if it is on the recommended path, but fails to be a maximizer for player \( i \)'s optimization problem (player \( i \) will deviate), the plan of action at vertex \( v \) will consist of all acts that maximize player \( i \)'s utility at \( v \).

**Theorem 3.** For any given game \( \Gamma \), and \( x \in \Omega \), the \( \sigma^*_x \)-equilibrium is non-empty and unique. In addition, \( \sigma^*_x \)-equilibrium has the following properties:

(i) The collection of equilibrium paths \( \{ \Omega^*_x \}_{v \in V} \) induced by \( \sigma^*_x \) solves, uniquely, the following recursive system: for \( v \in Z \), \( \Omega^*_x = \{ v \} \); and for \( v \in P_1 \), \( i \in N \), if \( v \sim x \), and if

\[
\begin{align*}
\mathrm{argmax}_{e \in E(v)} \min_{e \in \Omega^*_v} u_i(e), \quad x(v) \in \Omega^*_x(v),
\end{align*}
\]

\( \Omega^*_v = \{ (e, \omega) \in \Omega_v : \omega \in \Omega^*_v \text{ and } e \in \mathrm{argmax}_{e \in E(v)} \min_{e \in \Omega^*_v} u_i(e) \} \).

(ii) For all \( v \in V \) not on the path \( x \), \( \sigma^*_x(v) = \sigma^*(v) \), in particular, \( \Omega^*_v = \Omega^*_v \).

(iii) For all \( v \in V \) on the path \( x \), if \( x_v \in \Omega^*_x \), then \( \Omega^*_v = \{ x_v \} \).

**Proof.** The first statement of the theorem and property (i) follow by the definition of \( \sigma^*_x \)-equilibrium, and application of the standard backward induction argument. The details are thus omitted. Statement (ii) is also obvious because, when \( v \) is not on the recommended path, no vertices in the subgame \( \Gamma_v \) will be on the path \( x \). Therefore, for this subgame, we are facing the same situation as in the no-communication game studied in the previous section, and the \( \sigma^*_x \)-equilibrium coincides with the unique \( \sigma^*- \)equilibrium in that subgame.

We prove (iii) by induction. The statement holds obviously for \( l = 1 \). Assume that (iii) holds for all subgames that have a length less than or equal to \( l \), and suppose, on the contrary, that there exists an \( x \in \Omega \), and \( v \in P_1 \) such that \( x_v \in \Omega^*_v \), but \( x_v \notin \Omega^*_v \), and that the length of \( G(v) \) is \( l + 1 \). Note that \( x_v \in \Omega^*_v, \forall v \neq v' \sim x \). Therefore, the condition \( x_v \notin \Omega^*_v \) implies that there exists an \( e \in E(v), e \neq x(v) \) such that

\[
\begin{align*}
\max_{e \in E(v)} \min_{e \in \Omega^*_v} u_i(e) \\
\leq \max_{e \in E(v)} \min_{e \in \Omega^*_v} u_i(e),
\end{align*}
\]

where the equality follows from property (ii). This inequality contradicts the assumption that \( x_v \in \Omega^*_v \). \( \square \)
Property (i) in Theorem 3 implies that, the set of equilibrium paths will be a singleton when a well-designed recommendation $x$ is offered, and this singleton set is $x$ itself. Property (iii) says that all recommended paths $x$ that are induced by the $\sigma^*$-equilibrium will be followed. However, the converse of (iii) does not hold in general, i.e., we can construct a game that contains paths which are not induced by $\sigma^*$-equilibrium, that, once recommended, will be followed.

4.2. $\Omega^{**}$-equilibrium – the second refinement of subgame perfect equilibrium

For any given $v \in V$, denoted by $\Omega_v^{**} \subseteq \Omega_v$, the set of all paths $x$ in $\Omega_v$ that, once being recommended (for subgame $\Gamma_v$), will be followed, i.e.,

$$\Omega_v^{**} = \{ x \in \Omega_v : \text{for all } v' \preceq x, x_v \in \Omega_v^* \}. \quad (12)$$

This collection $\{ \Omega_v^{**} \}_{v \in V}$ of paths for all subgames in $\Gamma$ is referred to as $\Omega^{**}$-equilibrium of game $\Gamma$. It is obvious that, the set $\Omega_v$ of all $\sigma^*$-equilibrium paths is a subset of $\Omega_v^{**}$. Interestingly, we can further show that $\Omega^{**}$-equilibrium is still a refinement to the subgame perfect equilibrium, i.e., $\Omega_v^{**} \subseteq \Omega_{0,v}$. The following is the main result of this section:

**Theorem 4.** For any given game $\Gamma$, the set of $\Omega^{**}$-equilibrium paths of game $\Gamma$ contains the $\sigma^*$-equilibrium paths as a subset, and it refines the subgame perfect equilibrium. That is, for all $v \in V$,

$$\Omega_v^{**} \subseteq \Omega_{0,v} \subseteq \Omega_v.$$  

**Proof.** The relationship $\Omega_v^{**} \subseteq \Omega_{0,v}$ follows immediately from Theorem 3(iii) and by the definition of $\sigma^{**}$-equilibrium. The set $\Omega_v^{**} \subseteq \Omega_v$ is non-empty because $\Omega_v^* \neq \emptyset$.

We prove $\Omega_v^{**} \subseteq \Omega_{0,v}$ by induction with respect to $l$, the length of a game. For all $v \in Z$ which corresponds to $l = 1$, we have $\Omega_v^{**} = \Omega_{0,v} = \{ v \}$. Suppose $\Omega_v^{**} \subseteq \Omega_{0,v}$ holds for all subgames of $\Gamma$ that have a length of no more than $l$. Now, consider a subgame $\Gamma_v$ that has a length of $l + 1$. For any given $x \in \Omega_v^{**}$, we have $\Omega_v^* = \{ x_v \}$, for all $v' \preceq x$, by the definition of $\Omega_v^{**}$ and $\Omega_v^*$. In particular, for $e = x(v) \in E(v)$, we have $x_e \in \Omega_v^{**} \subseteq \Omega_{0,e}$. Assume that $v' \in P_t$, without loss of generality, we have

$$u_t(x) = \max_{e \in E(v)} \min_{x \in \Omega_v^*} u_t(\omega)$$  

$$\geq \max_{e \in E(v) \setminus \{ x(v) \}} \min_{x \in \Omega_v^*} u_t(\omega)$$  

$$= \max_{e \in E(v) \setminus \{ x(v) \}} \min_{x \in \Omega_v^*} u_t(\omega)$$  

$$\geq \max_{e \in E(v) \setminus \{ x(v) \}} \min_{x \in \Omega_{0,v}} u_t(\omega),$$
where the second equality is by Theorem 3(iii), and the second inequality is by Theorem 1. By Proposition 1, we conclude that \( x \in \Omega_{v,v} \). This is true for any arbitrary \( x \in \Omega_v^\sigma \). Therefore, \( \Omega_v^{**} \subseteq \Omega_{v,v} \), \( \forall v \in V \). \( \square \)

The following example shows that the \( \Omega^{**} \)-equilibrium can actually be a strict refinement of the subgame perfect equilibrium:

Example 3. Consider the game described in Fig. 2. This game has one \( \sigma^* \)-equilibrium path \( \Omega^* = \{(r,L)\} \), and two \( \Omega^{**} \)-equilibrium paths \( \Omega^{**} = \{(r,L);(r,R,l)\} \) together with three subgame perfect equilibrium paths \( \Omega_0 = \{l,(r,L);(r,R,l)\} \). Therefore, we have \( \Omega^* \subseteq \Omega^{**} \subseteq \Omega_0 \). The reason why the path connected by act \( l \) is not in \( \Omega^{**} \) is that, when \( l \) is recommended to player 1, player 1 will deviate to \( r \) (why?). As is noticed by player 1, once player 2’s decision node is reached, player 2 will choose to play \( L \) instead of \( R \). This is because the former gives player 2 a payoff of 3 while the latter gives a set of vague payoffs 4 or 0 with ‘certainty equivalent’ payoff of 0. Therefore, by deviating to \( r \), player 1 will increase his payoff.
5. $\Xi^*$-equilibrium with publicly announced plans

This section proposes a class of games (and the corresponding solution concepts) in which each player is required, or has the option, to make an announcement to the rest regarding his plan of action for each of his decision nodes (whenever they are reached) before they start to play the game. The players are not bound by their announcements in the sense that, once they start to play the game, they do not necessarily have to follow their announced plans.

5.1. Announcements as ‘credible threats’, or as ‘credible promises’

The existence of pre-announced plans of action may affect players’ decision making and belief formation. In contrast to games with public recommendation, games with public announcement of strategy systems allow players to decide, themselves, on what to announce and what to do after all players have made their announcements. In addition, from a technical perspective, an announced plan may correspond to a set of paths instead of a single path as described in the previous section. Therefore, players are playing an ‘active’ role in determining the announcement of a strategy system. In particular, a player may use his announced plan of action as an instrument to manipulate other players’ belief and acts for his own purposes. For example, by announcing their plan of action, a player can create ‘credible threats’ or ‘credible promises’ to other players. The following is an illustrating example:

Example 4. For the game described in Fig. 1a, there are two $\Omega^{**}\text{-equilibrium}$ paths $\Omega^{**} = \{l(r, L)\}$. That is, if either path $l$ or path $(r, L)$ is recommended, it will be followed by both players. However, the associated payoff $(1,1)$ with path $l$ is obviously Pareto dominated by the payoff $(2,2)$ associated with the path $(r, L)$. Consider the situation in which players can pre-announce their plan of action. Player 2 can make the following announcement: ‘Were the node $v_2$ reached, I will take action $L’$, and player 1 can announce: ‘I will take action $r’$. Each player knows the other player’s announcement. To player 1, player 2’s announcement is credible – player 2 has no incentive to deviate from his pre-announced act once he is reached; therefore, he (player 1) will follow his own pre-announced act $r$ which leads to the Pareto-optimal outcome $(2, 2)$. Moreover, any other sorts of announcements, such as ‘$R’$, or ‘either $L$ or $R’$, made by player 2 will result in a Pareto-dominated outcome $(1,1)$ (why?). Therefore, player 2’s only rational announcement is ‘$L’ – this forms a credible promise to player 1, and as a response, player 1’s only rational announcement is $r$. This is the unique equilibrium outcome of the game. Now, suppose the payoff associated with player 1’s act $l$ is $(1, 3)$ instead of $(1, 1)$, as is described in Fig. 1b. In that case, there exists unique equilibrium path $l$, which is rationalized by player 2’s announcing ‘either $L$ or $R’ – this is a credible threat to player 1, and player 1’s
announcing \( I \). Note that player 1, who is uncertainty averse, cannot benefit for sure by announcing ‘r’ instead of ‘l’. Similarly, announcing ‘L’ by player 2 is irrational since such an announcement may induce an inferior outcome for player 2. In both cases, we get refinement of \( \Omega^{**} \)-equilibrium.

5.2. \( \hat{\sigma} \)-Equilibrium with publicly announced strategy system \( \sigma \)

Generally speaking, the existence of announced plan \( \sigma \) can affect players’ decision making and belief formation through the following two channels: First, players do not deviate from their announced plan as long as following the plan does not contradict one’s utility maximizing principle; Second, a player will not commit to his announced plan if acts of players previous him contradict their announced plans.\(^{17} \) In consequence, the existence of pre-announced plans of action cannot eliminate Knightian uncertainties. Moreover, in deciding whether to deviate from his announced plan, a player must evaluate the consequences of such a deviation, and form a belief that is consistent with the above two assumptions. These are reflected in the following definition of \( \hat{\sigma} \)-equilibrium.

**Definition 7.** For any given \( \Gamma \), and \( \sigma \equiv (E(\sigma_i,v))_{i,v} \in \Xi \) a public announced strategy system, the \( \hat{\sigma} \)-equilibrium of \( \Gamma \) under announcement \( \sigma \) is a dynamic consistent strategy system \( \hat{\sigma} \in \Xi \) such that:

(a) There exists a belief system \( \hat{A} \in A \) that is consistent with \( \hat{\sigma} \in \Xi \): For all \( i, v \in V \), and \( v' \in P(v) \), \( \hat{A}_i(\hat{\sigma}_i,v,v') = M(\hat{\Omega}_v) \), where \( \hat{\Omega}_v \) consists of all paths that are generated by \( \hat{\sigma} \) within the subgame \( \Gamma_{v'} \).

(b) \( \hat{\sigma}_i = (E(\hat{\sigma}_i,v))_{v \in P_i} \), \( \forall v \in P_i, i \in N \), is such that

1. if there exist no \( \omega \in \Omega_\sigma(v^\omega) \) such that \( v \prec \omega \), then \( \hat{\sigma}_i(v) = \sigma_i^\omega(v) \);
2. if \( v \prec \omega \in \Omega_\sigma(v^\omega) \), then

\[
E(\hat{\sigma}_i,v) = \begin{cases} E(\sigma_i,v) \cap E^*(\hat{\sigma}_i,v) & \text{if } E(\sigma_i,v) \cap E^*(\hat{\sigma}_i,v) \neq \emptyset, \\ E^*(\hat{\sigma}_i,v) & \text{otherwise}, \end{cases}
\]

where

\[
E^*(\hat{\sigma}_i,v) = \arg \max_{e \in E(v)} \min_{\omega \in \hat{\Omega}_e} u_i(\omega).
\]

(c) For all \( v \in Z \), \( \hat{\sigma}_i(v) = \{v\} \).

\(^{17}\)The conditions under which a player will commit to his announced plans can be modified a little. For example, we may require that a player will commit only when all the following three conditions are met: first, players previous him do not contradict their announced plans; second, he also anticipates that players following him will not contradict their plans; and third, there exists a plan that does not contradict to the plan announced by himself, and is optimal for him to follow that plan. The corresponding equilibrium concept can be similarly modified. This, however, will not change the set of equilibrium solutions.
In the above definition, condition (b)(1) is equivalent to the requirement that if \( v \) is off the announced paths, then player at decision node \( v \) will use \( \sigma^* \)-equilibrium strategy system to play the subgame rooted from \( v \) once it is reached (note that if \( v \) is not on the announced paths, then so are all \( v' \in P(v) \)). The fact that \( v \) is off the announced paths but is nevertheless reached, implies that somebody did not follow their announced plan of action. Condition (b)(1) implies that, once a decision node \( v \) is reached, which deviates from the announced plan of action, everybody in the subgame rooted from \( v \) will behave as if they were playing a game without communication - nobody will commit to their announced plans of action within this subgame. This results in the \( \sigma^* \)-equilibrium outcome for that subgame.

Similarly, condition (b)(2) requires that, if \( v \) is on some of the announced paths, then player at decision node \( v \) will form a plan of action at \( v \) by selecting all acts from the announced acts \( E(\sigma, v) \) that are also in the best response set \( E^*(\hat{\sigma}, v) \), provided that these two sets have a non-empty intersection. In other words, we assume that the player at vertex \( v \) is willing to commit to his announced plan, if the acts of all players previous him do not contradict their announced plans, and if it is ‘rational’ for him to form a plan of action at \( v \) that does not contradict to the plan announced by himself (given his beliefs that players following him all use the strategy system \( \hat{\sigma} \)). However, when it is irrational for him to follow his announced plan (this occurs when there exists no pre-announced acts that are actually in the best response set), he will deviate from his announced acts, and will reduce to games with a \( \sigma^* \)-equilibrium outcome. Condition (b)(2) is consistent to condition (b)(1) in the sense that, if \( E(\sigma, v) \cap E^*(\hat{\sigma}, v) = \emptyset \), then there exists no \( v' \in E^*(\hat{\sigma}, v) \) that is on any of the announced paths, which implies that, by condition (b)(1), the \( \sigma^* \)-equilibrium will be the outcome for subgame \( \Gamma_v \), i.e., \( \hat{Q}_{v} = \Omega_{\sigma}^* \), \( \forall v \in E(v) \), and \( E^*(\hat{\sigma}, v) = E(\sigma, v) \).

**Theorem 5.** For any given game \( \Gamma \), and \( \sigma \in \Xi \), the \( \hat{\sigma} \)-equilibrium is non-empty and unique. In addition, \( \hat{\sigma} \)-equilibrium has the following properties:

(i) The equilibrium paths \( \{ \hat{Q}_{v} \}_{v \in V} \) induced by \( \hat{\sigma} \) solve uniquely the following recursive system: for \( v \in Z \), \( \hat{Q}_v = \{ v \} \); for \( v \in P_i \), \( i \in N \), if \( v \preceq \omega \in \Omega_{\sigma}(v^*) \), and \( E(\sigma, v) \cap \arg \max_{e \in E(v)} \min_{\omega \in \hat{Q}_{e}} u_i(\omega) \neq \emptyset \), then

\[
\hat{Q}_v = \left\{ (e, \omega) : \omega \in \hat{Q}_e \text{ and } e \in E(\sigma, v) \cap \arg \max_{e \in E(v)} \min_{\omega \in \hat{Q}_{e}} u_i(\omega) \right\};
\]

and if there does not exist an \( \omega \in \Omega_{\sigma}(v^*) \), s.t., \( v \preceq \omega \), then \( \hat{Q}_v = \Omega_{\sigma}^* \).

(ii) For all \( v \in V \), and \( v \preceq \omega \in \Omega_{\sigma}(v^*) \), if \( \sigma(v) \subseteq \sigma^*(v) \), then \( \hat{\sigma}(v) \subseteq \sigma(v) \); or equivalently, \( \hat{Q}_v \subseteq \Omega_{\sigma}(v) \).

**Proof.** The first statement of the theorem as well as property (i) follows from the definition of \( \hat{\sigma} \)-equilibrium and applications of the standard backward induction
argument. The details are thus omitted. To prove (ii), suppose, on the contrary, that there exists a \( v \in V \) and \( v \preceq \omega \in \Omega_d(v^*) \) such that \( \Omega_d(v') \subseteq \Omega^v, \ \forall v' \in P(v) \), and \( \omega \in \tilde{\Omega}_v, \ \omega \notin \Omega_d(v) \). By condition (b)(2) in Definition 7, we have either (1) \( \tilde{\Omega}_v \cap \Omega_d(v) = \emptyset \), or (2) \( E(\tilde{\sigma}_i, v) = E(\sigma_i, v) \cap E^v(\tilde{\sigma}_i, v) \subseteq E(\sigma_i, v) \). Relation (1) fails since, by (b)(1) of Definition 7, we have \( \tilde{\Omega}_v = \Omega^v \), in particular, \( \tilde{\Omega}_v \cap \Omega_d(v) = \Omega_d(v) \neq \emptyset \) since, by assumption, \( \Omega_d(v) \subseteq \Omega^v \). In addition, (2) implies that, for any arbitrary \( v \) along the announced paths, \( \tilde{\Omega}_v \subseteq \Omega_d(v) \), which contradicts to the assumption that there exists an \( \omega \in \tilde{\Omega}_v \) such that \( \omega \notin \Omega_d(v) \).

Property (ii) says that for announced strategy systems contained in the \( \sigma^* \)-equilibrium strategy system, the corresponding \( \tilde{\sigma} \)-equilibrium will never contradict to the announced strategy system.

5.3. \( \Xi^* \)-equilibrium: the third refinement of subgame perfect equilibrium

For games with pre-announced plans of action, the equilibrium is naturally defined as all stable and rational announcements made by the players. By ‘stable’, we mean that a strategy system that, once announced, will be followed. An announced strategy system is ‘rational’ if nobody regrets his announcement; that is, nobody can benefit for sure by substituting his announced plan with an alternative one given other players’ announced plan. The following definition of \( \Xi^* \)-equilibrium takes all these factors into consideration.

Let \( \tilde{\Xi} \) be the set of all stable announcements that consist of all strategy systems that, once announced, will be credibly followed, i.e.,

\[
\text{for all } v \in V, \tilde{\Xi}(v) \equiv \{ \sigma: \tilde{\sigma}(v') = \sigma(v'), \ \forall v' \in P(v) \}. \quad (14)
\]

**Definition 8.** For any given \( \Gamma \), the \( \Xi^* \)-equilibrium with publicly announced plans of action is a subset of stable announcements \( \Xi^* \subseteq \tilde{\Xi} \) such that, for all \( v \in V, \sigma(v) \in \Xi^*(v) \Leftrightarrow \) there does not exist a player \( i \in N(v) \) with strategy system \( \sigma'_i(v) \neq \sigma_i(v) \) and \( \sigma'_i(v) = \sigma_{-i}(v) \) such that \( \min_{\sigma \in \Omega} u_i(\sigma) > \max_{\sigma \in \Omega} u_{i}(\sigma) \), where \( \tilde{\Omega}_v \) consists of all paths that are generated by \( \tilde{\sigma} \) – the \( \tilde{\sigma} \)-equilibrium under announcement \( \sigma \) for the game \( \Gamma_v \), and where \( N(v) \) is the set of players in subgame \( \Gamma_v \).

From the definition of \( \Xi^* \)-equilibrium, the previously mentioned differences between games with public recommendation of paths and games with public announcement of strategy system becomes evident. Not all stable plans of action are in the \( \Xi^* \)-equilibrium: players will announce only those plans of action that are both stable and rational. It is in this sense we say that players are playing an ‘active’ role in determining the equilibrium announcement of strategy system, and hence the equilibrium outcome. In addition, each stable announcement corresponds to a set of credibly announced paths, among which each path can
be actually followed. This is in contrast to the single recommended path as specified in games with public recommendation.

To find the solution, we first need to find the set of all stable announcements, and then we must eliminate irrational plans from the stable set. The following is an illuminating example:

**Example 5 (A stable announcement could be irrational).** Consider the game described in Fig. 2. Player 1’s announcing l (with ‘either l’ or r”) is not credible because l will not be followed by player 1 under any circumstances. Player 1 will reason the following: ‘once l is announced, if I take action r, player 2’s best response is to take action L no matter what he announced, which will lead to the maximum payoff ’3’ for me. So I will take action r’. In this game, player 1’s credible announcements are those associated with playing r at his first decision node. This game has two stable announcements: (a) player 1 announces ‘r’ at v* ‘either l’ or r” at v’, while player 2 announces ‘L’ at v; (b) player 1 announces ‘r’ at v* with ‘l’ at v’, while player 2 announces ‘R’. System (a) is rational because nobody can benefit for sure by announcing other plans. System (b) is irrational since player 1’s announcing any other plan, such as ‘r” or ‘either l’ or r” at v’, will result in a better final outcome (r, L) for sure. Therefore, this game has a unique equilibrium path (r, L), which refines the Ω**-equilibrium \{(r, L); (r, R, l')\}.

Here is the main result of this section:

**Theorem 6.** For any given \( \Gamma \), we have:

(a) For all \( \sigma \in \mathcal{E} \), the corresponding \( \hat{\sigma} \)-equilibrium \( \hat{\sigma} \) is stable, i.e., \( \hat{\sigma} \in \mathcal{E} \).

(b) There exists a stable announcement \( \sigma \in \mathcal{E} \) that is contained in the \( \sigma^* \)-equilibrium. In particular, for \( \sigma = \sigma^* \), we have \( \hat{\sigma} = \sigma^* \).

(c) Every \( \Omega^* \)-equilibrium path corresponds to a stable announcement in \( \mathcal{E} \).

(d) Each stable announcement \( \sigma \in \mathcal{E} \) refines the subgame perfect equilibrium; i.e., for all \( v \in V \), \( \Omega_\sigma(v) \subseteq \Omega_{0,v} \).

**Proof.** Statement (a) follows obviously from the definition of \( \hat{\sigma} \)-equilibrium. The first statement in (b) is a direct corollary to Theorem 5(ii) and statement (a), while the second statement in (b) follows by backward induction. Statement (c) can be proved by construction. For any given \( x \in \Omega^* \), define \( \sigma_x \) by setting \( \sigma_x(v) = x(v) \), if \( v \neq x \); otherwise, \( \sigma_x(v) = \sigma^*(v) \). It is straightforward to verify that \( \hat{\sigma}_x = \sigma_x \).

We prove statement (d) by induction with respect to the length \( l \) of a game. For all \( v \in \mathcal{Z} \) that corresponds to \( l = 1 \), we have \( \Omega_\sigma(v) = \Omega_{0,v} = \{v\} \). Suppose \( \Omega_\sigma(v) = \hat{\Omega}_v \subseteq \Omega_{0,v} \) holds for all subgames of \( \Gamma \) that have a length of no more than \( l \). Now, consider a subgame \( \Gamma_v \) that has a length of \( l + 1 \). For any given \( \omega \in \Omega_\sigma(v) \), we have \( \omega_v \in \Omega_\sigma(v) = \hat{\Omega}_v \subseteq \Omega_{0,v} \) for all \( v' \mapsto \omega_v \) by definition of \( \hat{\Omega} \) and \( \hat{\Omega}_v \). In particular, for \( e = \omega(v) \in \mathcal{E}(v) \), we have \( \omega_e \in \hat{\Omega}_v \subseteq \Omega_{0,v} \). Assuming that \( v \in P_v \).
without loss of generality, we have

\[ u_i(\omega) = \max_{e \in E(\omega)} \min_{\omega' \in \hat{\Omega}_e} u_i(\omega') \]

\[ \geq \max_{e \in E(\omega) \setminus \{\omega(\omega)\}} \min_{\omega' \in \hat{\Omega}_e} u_i(\omega') \]

\[ \geq \max_{e \in E(\omega) \setminus \{\omega(\omega)\}} \min_{\omega' \in \Omega_{e,v}} u_i(\omega'), \]

where the first equality is by the assumption that \( \sigma \) is stable, and by the definition of \( \hat{\sigma} \); the first inequality is obvious, and the second inequality is by the fact that \( \hat{\Omega}_e = \Omega^*_{e,v} \subseteq \Omega_{0,v} \) if \( e \notin E(\hat{\sigma}, v) \); otherwise, for \( e \in E(\hat{\sigma}, v) \), by assumption, \( \Omega_{\hat{\sigma}(e)} \equiv \hat{\Omega}_e \subseteq \Omega_{0,v} \). By Proposition 1, we conclude that \( \omega \in \Omega_{0,v} \). This is true for any arbitrary \( \omega \in \Omega_{0,v} \). Therefore, \( \Omega_{\sigma(v)} \subseteq \Omega_{0,v}, \forall v \in V \). □

By definition, \( \Xi^* \subseteq \hat{\Xi} \), therefore, we can conclude that \( \Xi^* \)-equilibrium is also a refinement to the subgame perfect equilibrium. That is, \( \Xi^* \)-equilibrium is non-empty, then for all \( \sigma \in \Xi^* \), and for all \( v \in V, \Omega_{\sigma(v)} \subseteq \Omega_{0,v} \).

In general, for any given game, it is not clear if the corresponding \( \Xi^* \)-equilibrium is not empty. However, we can show that

**Theorem 7.** For any given game \( G \), if a strategy system \( \sigma \in \Xi \) is such that, for all \( v \in V \), there exists no stable plan \( \sigma' \in \hat{\Xi}(v) \) such that \( \min_{\omega \in \Omega_{\sigma(v)}(\omega)} u_i(\omega) > \max_{\omega \in \Omega_{\sigma(v)}(\omega)} u_i(\omega) \) for some \( i \in N(v) \), then \( \sigma \) belongs to the \( \Xi^* \)-equilibrium, i.e., \( \sigma \in \Xi^* \).

**Proof.** Suppose, on the contrary, that \( \sigma \in \Xi \) satisfies the stated condition, but \( \sigma \notin \Xi^* \). The latter implies, by Definition 8, that there exists \( v \in V, i \in N(v) \), and \( \sigma'(v) = (\hat{\sigma}(v), \sigma_{-i}(v)) \neq \sigma(v) \) such that \( \min_{\omega \in \Omega_{\sigma(v)}(\omega)} u_i(\omega) > \max_{\omega \in \Omega_{\sigma(v)}(\omega)} u_i(\omega) \). By Theorem 6(a), we have \( \hat{\sigma}'(v) \in \hat{\Xi}(v) \). This is a contradiction to the assumption. □

The condition described in the theorem is however not a necessary condition for the non-emptiness of the \( \Xi^* \)-equilibrium. This can be illustrated by the game described in Fig. 1b: The unique \( \Xi^* \)-equilibrium \( \{l\} \) in this game violates obviously the condition. Nevertheless, by Theorem 6(d), we see that the condition holds when the game contains unique subgame perfect equilibrium path. Finally, from Example 4, we see that, when \( \Xi^* \)-equilibrium exists, it may or may not contain the \( \sigma^* \)-equilibrium.
6. Concluding remarks

In summary, this paper presents three refinements to the universally-used notion of subgame perfect equilibrium for extensive form games of perfect information. These solutions are based on the decision-theoretical foundation of choices under Knightian uncertainty. In this paper, we assume that players do not use random devices to make choices. We do, however, notice that in the presence of Knightian uncertainty, an uncertainty averse decision maker has an incentive to randomize his choices (see Lo, 1996). Generalization to allow such randomness does not involve extra difficulties. Our assuming pure act space is to simplify the model for a better understanding of the proposed solution concepts. Conceptually, generalization to continuum act space and unbounded games can be also done in a similar fashion. However, the existence of solutions for these classes of games needs particular attention from the technical point of view, as is the case even for single player’s decision problem. See, for example, Martin (1975) for studying the existence of solutions for two-person, zero-sum games of perfect information. I leave this existence problem to be studied in a technical note. Similar solution concepts can be established by assuming general class of preferences such as the capacity-based (non-additive) expected utility of Gilboa and Schmeidler (1987). See Epstein (1995) for discussion of general preferences and solution concepts in normal form games. As mentioned in the introduction, $\sigma^*$-equilibrium as a unique ‘stable’ belief system is proved by Luo and Ma (1998). Finally, generalization of these solution concepts to games with imperfect information is still in progress.

For further reading

Nash (1951); Machina and Schmeidler, 1992; Selten, 1975.

Acknowledgements

This paper was initiated during a conversation with Luo Xiao. I have benefited from discussions with Larry Epstein, Curtis Eberwein, Mike Peters, Tan Wang and Licun Xue. I would particularly like to thank Luo Xiao for many motivating conversations. This paper was presented in Economic Theory Workshop at CIRANO, Montreal (Nov. 1994), International Conference in Economic Theory in Greece (May 1995), International Conference in Game Theory at

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18 Thanks to the referee for bringing my attention to this problem, and the work of Martin (1975).
Stony-Brook (July 1995), Advancement in Theory of Social Situations at McGill University (July 1995) and Hong Kong University of Science and Technology (October 1996). Financial supports from SSHRC and FCAR are gratefully acknowledged.

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