An analytic Riccati solution for two-target discrete-time control

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Abstract

This paper analytically solves the Riccati equation of discrete optimal control with two targets and one tool. This is accomplished by reducing the problem to a nonlinear univariate dynamic equation; this can be solved by a suitable transformation of variable. Beyond its direct application to two-target economic problems, the present approach may provide insight toward the eventual solution of the general Riccati equation. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

The discrete-time linear-quadratic control problem has been well studied both within and outside the economics profession. With a finite horizon it leads to a nonlinear matrix Riccati difference equation which must be iterated back in time from the horizon in order to obtain the optimal policy in any period; in the infinite horizon case, either the iterations must be continued until the Riccati matrix converges or a faster method must be used to find the steady-state Riccati

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matrix. As Amman and Neudecker (1997) point out, no general analytic solution has yet appeared in the literature for either the dynamic Riccati equation or the algebraic (steady-state) Riccati equation.

This paper presents an analytic solution for the dynamic and steady state Riccati equations for the case of two target variables and a single control variable with no cost on the control. This is accomplished by reducing the problem to a solvable nonlinear scalar dynamic equation. Aside from the contribution of Vaughan (1970), this is the first instance in which any nontrivial case of the Riccati problem has been solved. Vaughan considered the special case of a nonsingular transition matrix (a condition which, as Amman and Neudecker (1997) point out, often does not hold), and obtained a solution in terms of the eigenvalues of a matrix, which except for small problems must be found numerically. The present approach has two advantages over that of Vaughan: it is not limited to the case of a nonsingular transition matrix, and it renders the nature of the dynamics more transparent.

Having an analytic solution for the case of two target variables expands the range of applied economic problems whose dynamic optima can be theoretically analyzed or empirically tested. As Pitchford (1977) points out, two-state-variable problems in specific theoretical or empirical contexts can be more interesting than one-variable problems, which are popular due to their tractability, because two-variable problems allow for a richer set of interactions. Furthermore, the solution of the two-target case may lead to progress toward a solution for a wider class of problems or even for the general case.

2. Reduction of the two-target Riccati equation to a scalar equation

Consider the control problem of minimizing \( \sum_{t=1}^{T}(y_t^i K y_t) \) subject to \( y_t = A y_{t-1} + C u_t \), where \( y \) is a 2 \( \times \) 1 vector of target variables, \( K \) is a 2 \( \times \) 2 symmetric positive definite preference matrix, \( A \) is a 2 \( \times \) 2 transition matrix, \( C \) is a 2 \( \times \) 1 matrix of control multipliers, and \( u \) is a scalar control variable. It is well known (Chow, 1975) that the optimal policy is given by \( u_t^* = -(C' H_t C)^{-1} (C' H_t A) y_{t-1} \), with the 2 \( \times \) 2 Riccati matrix \( H_t \) evolving backward through time from time \( T \) according to this nonlinear matrix Riccati equation:

\[
H_{t-1} = K + A' H_t A - A' H_t C (C' H_t C)^{-1} C' H_t A, \quad H_T = K. \tag{1}
\]

To solve Eq. (1) in this case of two targets and one tool, postulate\(^3\)

\[
H_t = K + Q \ast w_t \tag{2}
\]

\(^3\) See Amman (1995) and Amman and Neudecker (1997).

\(^3\) Notationally the star (\( \ast \)) product of a scalar with a matrix denotes multiplying each element of the matrix by the scalar.
for some $2 \times 2$ matrix $Q$ and scalar variable $w_t$, with $w_T = 0$. Equating expressions for $H_{i-1} - K$ from Eq. (1) and (2), we have

$$Q \ast w_{i-1} = A'[H_i - H_iC(C'H_iC)^{-1}C'H_i]A. \tag{3}$$

Now the key is to note that in this case with $A$ being $2 \times 2$ and $C$ being $2 \times 1$, for any symmetric positive definite matrix $H_i$ we have the following identity:

$$A'[H_i - H_iC(C'H_iC)^{-1}C'H_i]A = A'[\text{adj}(CC')]A \ast [\det(H_i)/(C'H_iC)]. \tag{4}$$

Here $(C'H_iC)$ is a scalar, $|H_i|$ is the determinant of $H_i$, and $\text{adj}(CC')$ is the adjoint of $CC'$. Denoting $C' = [C_1 \ C_2]$, $\text{adj}(CC')$ has first row $[C_2^T - C_1C_2]$ and second row $[-C_1C_2 C_2^T]$. Eq. (4) can be confirmed by writing out all matrices in terms of their individual elements. The key features of Eq. (4) are that (i) the left-hand side is identical to the right-hand side of Eq. (3), and (ii) in the right-hand side of Eq. (4), it has proven possible to factor out all the time-varying parameters into a multiplicative scalar, so the matrix on the right-hand side of Eq. (4) is time invariant. Setting

$$Q = A'[\text{adj}(CC')]A \tag{5}$$

(where the elements of $Q$ are given individually in the appendix), Eq. (3) becomes

$$Q \ast w_{i-1} = Q \ast [\det(H_i)/(C'H_iC)] \tag{6}$$

and thus

$$w_{i-1} = \det(H_i)/(C'H_iC) = |K + Q \ast w_i|/(|C'KC| + (C'QC)w_i]. \tag{7}$$

The numerator of the right-hand side of (7) can be expanded element by element (recognizing that $K$ and $Q$ are both symmetric) to show that it equals $|K| + pw_t$, where

$$p = K_1Q_3 + K_3Q_1 - 2K_2Q_2. \tag{8}$$

Here the matrix $Q$ has first and second rows $[Q_1 \ Q_2]$ and $[Q_2 \ Q_3]$, and $K$ has first and second rows $[K_1 \ K_2]$ and $[K_2 \ K_3]$. Hence Eq. (7) becomes

$$w_{i-1} = |K| + pw_i]/(|C'KC| + (C'QC)w_i). \tag{9}$$

Thus the Riccati Eq. (1) has been simplified to Eqs. (2) and (9) with $w_T = 0$, where Eq. (9) contains nonlinear scalar dynamics. An explicit closed form solution of Eq. (9) will be derived, and stability analyzed, in the next section; to anticipate, Eq. (9) will turn out to be stable backwards through time unless $(C'QC) = 0$ and $|p/(C'KC)| \geq 1$. But let us focus here on the steady state. First, note that all four constants in Eq. (9) are non-negative. $|K|$ and $(C'KC)$ are
positive because \( K \) is positive definite. The appendix proves that \( p \geq 0 \). And \((C'QC) \geq 0\) because Eq. (5) can be used to show that

\[ C'QC = [A_2C_2^2 - A_3C_1^2 + (A_1 - A_4)C_1C_2]^2, \]

(10)

where the first and second rows of \( A \) are \([A_1, A_2]\) and \([A_3, A_4]\).

Since \( w_T = 0 \) we have from Eq. (9) that \( w_{T-1} > 0 \), and by induction all \( w_i > 0 \) for \( t < T \). So the relevant steady-state value of \( w_t \), denoted \( w_{\infty} \), is positive. In the case in which \((C'QC) \neq 0\), in Eq. (9) we set \( w_t = w_{t-1} = w_{\infty} \), cross-multiply, and obtain \( w_{\infty} \) as the positive (and real) value implied by the quadratic formula:

\[ w_{\infty} = \left\{ [p - (C'KC)] + \left[ [p - (C'KC)]^2 + 4(C'QC)|K|\right]^{1/2}\right\}/2(C'QC). \]

Thus the steady-state Riccati matrix follows from using Eq. (11) in Eq. (2) as \( t \to -\infty \):

\[ H_{\infty} = K + Q * w_{\infty}. \]

(12)

3. Dynamic solution

In this section we derive a closed-form solution for the scalar dynamic Eq. (9), in the nondegenerate case where \((C'QC) \neq 0\) so Eq. (9) is nonlinear and in the simpler linear case occurring when \((C'QC) = 0\). For the former case we nonlinearly transform the variable \( w_t \) to obtain a variable \( x_t \) which evolves linearly, as follows:

\[ x_t = 1/(c + w_t), \]

(13)

implying

\[ w_t = (1 - cx_t)/x_t. \]

(14)

To find the value of \( c \) which will result in linear dynamics for \( x_t \), use Eq. (14) in Eq. (9) to solve for \( x_{t-1} \) in terms of \( x_t \):

\[ x_{t-1} = \left\{ (C'QC) + [(C'KC) - c(C'QC)]x_t\right\}/\left\{ [c(C'QC) + p] + [c(C'KC) - c^2(C'QC) + |K| - pc]x_t\right\}. \]

(15)

Then \( x_t \) has linear dynamics provided the coefficient of \( x_t \) in the denominator of Eq. (15) is zero, implying (arbitrarily choosing the positive square root from the quadratic formula):

\[ c = \left\{ (C'KC) - p + r\right\}/2(C'QC), \]

(16)

where \( r \equiv [(C'KC) - p)^2 + 4|K|(C'QC)]^{1/2} > 0 \). (Note that \( r \) is real.)
Thus Eq. (15) simplifies to the linear dynamics

\[ x_{t-1} = \left( (C'QC) + [(C'KC) - c(C'QC)]x_t \right) / \left[ c(C'QC) + p \right]. \tag{17} \]

Using Eq. (16), the coefficient \([(C'KC) - c(C'QC)] / [c(C'QC) + p]\) of \(x_t\) in Eq. (17), which is the characteristic root \(\lambda\) of the linear process, simplifies to

\[ \lambda = \left( (C'KC) + p - r \right) / [(C'KC) + p + r]. \tag{18} \]

Given that we know \((C'KC) > 0\) and \(p \geq 0\), it is easily seen that \(-1 < \lambda < 1\), so \(x_t\) (and hence \(w_t\) and \(H_t\)) evolve stably backwards through time.

The solution of Eq. (17) for \(x_t\) is

\[ x_t = x_{-\infty} + \lambda^{T-t}(x_T - x_{-\infty}), \tag{19} \]

where (by \(w_T = 0\) and by Eq. (13)) \(x_T = 1/c\), and where \(x_{-\infty}\) is found by setting \(x_t = x_{t-1} = x_{-\infty}\) in Eq. (17) and using Eq. (16):

\[ x_{-\infty} = (C'QC)/r. \tag{20} \]

Notice that if \(x_{-\infty}\) from Eq. (20) is used in Eq. (14) the resulting expression for \(w_{-\infty}\) agrees with Eq. (11).

Thus the solution for \(w_t\) is given by using Eq. (19) in Eq. (14); in turn the solution for the Riccati matrix \(H_t\) is given by putting the solution for \(w_t\) into Eq. (2).

Here is a numerical example by which one may confirm that the solution agrees with the Riccati equation. Let

\[
C = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.
\]

The Riccati equation gives

\[
H_{T-1} = (1/53) \begin{bmatrix} 115 & 80 \\ 80 & 134 \end{bmatrix} \quad \text{and} \quad H_{T-2} = (1/541) \begin{bmatrix} 1235 & 1000 \\ 1000 & 1918 \end{bmatrix}.
\]

The analytic solution replicates these, using

\[
Q = \begin{bmatrix} 9 & 27 \\ 27 & 81 \end{bmatrix},
\]

\(|K| = 1, \quad C'KC = 53, \quad C'QC = 2601, \quad \text{and} \quad p = 117, \quad \text{leading to} \quad w_{T-1} = 1/53 \quad \text{and} \quad w_{T-2} = 17/541, \quad \text{with} \quad x_{-\infty} = 21.600 \quad \text{and} \quad \lambda = 0.171. \quad \text{The steady-state Riccati matrix is found using} \quad w_{-\infty} = 0.035451:\n\]

\[
H_{-\infty} = \begin{bmatrix} 2.319 & 1.957 \\ 1.957 & 3.872 \end{bmatrix};
\]

and this does satisfy the steady-state Riccati equation.
We now turn to the degenerate case in which \((C'QC) = 0\), where the dynamics for \(w_t\) in Eq. (9) become linear. It can be shown that this case occurs if and only if the condition for dynamic controllability (Turnovsky, 1974) is violated, in which case the system may or may not be stabilizable. The solution when \(p \neq (C'KC)\), so root of system (9) is not unity, is as follows (noting that \(w_T = 0\)):

\[
  w_t = \left[\frac{[K]/(C'KC - p)}{1 - (p/C'KC)^T^{-1}}\right].
\]

Then \(H_t\) is found by using Eq. (21) in Eq. (2).\(^4\) The steady-state Riccati matrix \(H_{-\infty}\), when we have the stable case of \(C'KC > p\), is found by letting \(t\) go to \(-\infty\) in Eq. (21), so

\[
  w_{-\infty} = \frac{[K]/(C'KC - p)}{1 - (p/C'KC)^T^{-1}}.
\]

and by using Eq. (22) in Eq. (2).

4. Conclusion and prospects for further analytical advances

This paper presented the analytic solution of the Riccati equation of discrete optimal control with two targets and one tool. When the underlying scalar dynamics are nonlinear, the solution is given by using Eqs. (19) and (14) in Eq. (2), supplemented by the expressions in Eqs. (5), (8), (16), (18) and (20), \(x_T = 1/c\), and the expression for \(r\) after Eq. (16). Then the steady-state Riccati matrix is given by using Eq. (11) in Eq. (2). When the underlying scalar dynamics are linear (because \(C'QC = 0\)), the solution is given by using Eq. (21) in Eq. (2); if the system is stabilizable, the steady-state Riccati matrix is given by using Eq. (22) in Eq. (2).

It is natural to ask whether the analytic solution in this paper can be parlayed into a solution for a broader class of problems – ideally a solution for all possible numbers of targets and tools, or perhaps for arbitrary numbers of targets but just one tool. The key insight which allowed the present analysis to succeed was that in the case of two targets and one tool the expression on the right side of Eq. (3) allows us to factor out all of its time-varying terms into a scalar factor, as shown in Eq. (4), thereby rendering the dynamics scalar. Unfortunately, this convenient property of the right-hand side of Eq. (3) does not hold in general – for instance, it does not hold for three targets and one tool. Moreover, it can be shown that for some problem sizes the relation in Eq. (2) cannot hold for any \(Q\): if it did we would have \(H_{T-1} - K = Q * w_{T-1}\) and \(H_{T-2} - K = Q * w_{T-2}\), so

\[^4\text{Equivalently, } H_t \text{ can be found by solving the implied linear matrix dynamics } H_{T-1} = [K'\{1 - (p/(C'KC))\} + Q'[K'//(C'KC)] + [p/(C'KC)]*H_t, \text{ which follows from Eqs. (2) and (9) with } (C'QC) = 0.\]
\(Q = (1/w_{T-1}) \times (H_{T-1} - K)\) and hence \((H_{T-2} - K) = (w_{T-2}/w_{T-1}) \times (H_{T-1} - K)\) implying that \((H_{T-2} - K)\) is a scalar multiple of \((H_{T-1} - K)\). But this fortuitous feature of the present problem size can be shown numerically not to hold when, for example, there are three targets and one tool.

Therefore, further insights will be needed in order to obtain results for larger problem size. However, it may be possible to build those insights around an analysis of the expression on the right-hand side of (3), or more generally around the notion of the possible presence of scalar dynamics in some way underlying the matrix dynamics.

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Appendix A. Proof that \(p \geq 0\)

By Eq. (8), \(p = K_1Q_3 + K_3Q_1 - 2K_2Q_2\). By Eq. (5), \(Q\) can be written

\[
Q = \begin{bmatrix}
(m + d)^2 & me + ms + de + ds \\
me + ms + de + ds & (e + s)^2
\end{bmatrix},
\]

where \(m = A_1C_2, d = -A_3C_4, e = A_2C_4\), and \(s = -A_4C_1\). Note that \(Q\) is positive semi-definite since for any scalars \(x_1\) and \(x_2\), \([x_1 x_2]Q[x_1 x_2]' = (x_1m + x_2e + x_1d + x_2s)^2 \geq 0\). Since \(K\) is positive definite by assumption, \(K_1 > 0\) and \(K_3 > 0\).

If \(Q_2 \geq 0\) and \(K_2 \leq 0\), or if \(Q_2 \leq 0\) and \(K_2 \geq 0\), it follows directly that \(p \geq 0\).

If \(Q_2 > 0\) and \(K_2 > 0\): Positive definiteness of \(K\) implies \(K_1K_3 > K_2^2\) so \(K_1^{1/2}K_3^{1/2} > K_2\). Also, expansion of \([K_1^{1/2} - K_1^{1/2}]Q[K_3^{1/2} - K_3^{1/2}]' \geq 0\) (using the fact that \(Q\) is positive semi-definite) and substituting in \(p\) gives \(p \geq 2Q_2(K_1^{1/2}K_3^{1/2} - K_2)\) where the right-hand side is strictly positive; so \(p > 0\).

If \(Q_2 < 0\) and \(K_2 < 0\): positive definiteness of \(K\) implies \(K_1K_3 > K_2^2\) so \(K_1^{1/2}K_3^{1/2} > -K_2\). Expansion of \([K_1^{1/2} K_1^{1/2}]Q[K_3^{1/2} K_3^{1/2}]' \geq 0\) (again using the fact that \(Q\) is positive semi-definite) and substituting in \(p\) gives \(p \geq -2Q_2(K_1^{1/2}K_3^{1/2} + K_2)\) where the right-hand side is strictly positive; so \(p > 0\). \(\square\)

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