Guest editorial

Complex Nonlinear Dynamics and Computational Methods

W.D. Dechert\textsuperscript{a,*}, Cars H. Hommes\textsuperscript{b}

\textsuperscript{a}Department of Economics, University of Houston, Houston, TX, USA
\textsuperscript{b}University of Amsterdam, Amsterdam, Netherlands

In the summer of 1997, Berc Rustem asked us whether we would be interested in editing a special issue of the Journal of Economic Dynamics & Control on Complex Nonlinear Dynamics and Computational Methods. We were enthusiastic about the idea and soon started soliciting papers on these themes. This issue contains 19 contributions by some experts in the field and should provide the reader with an impression of the current ‘state of the art’. In order to put these contributions somewhat into historical perspective, let us briefly review some recent developments in the closely related themes, Complex Nonlinear Dynamics and Computational Methods.

Complex Nonlinear Dynamics is a topic that has played an important role in the controversy about the main sources of economic fluctuations in the last two decades or so. (See e.g. the books by Day, 1994; Dechert, 1996; Goodwin, 1990; Hommes, 1991; Lorenz, 1993; Medio, 1992; Puu, 1997; Rosser, 1991 and de Vilder, 1995 for more discussions concerning this controversy.) Is the main source of economic fluctuations to be found in exogenous random shocks to an inherently stable (linear) economic system, or can nonlinear economic ‘laws of motion’ explain a significant part of observed fluctuations in output, growth rates, capital stocks, inventories, stock prices, exchange rates and other economic or financial variables? This controversy dates back to the 1930s and the 1940s, when Frisch, Slutsky and Tinbergen, a.o., proposed their propagation mechanism subject to exterior random impulses, as a way of modeling business cycles. In contrast, e.g., Kaldor, Goodwin and Hicks argued that the...
multiplier–accelerator mechanism was the main source of business fluctuations, and that this endogenous mechanism could generate business cycles even in the absence of any external shocks. These early endogenous business cycle models were heavily criticized however, for at least three reasons:

C1 The cycles generated by the nonlinear models were much too regular compared to real business cycle data.

C2 The early business cycle models were viewed to be ‘ad hoc’, since they were not derived from economic optimization principles.

C3 In the early business cycle models agents behave ‘irrational’, because along the cycles their expectations are ‘systematically wrong’.

In the 1960s and 1970s, this critique culminated in the Rational Expectations Hypothesis (Muth, Lucas, Sargent, a.o.) and the development of the Real Business Cycle models (Kydland, Prescott, a.o.), where random productivity shocks are viewed as the main exogenous source driving business cycles.

The discovery of the phenomenon of deterministic chaos in mathematics, physics, meteorology and other applied sciences in the 1970s however, sheds important new light on the business cycle debate. The fact that simple nonlinear deterministic models can generate seemingly random, chaotic time paths implies that it should be possible, at least in principle, to explain a significant part of observed regularities and irregularities in business cycles by simple nonlinear deterministic economic laws, perhaps buffeted with small shocks. A search for chaotic business cycle models started. The earliest examples, although of scientific interest, were generally viewed to be ‘ad hoc’ since they were not derived from optimization principles. In the early 1980s however, e.g. Benhabib and Day (1982), Grandmont (1985) and Boldrin and Montrucchio (1986) derived chaotic business cycle models from utility and profit maximization principles within the general equilibrium paradigm of perfectly competitive markets and rational expectations; see e.g. the special issue of the Journal of Economic Theory (Grandmont, 1986) for a collection of articles along these lines. These models thus counteracted the early critique C1–C3 on endogenous business cycle models.

Simultaneously, a search for chaos in economic and financial data was initiated by, a.o., Brock (1986), Brock and Sayers (1988) and Scheinkman and LeBaron (1989); see also the review by Barnett and Serletis in this issue. This search for chaos in economic data turned out to be difficult however. Summarized in one sentence, the conclusion seems to be that the evidence for purely deterministic chaos in economic and financial time series is low, whereas the evidence for nonlinearity is strong but it is not clear where exactly this nonlinearity is coming from. In obtaining this conclusion, the development of the BDS-test, a statistical test for IID based upon methods from chaos theory such as the correlation dimension, developed by Brock et al. (1986) played a stimulating role. The BDS-test can be used as a specification test, and nowadays it is one of the most frequently used general tests for nonlinearity. (It is remarkable that
the original working paper by Brock et al., 1986 was never published; an extended and revised version was published one decade later in Brock et al., 1996.) It is important to note that the lack of success for finding deterministic chaos in economic and financial data may be due to a fundamental problem already noted by Brock and Sayers (1988, p. 72): ‘measurement error and aggregation error may be responsible for the rejection of the null hypothesis of deterministic chaos’. Although the methods for distinguishing between random and chaotic systems have been improved over the past decade, this early statement may still be true. In fact, the evidence that distinguishing between deterministic chaos and randomness based upon time series observations may be principally impossible, due to exponential noise amplification in chaotic systems, is accumulating; see e.g. Takens (1996) for some recent advances on this point. Noisy chaos may thus be a null hypothesis which cannot be rejected by economic and financial data.

In the early 1990s however, the interest in complex nonlinear dynamics in economics seemed to have faded somewhat. There seem to be at least two important reasons for this decline in interest:

1. For more than a decade there had been a focus on extremely simple, mainly one-dimensional models.
2. There has been hardly any empirical validation, such as calibration or estimation of the theoretical nonlinear (chaotic) business cycle models.

These two reasons are in fact connected to each other. Time-series properties of a noisy chaotic one-dimensional model are quite different from what is observed in real data. For example, a simple delay plot of $x_t$ versus $x_{t-1}$ for a time series from a noisy chaotic one-dimensional model will be quite different from the corresponding delay plot of an economic or financial series. In the last few years there seems to be a revival of applications of nonlinear dynamics to economics and finance however. There are a number of fundamental reasons for this revival.

It seems that the profession is getting ready to move towards two- and higher-dimensional nonlinear business cycle models, and several examples have recently been developed and explored. Recent examples include, a.o., exchange rate models by De Grauwe et al. (1993), higher-dimensional versions of the overlapping generations model by Medio and Negroni (1996) and de Vilder (1996) and nonlinear evolutionary systems by Brock and Hommes (1997a,b,1998). These higher dimensional nonlinear economic models exhibit rich bifurcation routes to chaos and strange attractors. The complicated dynamics in these high-dimensional economic models are caused by the occurrence of homoclinic orbits, a notion introduced already by Poincaré more than a century ago. This fundamental concept is intimately related to the nonlinear economic mechanisms causing endogenous fluctuations. Higher-dimensional models also have the potential to generate complicated time series with statistical properties
similar to those observed in real data. For example, Brock and LeBaron (1996), Brock and Hommes (1997b) and a very nice recent paper by de Fontnouvelle (1999) may be seen as an attempt to build nonlinear economic models generating empirical ‘stylized facts’.

A second reason for the revival of interest in nonlinear economic dynamics is the bounded rationality approach to economics; see e.g. Sargent (1993,1998) for stimulating surveys. Adaptive learning processes constitute highly nonlinear systems and may thus be a natural source for local instability and complicated dynamics, see e.g. Grandmont and Laroque (1991), Bullard (1994) and Schonhofer (1996). In particular, the fact that it is hard to distinguish between a random time series and a (noisy) chaotic time series sheds important new light on the expectations hypothesis. This point is discussed extensively in Grandmont’s Presidential address at the world meeting of the Econometric Society, at Barcelona 1990, published in Grandmont (1998), where he argues that “one might envision situations in which agents think that they are living in a world that is relatively simple, although subjected to random (e.g. white noise) shocks, but in which deterministic ‘learning equilibria’ are complex (‘chaotic’) enough to make the agents’ forecasting mistakes still ‘self-fulfilling’ in a well-defined sense”, Grandmont (1998, p. 744). The chaotic consistent expectations equilibria in Hommes and Sorger (1998), which are self-fulfilling in a linear statistical sense, that is expectations are correct in sample average and sample autocorrelations, may be seen as a simple example of Grandmont’s self-fulfilling mistake. Chaotic fluctuations under bounded rationality and learning may also be seen as what Sargent calls an ‘imperfect rational expectations equilibrium’ in his new book Sargent (1998, p. 68).

A third reason for the revival of interest in nonlinear systems is the computationally oriented literature on ‘complexity in economics and finance’, where markets are seen as evolutionary systems between competing trading or expectations strategies. This view is e.g. advocated by the Santa Fe Institute, and papers along these lines may be found in the Santa Fe volumes Anderson et al. (1988) and Arthur et al. (1997). Heterogeneity, social interactions and evolutionary change play an important role in this ‘complexity approach’ and are indeed very natural nonlinear effects which may play a key role in explaining fluctuations in real markets. The invited contributions to this special issue by Brock and by LeBaron discuss the complexity and computational approach to economics and finance extensively.

Until now, we have mainly discussed Complex Nonlinear Dynamics, without referring explicitly to Computational Methods. These two fields are closely related however. Most of the contributions in this issue deal with complex nonlinear dynamics, and almost all of them are using computational methods extensively, for analyzing the dynamical behavior. Computational methods are playing an increasingly important role in economic analysis in general, and in economic dynamics in particular. Especially in two or higher dimensional
systems numerical methods are getting increasingly important, because analytic results are harder to obtain. We are entering the 21st century, where sophisticated computational analysis will be an increasingly important complement, or as some believe perhaps even an almost complete substitute, for ‘pencil and paper analytics’.

Let us now briefly discuss the contents of this special issue. It is divided into six sections: Section I: Invited contributions; Section II: Heterogeneous agents; Section III: Local and global bifurcations in 2-D systems; Section IV: Macro dynamics; Section V: Adaptive learning; and Section VI: Computational methods.

The special issue starts with an invited contributions section with contributions by Brock, by LeBaron and by Barnett and Serletis. (A fourth invited contribution by Judd, Kubler and Schmedders is contained in the final section on Computational Methods.) The issue starts with a future perspective on “complex systems approaches” and nonlinear methodology in Economic Science, by one of its pioneers Buz Brock. (See Dechert, 1999 for a very nice survey of Brock’s work.) Brock discusses the debate in economics on the costs and benefits of rational expectations modeling versus boundedly rational/evolutionary modeling. An important argument is to use computer assisted methods to allow both approaches to compete in either a ‘nested’ testing setting or, perhaps, in a ‘Bayesian’ manner to give a decomposition of the variability and patterns seen in the data. It is argued that recent advances in computational techniques as well as lower costs of access to more extensive data, will propel the further development of nonlinearist and complexity-based types of research, simply because the cost is dropping. Brock also sketches the ‘Resilience Network’ view, which is a general ‘systems theoretic’ framework, with a temporal hierarchy of time scales and lots of nonlinearity, heterogeneity and adaptability added.

The second invited paper, by Blake LeBaron, reviews the new computational approach in finance concerning computer simulated markets with individual adaptive agents. LeBaron argues: “Modeling economic markets from the bottom up with large numbers of interacting agents is beginning to show promise as a research methodology that will greatly impact how we think about interaction in economic models”. An interacting agents approach to financial markets relies heavily on computational tools to push the analysis beyond the restriction of analytical methods. LeBaron surveys some of the earliest contributions in this area, and his survey may be seen as a tutorial to those interested in getting started in this rapidly growing field.

The third invited paper by Barnett and Serletis is a review of the empirical literature on the efficient market hypothesis and testing for chaos and nonlinearity in economic and financial data. These issues are closely related, since if asset prices would be chaotic the implication might be that profitable, nonlinearity based trading rules exist, at least in the short run and provided the
data generating mechanism is known. The review emphasizes how hard it is to look for chaos, and controversies that have arisen about the tests for chaos and nonlinearity and their power are discussed. The controversy may be hard to settle, because of the fact that (dynamic) noise makes it difficult, if not impossible, to distinguish between noisy (low-dimensional) chaos and pure randomness.

Section II on heterogeneous agents contains three theoretical and one empirical contribution. The papers by Brock and de Fontnouvelle, Goeree and Hommes and Gaunersdorfer investigate evolutionary markets consisting of interacting heterogeneous agents both from a theoretical as well as from a computational perspective. The evolutionary models analyzed here may be seen as stylized versions of the computationally oriented artificial markets literature, as e.g. done in the Santa Fe institute. Heterogeneity of agents in an evolutionary framework introduces a very natural (multiplicative) nonlinearity in economic modeling. Brock and de Fontnouvelle investigate an overlapping generations monetary economy with heterogeneous agents and evolutionary updating of prediction strategies. In particular, they introduce a so-called large type limit (LTL), that is, a deterministic approximation when the number of different trading types goes to infinity. Goeree and Hommes investigate the cobweb model with costly rational expectations versus freely available naive expectations, for general nonlinear (monotonic) demand and supply. Gaunersdorfer investigates the discounted value asset pricing model, and shows that strange attractors arise in this 3-D dynamic model. Her paper is a nice example how to use a mixture of theoretical and computational tools in the analysis of a 3-D dynamic model. In all these adaptive heterogeneous agent equilibrium models the evolutionary dynamics exhibits a rational route to randomness, when the intensity of choice tends to infinity. Stated differently, a bifurcation route to strange attractors occurs when the traders’ sensitivity to differences in fitness becomes large. In the fourth contribution to this section, Chavas investigates whether there is empirical evidence for heterogeneity in expectations among producers in the competitive beef market. The beef market exhibits cyclical patterns and significant biological lags in the production process. US beef market data are used and beef price equations derived from optimization principles are estimated under different expectations regimes. An important feature of the testing methodology is that rational expectations is nested as a special case, as suggested in Brock (1997). The empirical results indicate heterogeneous expectations in the beef market, with a significant fraction of beef producers having ‘naive’ expectations (47%) and ‘quasi-rational’ expectations (35%), and only a minority having rational expectations (18%).

The contributions by Bischi, Gardini and Kopel and Tuinstra in Section III focus on new methodological issues, in particular local and global bifurcations in two-dimensional systems. Bischi, Gardini and Kopel use the concept of critical
curves in the analysis of the global dynamical behaviour for two-dimensional noninvertible mappings, and, as an example use this concept in the analysis of a market share attraction model from marketing theory. A critical curve is in fact a two-dimensional generalization of the notion of critical value (i.e. local maximum or minimum value) of a one-dimensional map. Critical curves play an important role in the qualitative changes of attractors and their basins of attraction, as a parameter varies. These theoretical issues are important, because in general nonlinear dynamic economic models will be noninvertible systems. Tuinstra investigates non-linear, symmetric tâtonnement processes on the simplex. For the multiplicative discrete time price adjustment process, the simplex is a natural normalization rule, because it is invariant under the dynamics. The theory of bifurcations for symmetric systems is used to investigate bifurcation routes to chaos and coexisting attractors for a symmetric price adjustment process. Symmetric systems may be seen as an ‘organizing center’ for general asymmetric systems. Indeed Tuinstra shows that nearby asymmetric price adjustment processes exhibit similar bifurcation routes to complexity.

Section IV on Macro Dynamics contains three contributions by Yokoo, Chiarella and Flaschel and Böhm and Kaas on quite different types of dynamic macro economic models. Yokoo investigates a Diamond-type overlapping generations model, extended to allow for government intervention. The dynamics is described by a second order, or equivalently a two-dimensional difference equation, in the capital–labor ratio. Using a singular perturbation method, conditions for homoclinic orbits associated to the golden rule saddle point steady state are identified. For a parametric example with a CES-production function, the occurrence of strange attractors associated with homoclinic bifurcations is demonstrated. An example with three coexisting attractors (the golden rule steady state, a stable 8-cycle and a strange attractor) is presented, illustrated by a nice color picture of the three basins of attraction and their fractal basin boundaries. Chiarella and Flaschel consider an open monetary growth model with sluggish price and quantities, where Rose’s employment cycle, Metzler’s inventory cycle, Tobin’s internal nominal dynamics and Dornbusch’s external nominal dynamics are interacting. The complete model consists of an eight-dimensional autonomous system of differential equations. This is one of the first examples (see also e.g. Goodwin, 1990) in nonlinear economic dynamics of a higher D system of differential equations. Chiarella and Flaschel present numerical evidence of period doubling bifurcation routes to chaos in their macro-model. Böhm and Kaas analyze the simple neoclassical one-sector growth model with differential savings in the sense of Kaldor–Pasinetti. It is well known that, under standard neoclassical assumptions, the one sector neoclassical growth model can only generate time path converging to a stable steady state. Hence, additional economic features, such as nonconvexities in production or an extension to two sectors, are necessary in order to generate endogenous fluctuations. Böhm and Kaas show that in the one-sector growth model, if
income distribution varies sufficiently and if shareholders save more than workers, the economy exhibits an unstable steady state and periodic and chaotic fluctuations can arise. Nice color pictures of the so-called ‘cycle cartograms’ illustrate how the long run dynamical behavior depends upon two of the parameters. Each point in the two-parameter plane is represented by the color corresponding to the period of the long run dynamical behavior.

Section V on adaptive learning contains three papers by Arifovich and Gençay, Heinemann and by Barucci. Arifovich and Gençay investigate genetic algorithm learning in an exchange rate model with an overlapping generations structure with two currencies and a free-trade, flexible exchange rate system. Agents update their savings and portfolio decision using the genetic algorithm. In the case of homogeneous rational expectations the model has a constant exchange rate, but the exchange rate equilibrium between the two currencies is indeterminate. Under genetic algorithm learning the exchange rate dynamics becomes chaotic however, due to profit seeking portfolio selection. Time-series properties of the chaotic exchange rates are investigated, and the exchange rate fluctuations exhibit the well known GARCH effects. The papers by Heinemann and Barucci consider adaptive learning in representative agent models. Heinemann investigates adaptive learning of rational expectations equilibria by the use of neural networks, focusing on the cobweb model. Necessary conditions for convergence of the neural network learning process to a, possibly approximate, rational expectations equilibrium are derived. Heinemann points out that approximate rational expectations equilibria are in general only locally optimal, because they are local minimizers of the mean squared prediction error. Approximate rational expectations equilibria learned with the help of a misspecified neural network may differ substantially from the correct rational expectations equilibria. Barucci investigates learning with exponentially fading memory with a learning step not vanishing in the limit. The dynamics under bounded rationality can be parameterized with respect to the memory of the learning process. The stabilizing and destabilizing role of memory is investigated in the pure exchange overlapping generations model and also in models with predetermined variables. For the OLG-model, more memory is stabilizing in a local sense, but in a global sense more memory does not necessarily simplify the structure of attractors. In the model with predetermined variables, the role of memory is ambiguous.

The final Section VI on computational methods contains an invited contribution by Judd, Kubler and Schmedders along with papers by Akdeniz, by Sefton, and by Semmler and Sieveking.

In the invited paper by Judd et al., they extend the ideas introduced by Judd (1992) to infinite-horizon asset pricing models with heterogeneous agents as well as incomplete markets. As an application, the paper is interesting because it allows us to study the qualitative properties of equilibria in such models. But what makes this contribution a real gem is that it describes the computational
methodology in detail, including a discussion about the pros and cons of the various computational techniques used. There is also a brief discussion on one of Judd’s ideas for evaluating the effectiveness of a computational result. (Judd likes to evaluate the error of the numerical approximation at grid points that were not used in the estimation. Then if the maximum error is say $10^{-6}$, he likens that to an error of at most a $1$ error in a $1,000,000$ investment.) In this paper, the authors choose to solve functional Euler equations using cubic splines to estimate the solution functions. (Splines are better for sharply curved functions than polynomials since the latter have to be of a very high degree in order to achieve the same numerical accuracy as the splines. From a computational point of view, the fewer the number of parameters that need to be estimated, the better.) They then solve for the optimal parameters with a novel iterative collocation scheme. For the collocation part, they choose a mesh of grid points and for the iteration part, they solve for this period’s parameter based on next period’s parameters as given. They continue the iteration until they find a fixed point of this algorithm. Here again the authors fully explain the steps that they take to get the iterative scheme to converge. They begin with a homotopy method and switch to a Newton method as the solution gets close to the end. They also describe the application of a computational speed-up which was designed for linear systems but which they were successful in using with nonlinear equations.

Akdeniz extends the method of Judd (1992) to solve a stochastic growth model with 25 firms for the share prices of the firms. He then used the model to simulate stock market data for 240 time periods. With this data set, he tests the Fama and French (1995) thesis that the size of the firm (as well as the firm’s beta) is one of the significant determinants of stock price risk. When the entire sample is used, the size of the firm is a significant factor in explaining risk. However, as Akdeniz and Dechert (1997) showed, CAPM conclusions depend on where in the business cycle the data is drawn from. So Akdeniz selects a sub sample of data that occurs at the same point in the business cycle, and by using the same methodology of Fama and French on this data, he shows that their conclusions do not hold. Thus, Akdeniz shows that to properly test the CAPM on data from a dynamic economy, it is important to control for where in the business cycle the data is drawn.

Sefton describes a technique to solve a consumer life-cycle problem. The primary technique is value function iteration, but with a number of adaptations to improve the speed and accuracy of the results. In particular, he describes in detail the selection of a combination of coarse and fine grids and the use of splines to interpolate the results. Since this is a general equilibrium model, prices in each period must be solved for, and this is done with an iteration scheme. The author discusses some of the issues that are involved in choosing between solving the Bellman equation (via functional iteration) and the Euler equation (via a Newton scheme). One difficulty that he raises is that for general
equilibrium models, using a linear-quadratic approximation to the value function does not work particularly well. Although the point is not raised by the author, it is likely that a combination of techniques works best to solve dynamic optimization problems. The first stage should be value function iteration to get a good initial value for a Newton-like routine to solve the Euler equations. In that regard, the author’s contribution is a good description of how to solve a complex dynamic programming problem with value function iteration.

Semmler and Sieveking study sustainable debt problems in an optimal control model for an open economy which borrows from abroad in order to finance consumption. The country may exploit a renewable resource to service their debt. The optimal control model has two control variables, the extraction rate and consumption, and two state variables, the stock of the resource and debt. Semmler and Sieveking present a numerical method to compute the so-called critical debt curve, determining a critical debt as a function of the resource stock, above which debt tends to infinity and below which debt can be steered to zero. The critical debt curve is computed using an ordinary differential equation of steepest descent. Simulations of the debt-resource dynamics are also included, suggesting the existence of limit cycles for intermediate discount rates.

When we started editing a special issue on Complex Nonlinear Dynamics and Computational Methods, it was hard to guess the number of submissions. In fact, we did not expect so many high-quality submissions. We were wrong and the result is now a triple issue. We would like to thank all authors for submitting their excellent contributions. Special thanks are due to all referees, for their evaluation of the submissions. Their quick and clear reports made the quality control for the editors much easier. We hope that the reader will enjoy this issue as much as we did already in the past months.

This special issue will not settle the business cycle controversy, discussed in the beginning of this introduction. Hopefully, the issue will contribute in stimulating research in nonlinear dynamics and computational methods and be helpful in learning about computational and theoretical techniques that should become part of the standard tool box of economic dynamics. The truth behind the real forces of business cycles is probably somewhere in between the two alternatives, nonlinear endogenous forces buffeted by exogenous shocks. This is precisely what makes the research in this area both difficult and interesting.

For further reading

Brock et al. (1991); Judd (1997).
References


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