Endogenous fluctuations in a simple asset pricing model with heterogeneous agents

Andrea Gaunersdorfer*

Department of Business Studies, University of Vienna, Brünner Straße 72, 1210 Vienna, Austria
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Abstract

In this paper we study the adaptive rational equilibrium dynamics in a simple asset pricing model introduced by Brock and Hommes (System Dynamics in Economic and Financial Models, Wiley, Chichester, 1997, pp. 3–44; Journal of Economic Dynamics and Control, 22, 1998, 1235–1274). Traders have heterogeneous expectations concerning future prices and update their beliefs according to a risk adjusted performance measure and to market conditions. Further, also their expectations about conditional variances of returns vary over time. We show that even for the simple case where agents can only choose between two different predictors complicated dynamics arise and we analyse the bifurcation routes to chaos. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

For the last 40 years economic and finance theory have been based on the assumption of rational behavior. It is assumed that agents have homogeneous expectations and are fully rational in the sense that they immediately take all available information into consideration, optimize according to a model which is common knowledge and are able to make arbitrarily difficult logical inferences. One paradigm of modern finance is the efficient market hypothesis (EMH). EMH postulates that the current price contains all available information and past prices cannot help in predicting future prices. Sources of risk and economic fluctuations are exogenous. Therefore, in the absence of external shocks prices would converge to a steady-state path which is completely determined by fundamentals and there are no opportunities for consistent speculative profits.

However, there is evidence that markets are not always efficient and a lot of phenomena observed in real data cannot be explained by the EMH. For example, trading volume and volatility of returns in real markets are large and show significant autocorrelation, higher returns than expected and calendar effects are observed. It seems that psychology and heterogeneous expectations play an important role in real markets.

Indeed, news on financial markets is always accompanied by pictures of people engaged in wild activity which gives the impression that financial markets are dominated by actions of people. Traders often see markets as offering speculative opportunities and believe that technical trading is profitable. In their opinion, something like a market psychology exists and herd effects unrelated to market news can cause bubbles and crashes. Markets are seen as possessing their own moods and personalities.

In the 1930s, Keynes already argued that agents do not have sufficient knowledge of the structure of the economy to form correct mathematical expectations. Under this assumption, it is impossible for any formal theory to postulate unique expectations that would be held by all agents. In the Keynesian view, prices are not only determined by fundamentals but part of observed fluctuations is endogenously caused by nonlinear economic forces and market psychology. This implies that technical trading rules need not be systematically wrong and may help in predicting future price changes. Empirical work which has shown that such trading rules may indeed outperform traditional stochastic finance models includes, for example, Brock et al. (1992), and Gençay and Stengos (1997).

By now, there is enough statistical evidence both, to question the EMH and to consider traders’ viewpoints when modelling the price behavior on financial markets (see e.g. Shiller, 1991). Haugen (1998a,b,c) gives ‘a collection of the evidence and arguments, which constitute a strong and persuasive case for a noisy stock market that over-reacts to past records of success and failure on the part of business firms, and prices with great imprecision’.
Also developments in the theory of nonlinear dynamical systems have contributed to new approaches in economics and finance. Simple deterministic models may generate complex (chaotic) dynamics, similar to a random walk. Though empirical evidence for the existence of chaotic dynamics seems to be rather weak, chaos offers a pragmatic shift in thinking about methods to study economic activity.\footnote{Data quality and weakness of statistical tests makes it difficult to detect chaos in real data. A presentation of techniques to detect chaos in empirical data can be found, for example, in Brock et al. (1991).} Introducing nonlinearities in the models may improve our understanding of how economic processes work. Chaos is suggestive of pathways to complex dynamics, it stimulates the search for a mechanism that generates the observed movements in real financial data and that minimizes the role of exogenous shocks.

Recently, finance literature has been searching for alternative theories that can explain observed patterns in financial data. A number of models were developed which build on boundedly rational, non-identical agents. Financial markets are considered as systems of interacting agents which continually adapt to new information. Heterogeneity in expectations can lead to market instability and complicated dynamics.

One approach to explain movements in financial returns are models with informed and uninformed traders or with ‘irrational noise traders’ (see for example Grossman, 1989; Black, 1986; Shleifer and Summers, 1990; De Long et al., 1990a,b; Allen and Gorton, 1993). Brunnermeier (1998) provides a survey of the literature on the informational aspects of price processes. The literature on behavioral finance (see Thaler, 1991, part V, and the papers in Thaler, 1993, part III) emphasizes the role of quasi-rational, overreacting, and biased traders.

Another view is that agents are intelligent, but since agents do not have complete knowledge about the underlying model and do not have the computational abilities assumed in rational expectations models, equally informed traders may interpret the same information differently. This results in heterogeneous beliefs about the market, traders evaluate their forecasts and trade on those predictors which perform best. Prices are therefore driven endogenously and agents’ expectations co-evolve in a world they co-create.

A number of structural models with a few strategy types have been introduced emphasising this heterogeneity in expectations formations. Typically, they include fundamental and technical traders (chartists). Examples are Beja and Goldman (1980), Chiarella (1992), Frankel and Froot (1988), Ghezzi (1992), Day and Huang (1990), De Grauwe et al. (1993), Sethi (1996), Franke and Sethi (1993), Lux (1994, 1995), Lux and Marchesi (1998). Computational models relying on artificial intelligence methods where many traders interact in an artificial financial market are Arthur et al. (1997a,b) and LeBaron (1995).
Kurz (1997) presents an equilibrium concept, which he calls rational belief equilibrium. He introduces heterogeneous beliefs about probability distributions which give rise to endogenous uncertainty. Agents concentrate on the empirical consistency of their forecasting model. Grandmont (1998) and Hommes and Sorger (1998) develop models where expectations may become self-fulfilling. Related approaches are also evolutionary models such as Blume and Easley (1992) and learning models (see for example Sargent, 1993). Barucci and Posch (1996) show the emergence of complex beliefs dynamics in linear stochastic models as the outcome of boundedly rational learning.

Brock and Hommes (1997a), hereafter BH, consider Adaptive Belief Systems to study heterogeneous expectations formations. Agents adapt their predictions by choosing among a finite number of predictor or expectations functions which are functions of past information. Each predictor has a performance measure attached which is publicly available to all agents. Based on this performance measure agents make a (boundedly) rational choice between the predictors. This results in the Adaptive Rational Equilibrium Dynamics (ARED), an evolutionary dynamics across predictor choice which is coupled to the dynamics of the endogenous variables. BH show that the ARED incorporates a general mechanism which may generate local instability of the equilibrium steady state and complicated global equilibrium dynamics.

Brock and Hommes (1997b, 1998) apply this concept to a simple asset pricing model, building on a theoretical framework formulated by Brock and LeBaron (1996), where traders in a financial market use different types of predictors for their price forecasts of a risky asset (see also Brock (1997), for a general formulation of the model and possible generalizations). In this model, under homogeneous, rational expectations and the assumption of an independently identically distributed dividend process, prices are constant over time. However, introducing heterogeneous price expectations changes the situation substantially. Market dynamics may then be characterized by an irregular switching of periods where prices are close to the (EMH) fundamental, phases of ‘optimism’ where prices rise far from the fundamental — traders become excited and extrapolate trends by simple technical analysis — and ‘pessimistic’ periods characterized by a sharp decline in prices. This irregular switching is caused by the rational choice between predictors as described above. BH call it ‘the market is driven by rational animal spirits’. They show that increasing the ‘intensity of choice’ to switch between predictors may result in a bifurcation route to complicated price fluctuations where price dynamics takes place on a chaotic (strange) attractor. The model nests the usual rational expectations type of model, e.g. a class of models that are versions of the EMH. But these rational expectations beliefs are costly and ‘compete’ with other types of beliefs in generating net trading profits as the system evolves over time.

The main purpose of this paper is to extend this model in two ways. We adjust the performance measure of the predictors by a term capturing risk aversion. It
turns out that predictor choice is then determined by squared prediction errors of price forecasts. Further, agents do not only update their conditional expectations of prices in every period but also their beliefs about conditional variances of returns. In every period traders estimate variances as exponential moving averages of past returns. We focus on a simple version of the model with two types of traders, fundamentalists and trend extrapolators (trend chasers). To prevent prices to diverge to infinity we introduce a ‘stabilizing force’. This stabilizing force may be interpreted as technical traders do not use the trend chasing predictor if prices are too far away from the fundamental value, even if that predictor performed best in the recent past. Predictor choice and therefore the fractions of the two types of traders are thus not only determined by past performance of the predictors but also conditioned on market conditions. The question is whether the ‘rational route to randomness’ is similar to that in BH (1997a,b, 1998). We give a detailed bifurcation analysis applying local bifurcation theory to detect primary and secondary bifurcations of the steady states and using the LOCBIF bifurcation package (Khibik et al., 1992). Further, we use numerical tools such as phase portrait analysis, bifurcation diagrams, Lyapunov characteristic exponents, and compute invariant manifolds to demonstrate the emergence of strange attractors.

This paper is organized as follows. In Section 2 we briefly recall the asset pricing model of BH and explain our extensions to the model. We study the local behavior of the system near steady states in Section 3 and the global dynamics in Section 4. By numerical analysis we show the existence of horseshoes and strange attractors for a wide range of parameter values. Section 5 concludes. Proofs and details of derivations are given in an appendix.

2. The model

We briefly recall the model used in BH (1997b, 1998). They consider an asset pricing model with one risky asset and one risk-free asset available with gross return \( R_t \). \( p_t \) denotes the price (ex-dividend) of the risky asset and \( \{ y_t \} \) the dividend process which is assumed to be independently and identically distributed (IID). The dynamics of wealth is described by

\[
W_{t+1} = RW_t + R_{t+1}z_t,
\]
where \( z_t \) is the number of purchased shares of the risky asset at time \( t \) and \( R_{t+1} = p_{t+1} + y_{t+1} - Rp_t \) is the excess return per share. Bold face type denotes random variables. We write \( E_t, V_t \) for the conditional expectation and conditional variance operators at time \( t \), based on a publically available information set of past prices and dividends, \( \mathcal{F}_t = \{ p_t, p_{t-1}, \ldots, y_t, y_{t-1}, \ldots \} \). The ‘beliefs’ of investor type \( h \) about these conditional expectation and variance are denoted by \( E_{ht} \) and \( V_{ht} \).

Assuming that investors are myopic mean variance maximizers the demand for shares \( z_{ht} \) by type \( h \) solves

\[
\max \left\{ E_{ht} W_{t+1} - \frac{a}{2} V_{ht} W_{t+1} \right\}, \quad \text{i.e. } z_{ht} = \frac{E_{ht} R_{t+1}}{a V_{ht} R_{t+1}},
\]

where \( a \geq 0 \) characterizes risk aversion. Let \( z_{st} \) and \( n_{ht} \) denote the supply of shares per investor and the fraction of investors of type \( h \) at time \( t \), respectively. Equilibrium of supply and demand implies

\[
\sum_h n_{ht} z_{ht} = z_{st}.
\]

Assuming constant supply of outside shares over time we may, without loss of generality, stick to the (equivalent) special case \( z_{st} = 0 \) (for the general case see Brock (1997)).

To obtain a benchmark notion of ‘fundamental solution’ consider the case where there is only one type. Then Eq. (2) becomes \( R \tilde{p}_t = E_t \tilde{p}_{t+1} + \bar{y} \); \( E_t \bar{y}_{t+1} = \bar{y} \) is constant since \( \{ y_t \} \) is assumed to be IID. The fundamental solution \( \tilde{p}_t^* \equiv \bar{p} \) is the only solution which satisfies the ‘no bubbles’ condition

\[
\lim_{t \to \infty} (E_t \tilde{p}_t^*/R) = 0, \quad \text{i.e. } \bar{p} = \bar{y}/(R - 1).
\]

Note that \( \bar{p} \) is the expectation of the discounted sum of future dividends. In what follows we express prices in deviations from the benchmark fundamental, \( x_t = p_t - p_t^* = p_t - \bar{p} \).

Beliefs are assumed to be of the form

\[
E_{ht}(p_{t+1} + y_{t+1}) = E_t(p_{t+1} + y_{t+1}) + f_h(x_{t-1}, \ldots, x_{t-L}) = R p_t^* + f_{ht}
\]

for some deterministic function \( f_h \). Note that \( f_{ht} \equiv f_h(x_{t-1}, \ldots, x_{t-L}) = E_{ht} x_{t+1} \) is the conditional expectation of type \( h \) for the price deviation from a commonly shared fundamental. As in BH (1997b, 1998) we only consider beliefs of the simple form \( f_{ht} = g_h x_{t-1} + b_h \). The equilibrium equation (2) now becomes

\[
Rx_t = \sum_h n_{ht} f_{ht}.
\]

Fractions are determined as discrete choice probabilities (cf. Anderson et al., 1993)

\[
n_{ht} = \exp(\beta U_{h,t-1})/Z_t, \quad Z_t = \sum_h \exp(\beta U_{h,t-1}),
\]
where $U_n$ is some ‘fitness function’ or ‘performance measure’. The parameter $\beta$ is called the intensity of choice. It measures how fast agents switch between different predictors, i.e. it is a measure of traders’ rationality. For $\beta = 0$ fractions are fixed over time and are equal to $1/N$, where $N$ is the number of different types of traders. If $\beta = \infty$ all traders choose immediately the predictor with the best performance in the recent past. Thus, for finite, positive $\beta$ agents are boundedly rational in the sense that fractions of the predictors are ranked according to their fitness. The parameter $\beta$ plays a crucial role in the bifurcation route to chaos.

Let

$$\rho_t := E_t R_{t+1} = E_t x_{t+1} - Rx_t = x_{t+1} - Rx_t$$

denote the rational expectations of excess returns$^3$ and let

$$\rho_{ht} := E_{ht} R_{t+1} = E_{ht} x_{t+1} - Rx_t = f_{ht} - x_{t+1} + \rho_t$$

be the conditional expectations of type $h$. Define risk adjusted realized profits

$$\pi_{ht} := \pi(\rho_t, \rho_{ht}) := \rho_t z(\rho_{ht}) - \frac{\alpha}{2} z(\rho_{ht})^2 V_{ht} R_{t+1}, \quad (4)$$

where the demand for shares

$$z(\rho_{ht}) = \frac{\rho_{ht}}{a V_{ht} R_{t+1}}$$

is the solution of the maximum problem$^4$

$$\max_z \left\{ \rho_{ht} z - (\alpha/2) z^2 V_{ht} R_{t+1} \right\}$$

and

$$V_{ht} R_{t+1} = V_{ht} (x_{t+1} - Rx_t + \delta_{t+1})$$

$$= V_{ht} (x_{t+1} - Rx_t) + V_{ht} \delta_{t+1}$$

is the belief of type $h$ about the the conditional variance. $\delta_{t+1}$ is a martingale difference sequence (see BH, 1998, Eq. (2.11)). We assume $\text{Cov}(x_{t+1} - Rx_t, \delta_{t+1}) = 0$, $V_{ht} \delta_{t+1} = \sigma_h^2$ to be constant, and $V_{ht} (x_{t+1} - Rx_t) = V_d (x_{t+1} - Rx_t) = : \tilde{\sigma}_h^2$. Thus, we assume that agents have homogeneous expectations on conditional variances of returns. Nelson (1992) (see also Bollerslev et al., 1994) provides some justification for this assumption. He shows that conditional variances are much easier to estimate thanconditional means, hence there shouldbe more disagreement about the mean than about the variance among the traders. BH

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$^3$ It can be shown that under the given assumptions $x_{t+1}$ is a deterministic function of $(x_t, x_{t-1}, \ldots)$ (see Brock, 1997).

$^4$ Notice that this maximum problem is equivalent to (1), up to a constant, so the optimal choice of shares of the risky asset is the same.
(1997b, 1998) study the case where the belief about conditional variances of returns is constant over time, $V_{ht} = \sigma^2$. Here we allow for time varying beliefs about variances. This is an important generalization of the model. Agents observe price behavior to update their beliefs about prices. It seems natural that they use the observed data also to update their estimations of the variances of returns. We assume that traders estimate $\tilde{\sigma}^2_t$ as exponential moving averages

$$\tilde{\sigma}^2_t = w_\sigma \tilde{\sigma}^2_{t-1} + (1 - w_\sigma)(x_{t-1} - Rx_{t-2} - \tilde{\mu}_{t-1})^2,$$

$$\tilde{\mu}_t = w_\mu \tilde{\mu}_{t-1} + (1 - w_\mu)(x_{t-1} - Rx_{t-2}),$$

where $w_\sigma, w_\mu \in [0,1]$. $\tilde{\mu}_t$ defines the exponential moving averages of returns.

Let us return to the discussion of the performance measure $U_{ht}$. The first term of $\pi_{ht}$ in (4) denotes realized profits for type $h$, the second term captures risk aversion. $\pi_t$ is defined by analogy, dropping index $h$. BH (1997b, 1998) concentrate on the case without risk adjustment, i.e. they drop the second term in (4). In this paper we study the model with risk adjustment. The fractions of the different types at the end of period $t$ will be determined by the ‘fitness function’ $\pi_{h,t-1}$. Subtracting off the same term $\pi_{t-1}$ for all types does not change the discrete choice fractions. For updating the fractions we therefore use the difference in (risk adjusted) profits of type $h$ beliefs and rational expectations beliefs:

$$U_{ht} = \min \{ \rho_{t-1}, \rho_{ht,t-1} \} : = \pi_{h,t-1} - \pi_{t-1} = -\frac{1}{2d(\tilde{\sigma}^2_{t-1} + \sigma^2_\beta)}(x_t - f_{h,t-1})^2.$$

Thus, in contrary to BH (1997b, 1998) where the fitness measure of each type is determined by past realized profits, using risk adjusted profits, squared prediction errors determine the predictor choice. Notice that also Arthur et al. (1997a,b) look at squared forecast errors to form expectations about future prices in their artificial market.

The timing of expectations formations is important. In updating fractions of beliefs in each period, the most recently observed prices and returns are used. Agents take positions in the market in period $t$ based on forecasts they make for period $t + 1$. Which predictors they use depends on the performance of this predictor in period $t - 1$. That is, at the end of period $t - 1$ (beginning of period $t$), after having observed price $x_{t-1}$, the fractions $n_{ht}$ and the expectations $\rho_{ht,t-1}$ and $\tilde{\sigma}^2_{t-1}$ are formed. Note that $f_{h,t-1}$ and $\tilde{\sigma}^2_{t-1}$ depend on $x_{t-2}, x_{t-3}, \ldots$. In period $t$ price $x_t$ is determined, which defines $\rho_{t-1}$, etc.

For the class of fundamentalists we will make two further adaptions of the fitness measure, by introducing costs and a ‘stabilizing force’. We define $\tilde{U}_{ht} := U_{ht} - C + zx_t^2$ for fundamentalists (traders who believe that prices will return to their fundamentals) and $\check{U}_{ht} := U_{ht}$ for all other types of traders. Fundamentalists have all past prices and dividends in their information set, however, they do not know the fractions of the other belief types. They act as if
all agents were fundamentalists which means that they are not perfectly rational. Costs of the fundamental predictor, $C$, might be positive since it takes some effort to understand how the market works and to believe that it will price according to the fundamental. Thus, the performance function (realized profits) of the fundamental predictor has to be reduced by costs $C \geq 0$.

$\alpha \geq 0$ defines an exogenous stabilizing force which, when the derivation from the fundamental price becomes too large, should drive prices back to the fundamental. As we will see, the evolutionary dynamics gets easily dominated by trend followers and prices will grow exponentially without bound. Introducing this stabilizing parameter means that fundamentalists get more weight as prices move further away from the fundamental and the evolutionary dynamics will remain bounded if $\alpha$ is large enough. Thus, fractions are not only determined by the predictor performance but also by market conditions. If prices are too far away from the fundamental price, technical traders might not use the trend chasing trading rule. Even when its prediction in the recent period was better than the fundamental trading rule, they do not believe that this predictor will also perform better in future periods. Arthur et al. (1997a,b) also introduce condition/forecast rules for predictors that contain both, a market condition which determines if a certain predictor is used and a forecasting formula for next period’s price and dividend. At this point our stabilizing force is exogenously specified and ad hoc. De Grauwe et al. (1993) formulate weights for chartists in an analogous way. In future work one might incorporate ‘far from equilibrium’ forces driving prices back to the fundamental, such as e.g. futures markets or long-term traders on fundamental. Adding those more realistic economic forces might give similar dynamics as the stylized model above. It seems useful to get more insight into the dynamics of this stylized model before adding another layer of complexity.

The evolution of equilibrium prices, fractions, and beliefs about conditional variances is summarized by the Adaptive Rational Equilibrium Equation:

$$R_{xt} = \sum_{h=1}^{H} n_{ht} f_{ht},$$  \hfill (5)

$$n_{ht} = \exp(\beta \hat{U}_{h,t-1}/Z_{t}),$$  \hfill (6)

$$\hat{\sigma}_t^2 = w_{o} \hat{\sigma}_{t-1}^2 + (1 - w_{o}) (x_{t-1} - R_{x,t-2} - \tilde{\mu}_{t-1})^2,$$  \hfill (7)

$$\tilde{\mu}_t = w_{p} \tilde{\mu}_{t-1} + (1 - w_{p}) (x_{t-1} - R_{x,t-2}).$$  \hfill (8)

Considering (5) and (6), prices in period $t$ are determined by $\hat{U}_{h,t-1}$, i.e. by $\delta \left( \frac{\rho_{t-2}}{\rho_{h,t-2}} \right) = - (x_{t-1} - f_{x,t-2})^2/(2a \hat{\sigma}_{t-2}^2)$. Introducing $\sigma_t := \hat{\sigma}_{t-1}$ and $\mu_t := \tilde{\mu}_{t-1}$ and replacing (7) and (8) by the corresponding equations for $\sigma_t^2$ and $\mu_t$ reduces the dimension of the system by one (see below).

We restrict to a simple but typical case with only two types of traders, fundamentalists (type 1) and trend chasers (type 2), i.e. $f_{1t} \equiv 0$ and $f_{2t} = gx_{t-1}$.
(g > 0). This gives the following dynamical system:

\[ R x_t = n_2 g x_{t-1}, \]

\[ n_{1t} = \exp \left[ \beta \left( -\frac{1}{2a(\sigma_{i-1}^2 + \sigma_3^2)} x_{i-1}^2 + 2x_{i-1}^2 - C \right) \right] / Z_t, \]

\[ n_{2t} = \exp \left[ -\frac{\beta}{2a(\sigma_{i-1}^2 + \sigma_3^2)} (x_{i-1} - g x_{i-3})^2 \right] / Z_t, \]  

(9)

\[ \sigma_i^2 = w_\sigma \sigma_{i-1}^2 + (1 - w_\sigma)(x_{t-2} - Rx_{t-3} - \mu_{t-1})^2, \]

\[ \mu_t = w_\mu \mu_{t-1} + (1 - w_\mu)(x_{t-2} - Rx_{t-3}), \]

which is a third-order difference equation. Setting \( y_t := x_{t-1} \) and \( z_t := x_{t-2} \) and defining

\[ m_t := 1 - 2n_{2,t+1} = n_{1,t+1} - n_{2,t+1}, \]

\[ = \tanh \left[ \frac{\beta}{2} \left( \frac{g}{2a(\sigma_{i-1}^2 + \sigma_3^2)} x_{i-1}^2 (g x_{i-2} - 2x_i) + 2x_{i-1}^2 - C \right) \right], \]

(9) is equivalent to

\[ x_t = \frac{g}{2R} x_{t-1} \left( 1 - \tanh \left[ \frac{\beta}{2} \left( \frac{g}{2a(\sigma_{i-1}^2 + \sigma_3^2)} z_{i-1} (g z_{i-1} - 2x_{i-1}) + 2x_{i-1}^2 - C \right) \right], \]

\[ y_t = x_{t-1}, \]

\[ z_t = y_{t-1}, \]

(10)

\[ \sigma_i^2 = w_\sigma \sigma_{i-1}^2 + (1 - w_\sigma)(y_{t-1} - R z_{t-1} - \mu_{t-1})^2, \]

\[ \mu_t = w_\mu \mu_{t-1} + (1 - w_\mu)(y_{t-1} - R z_{t-1}). \]

This system is five dimensional, in contrast to BH (1997b, 1998) where the corresponding case with two types of traders is three dimensional. Allowing time-dependent beliefs about conditional variances of returns introduces two additional dynamical variables, the average return \( \mu_t \) and the estimate of the conditional variance \( \sigma_i^2 \). The model differs also in the way fractions are determined. We adjust the performance measure by a term of risk aversion and condition the fractions of the different types on market conditions by introducing the parameter \( \alpha \). This allows for a two-parameter bifurcation analysis with respect to the intensity of choice \( \beta \) and the strength of the stabilizing force \( \alpha \).
3. Steady states and local bifurcations

In this section we study the local behavior near the steady states of the system. Our bifurcation analysis is supported by the program package LOCBIF (Khibik et al., 1992). The LOCBIF program is based on numerical methods to compute steady states, periodic orbits and bifurcation curves in the parameter space which are described in Kuznetsov (1995, Chapter 10).

We first restrict to the case with constant beliefs about variances, \( w_\sigma = w_\mu = 1 \), and denote \( \sigma^2 \equiv \sigma^2 \). In this case we can restrict to the three-dimensional system defined by the first three equations of (10).

\textbf{Proposition 1} (Existence and stability of steady states of (10)). Let \( C > 0 \).

1. For \( 0 < g < R \), \( E_1 = (0,0,0) \) is the unique, globally stable steady state (fundamental steady state).
2. For \( R < g < 2R \) a pitchfork bifurcation occurs at \( \beta^* := (1/C) \log[g/(g - R)] \). \( E_1 \) is stable for \( \beta < \beta^* \) and unstable for \( \beta > \beta^* \).

   - (a) For \( \alpha < \alpha^* := -(1/2\sigma^2)g(g - 2) \) the pitchfork bifurcation is subcritical, i.e. there exist two unstable non-fundamental steady states for \( \beta < \beta^* \).
   - (b) For \( \alpha > \alpha^* \) the pitchfork bifurcation is supercritical, i.e. there exist two stable non-fundamental steady states for \( \beta^* < \beta < \beta^\dagger \).

3. For \( g \geq 2R \) there exist three steady states. The fundamental steady state \( E_1 \) is unstable for all \( \beta > 0 \). The non-fundamental steady states are stable for \( \beta < \beta^\dagger \) (see footnote 5).

\textbf{Remark}. Result 3 also holds for the case \( C = 0 \).

\textbf{Proof}. See the appendix.

Results 1, 2(b), 3 coincide with the results in Proposition 2 in BH (1998). Thus, different performance measures may yield similar primary bifurcations towards instability of the evolutionary dynamics. Result 2(a) shows that when there are costs for the fundamental predictor prices might explode if the stabilizing force \( \alpha \) is small. Near the fundamental steady state both predictors yield small forecasting errors. Since the fundamental predictor is costly, agents choose the trend chasing forecasting rule which drives prices away from the fundamental. Then the forecasting error of the fundamental predictor becomes large, so there

\footnote{At \( \beta = \min\{\beta_0, \beta_r\} \), defined in Proposition 2, a Hopf or flip bifurcation occurs and the two non-fundamental steady states become unstable.}
is no force which brings this upward (or downward) trend to an end. This will not happen in real markets. When prices are too far from ‘fundamentals’ there must be some forces which drive them back, though it is not really clear what kind of forces these are. If there are more types of traders in the market this might also prevent prices to explode (cf. our discussion in the previous section).

The next proposition shows that the non-fundamental steady states become unstable either by a Hopf or a flip bifurcation when the intensity of choice \( \beta \) is increased.

Proposition 2 (Secondary bifurcations). Let \( g > R \) and \( \beta > \beta^* \). When \( \beta \) is increased, the non-fundamental steady states become unstable either by a Hopf or by a flip bifurcation.

1. For small \( \alpha \), more precisely, for \( \alpha^* < \alpha < \alpha^{**} = g^2 / 2\alpha^2 \), the non-fundamental steady states undergo a Hopf bifurcation at a certain value \( \beta_H(\alpha) \) (defined by Eq. (20) in the appendix).

2. For \( \alpha > \alpha^{**} \) the non-fundamental steady states undergo a flip bifurcation at

\[
\beta_F := \frac{1}{C} \left( \frac{g}{g - R} + \log \frac{R}{g - R} \right)
\]

Proof. See the appendix.

Fig. 1 shows a detailed bifurcation diagram on the \((\beta, \alpha)\)-plane (for \( a = 10, \sigma^2 = 0.1, C = 1, R = 1.01, g = 1.2 \)) which we generated by the use of the LOCBIF program.

Depending on \( \alpha \), there are different bifurcation routes possible as \( \beta \) increases. If the non-fundamental steady states are destabilized by a Hopf bifurcation (i.e. \( \beta_H(\alpha) < \beta_F \)) a closed invariant curve filled by quasiperiodic orbits is created. For certain \( \beta \)-values phase locking phenomena occur that create and destroy stable and unstable periodic orbits. These periodic orbits are located on the closed invariant curve. Actually, an infinite number of phase locking windows having the form of small tongues — so-called Arnold tongues — are rooted in the Hopf bifurcation curve (see Kuznetsov (1995, Chapter 7.3) for a mathematical treatment of this phenomenon). Their origin points correspond to eigenvalues \( \lambda \) of the Jacobian of the non-fundamental steady states with \( \arg \lambda = \arg (A + Bi) = 2\pi p/q \) for rational \( p/q \), reduced, lying on the unit circle. Arnold tongues define parameter regions where points of period \( q \) occur. They are delimited by

\[ We use these parameter values for all figures in this paper (in examples where the variance is time dependent, \( \sigma^2 \) has to be replaced by \( \sigma^2_d \)).
saddle-node bifurcation curves of period $q$ points, corresponding to a collision between stable and unstable periodic orbits of $F$ (the map which defines the dynamical system (10)). In Fig. 1, a 1:10 Arnold tongue is drawn and origin points of some further Arnold tongues are plotted. As $\beta$ increases, $p/q$ increases.

$B = (\beta^*, z^*)$ is the intersection point of the pitchfork bifurcation curve, the Hopf bifurcation curve (20), and the line $g(g - 2) + 2a\sigma^2x = 0$ (see Fig. 1). At $B$ the pitchfork bifurcation changes from sub- to supercritical and expression (13), which determines the non-fundamental steady states, is not defined. When approaching $B$ along the Hopf bifurcation curve one eigenvalue of the Jacobian of the non-fundamental steady states converges to $-\frac{1}{2}$ and the two complex conjugate eigenvalues on the unit circle converge to 1, i.e. $\arg\lambda_{2,3} = 0$ (for details see the appendix). A situation where a double eigenvalue 1 occurs would correspond to a 1:1 resonance (see for example Kuznetsov, 1995, Chapter 9.5.2). However, in our model $B$ is a singularity on the $(\beta, z)$-plane, the dynamical system only consists of fixed points for $z = z^*$. In the appendix we show that when approaching $B$ from different directions the eigenvalues have different limits.

At point $C = (\beta_F, z^{**})$ a codimension two bifurcation occurs. The Jacobian of the non-fundamental steady states has one eigenvalue $\lambda_1 = -1$ and two complex
conjugate eigenvalues $\lambda_{2,3} = A \pm Bi$ lying on the unit circle. Since
$\lambda_1 = -\lambda_2 \lambda_3/(\lambda_2 + \lambda_3) = -(A^2 + B^2)/2A$ (c.f. appendix, proof of Proposition 2),
$A = 1/2$ and $B = \sqrt{3}/2$, thus $\arg \lambda_{2,3} = \arctan \sqrt{3} = 2\pi/3$. Point C is the origin of
a 1:6 Arnold tongue.\footnote{In the class of all dynamical systems C would be a point of codimension three. In our special family of systems, however, the 1:6 resonance follows from the simultaneous occurrence of the Hopf and flip bifurcations.}

Between the points C and B any resonance $p:q$ with $0/1 < p/q < 1/6$ can be found, which implies in particular all 1:q ($q > 6$) resonances. $p/q$ increases monotonically as $\beta$ is increased. The computation of these points is described in the appendix.

For values $a > a^{**}$ we either have $\beta_V < \beta_H(a)$ or $\beta_H(a) \not\in \mathbb{R}$ and the secondary bifurcation of the non-fundamental steady states is a flip bifurcation. In Fig. 1, besides the Hopf bifurcation curve of the non-fundamental steady state described in Proposition 2, a Hopf bifurcation curve of the 2-cycle (which was created by the flip bifurcation) starts in point C. Thus, for the chosen parameter values, the 2-cycle becomes unstable by a Hopf bifurcation which results in two attracting invariant cycles. After a saddle-node bifurcation of the sixth iterate of $F$ a stable (and an unstable) 6-cycle is created, lying in an Arnold tongue which originates in C. From the bifurcation diagram in Fig. 2(a) we conclude that this 6-cycle is destabilized by a period doubling bifurcation.

Let us now return to the general case of time varying beliefs on conditional variances of returns, i.e. $w_\sigma, w_\mu \in (0, 1)$. The equilibria of the five-dimensional system (10) are the fundamental steady state $E_1 = (0,0,0,0,0)$ and the non-fundamental steady states $E_2 = (x^*, x^*, x^*, 0, (1 - R)x^*)$ and $E_3 = -E_2$, where $x^*$ is given by (13), replacing $\sigma^2$ by $\sigma_\delta^2$.

**Proposition 3.** In the case of time varying beliefs about conditional variances of returns ($0 < w_\sigma, w_\mu < 1$) the primary bifurcation of the fundamental steady state and the secondary bifurcations of the non-fundamental steady states are identical as in the case with constant beliefs about variances ($w_\sigma = w_\mu = 1$).

**Proof.** See the appendix.

This means that the results of Propositions 1 and 2 also hold for $w_\sigma, w_\mu \in (0,1)$. That is, the primary and secondary bifurcations of the case with constant beliefs about variances are exactly the same as in the case with time varying beliefs about variances. Higher-order bifurcations might be different as can be seen from Fig. 2(a) and (b). For example, for the chosen parameter values the 2-cycle which is generated by the flip bifurcation undergoes a further period doubling before it is destabilized by a Hopf bifurcation.
Numerical simulations show that increasing $\beta$ further leads to the occurrence of strange attractors. We will study the global dynamics and the occurrence of strange attractors in the next section.

4. Global dynamics

Proof of Proposition 1 shows that for large $\beta$-values the fundamental steady state is unstable with one unstable and two stable (four in the case of time
The stable and unstable manifolds of a steady state $x_0$ of a dynamical system corresponding to the map $F$ are defined as $M_x: \lim_{n \to \infty} F^n(x) = x_0$ and $N_x: \lim_{n \to -\infty} F^n(x) = x_0$, resp., where $F^n$ denotes the $n$th iterate of $F$.

For a brief introduction to the concept of homoclinic orbits see BH (1997a) and BH (1997b, 1998), Goeree and Hommes (1999) in this issue also recall this notion. An extensive mathematical treatment of homoclinic bifurcation theory can be found in Palis and Takens (1993).

A map $F$ has a horseshoe if there exist rectangular regions $R$ such that $F^n(R)$ is folded over $R$ in the form of a horseshoe, for some $n \in \mathbb{N}$.

Proposition 4. Let $C > 0$ and $\beta = \infty$. If

$$g > R \quad \text{and} \quad \alpha > \frac{R^2}{2a \sigma^2 g^3(2g - R^2)},$$

the unstable manifold of the fundamental steady state $E$ is bounded and all orbits converge to the saddle point $E$.

Proof. See the appendix.

This proposition shows that for $\beta = \infty$ orbits remain bounded for a wide range of parameter values. Since the fundamental steady state is unstable prices move away, above or below the fundamental, but will return close to it. Thus, the unstable and stable manifolds have to intersect in a so-called homoclinic point. This suggests that for large but finite $\beta$ the system is close to having a homoclinic orbit. Homoclinic points imply very complicated behavior and possibly the existence of strange attractors. Smale (1965) showed that a homoclinic point implies the existence of (infinitely) many horseshoes which leads to a situation called topological chaos (i.e. there is a closed invariant set that contains a countable set of periodic orbits, an uncountable set of non-periodic orbits, among which there are orbits passing arbitrarily close to any point of the invariant set, and exhibits sensitive dependence on initial conditions).
For further analysis of the global geometric properties of the unstable manifold we restrict to the case with constant beliefs about variances ($w_\sigma = w_\mu = 1$). Recall that the dynamical system is three dimensional in that case. We rewrite system (10) in the $(x, y, m)$-space, which yields

$$x_t = \frac{g}{2R} x_{t-1}(1 - m_{t-1}),$$

$$y_t = x_{t-1},$$

$$m_t = \tanh \left[ \frac{\beta}{2} \left( \frac{g^2}{2a\sigma^2} y_{t-1} \left( y_{t-1} - \frac{1}{R} x_{t-1}(1 - m_{t-1}) \right) + \frac{a^2}{4R^2} x_{t-1}^2(1 - m_{t-1})^2 - C \right) \right]. \tag{11}$$

We will use the notation $(x_{t+1}, y_{t+1}, m_{t+1}) = F_{\beta}(x_t, y_t, m_t)$.

Next we consider the stable and unstable manifolds of the fundamental steady state $E$. The stable manifold of $E$ is tangent to the plane $S = \{x = 0\}$ since the stable eigenspace is spanned by the eigenvectors $(0,1,0)$ and $(0,0,1)$. For $\beta = \infty$ the stable manifold of $E$ contains the plane $S$ since every point in $S$ is mapped onto $E$. In this case agents switch infinitely fast between the predictors, the difference of the fractions is given by (22), resp. by

$$m_t = \begin{cases} 
+1 \text{ if } \frac{g^2}{2a\sigma^2} y_{t-1} \left( y_{t-1} - \frac{1}{R} x_{t-1}(1 - m_t) \right) + \frac{a^2}{4R^2} x_{t-1}^2(1 - m_t)^2 > C, \\
-1 \text{ if } \frac{g^2}{2a\sigma^2} y_{t-1} \left( y_{t-1} - \frac{1}{R} x_{t-1}(1 - m_t) \right) + \frac{a^2}{4R^2} x_{t-1}^2(1 - m_t)^2 \leq C. 
\end{cases} \tag{12}$$

Any point lying on the plane $\{m = 1\}$ is mapped onto the plane $S$. Let $A_0$ be the point on the unstable manifold where all agents switch from being trend chasers to fundamentalists, i.e. the $m$-coordinate of $A_0$ is $-1$, but on the the right-hand side of (12) equality holds. The line segment $EA_0$ lies on the unstable manifold of $E$. Define $A_1 := F_x(A_0), A'_1 := (x(A_1), y(A_1), 1)$, the point with the same $x$ and $y$ coordinates as $A_1$ lying on the plane $\{m = 1\}$ ($x(A_1)$ denotes the $x$-coordinate of $A_1$, etc.), $F_x(A_1) = A_2, F_x(A_1') = A'_2, F_x(A_2) = A_3$. The segment $A_2A_3$ is mapped onto $(0,0,1) = : A_4$ and $A_4$ is mapped to $E$ (see Fig. 3). Thus, the first five iterates of $EA_0$ are

$$F_x(EA_0) = EA_1,$$

$$F_x^2(EA_0) = EA_1 \cup A'_1 A_2,$$

\[\text{Note, that when the system is studied in the (x,y,z)-plane, (0,1,0) is a generalized eigenvector.}\]
Fig. 3. Unstable manifold for case 1 ($\alpha = 0.65$, $\beta = 500$): first 6 iterates of the segment $EA_0$.

$$F^3_\infty(EA_0) = EA_1 \cup A'_1 A_2 \cup A'_2 A_3,$$

$$F^4_\infty(EA_0) = EA_1 \cup A'_1 A_2 \cup A'_2 A_3 \cup A_4,$$

$$F^5_\infty(EA_0) = EA_1 \cup A'_1 A_2 \cup A'_2 A_3 \cup A_4 \cup E.$$

$F^\infty_\infty(EA_0)$ defines the unstable manifold for $\beta = \infty$. Our analysis implies that for large but finite $\beta$ the system must be close to having a homoclinic orbit between the stable and the unstable manifolds. But to see whether this implies complicated dynamics one needs complicated horseshoe constructions.

A numerical analysis of the unstable manifold for finite $\beta$-values is presented in the appendix. We observe two cases. For sufficiently large $\alpha$-values (case 2, see Fig. 7) the unstable segment $EA_0$ is expanded and folded over (close to) itself by the 6th iterate of $F_\beta$, which suggests that $F^6_\beta$ has a full horseshoe and thus suggests the occurrence of chaotic dynamics. For smaller $\alpha$-values (case 1, see Fig. 3) our analysis does not show the existence of horseshoes. However, numerical simulations suggest that chaos arises in that case also.
To show that chaos arises indeed we compute the largest Lyapunov characteristic exponents \( LCE \) for the system. LCE measure the asymptotic exponential rate of divergence (resp. convergence) of two trajectories starting close to each other. Hence a positive LCE means that nearby trajectories separate exponentially as time goes by and the system exhibits sensitive dependence on initial states and therefore chaos. When \( \beta \) becomes sufficiently large we find positive LCE for both cases, 1 and 2, and also for the model with time varying beliefs about conditional variances. Thus, a high intensity of choice gives rise to chaotic dynamics. Fig. 4 shows the largest LCE for examples corresponding to case 1.

Figs. 5 and 6 show phase portraits in the \((x_t, x_{t-1})\)-plane and time series of strange attractors. We observe that the shape of the attractors may differ for the cases with constant and time varying beliefs about variances as long as the intensity of choice is not too high. For large \( \beta \) the projections of the attractors look very similar to the projection of the unstable manifold of \( E \) on the \((x_t, x_{t-1})\)-plane (cf. Figs. 3 and 7) for all cases.

Our analysis also gives insight into the underlying economic mechanism. When prices are close to the fundamental both predictors give good forecasts. Since the fundamental predictor is costly most agents choose the trend chasing predictor which causes prices to move away from the fundamental value. At a certain point the stabilizing force \( z \) gives enough weight to the fundamentalists to push prices back to the fundamental. This leads to an irregular switching between periods where prices are close to and periods where prices are far above or below the fundamental. Fig. 6(a) shows a time series with such an irregular switching between periods where prices are close to the fundamental steady state, followed by an upward trend, after some unstable phase away from the fundamental value returning back and getting stuck near the locally unstable fundamental steady state and later moving away again. Fig. 6 also shows time series of returns. We observe periods with high and low volatilities.

For prices to return to the fundamental value the intensity of choice has to be high enough. In our numerical examples we observe that, in the case where the beliefs about variances vary over time, the intensity of choice \( \beta \) has to be even larger, compared to the case with constant beliefs about variances (cf. Fig. 5). This stems from the fact that the same parameter values are used for \( \sigma^2 \) and \( \sigma^2_\delta \), respectively. Therefore, the total variance in the case of time varying variances, \( \sigma^2_i + \sigma^2_\delta \), is larger and hence the performance \( U_{hi} \) is lower, which has a similar effect as decreasing \( \beta \).

Note that the system exhibits a symmetry with respect to the \( m \)-axes and the positive octant of the state space is invariant. That is, taking initial conditions

\footnote{\( BH \) (1998) briefly explain the concept of Lyapunov exponents, see also the references there. Medio (1992, Chapter 6), also gives an introduction into that concept.}
where prices lie below the fundamental value results in a dynamics which is symmetrical, with negative price deviations. However, adding some noise to system, as prices come close to the fundamental the market might switch between 'optimistic' periods where prices are higher than the fundamental and 'pessimistic' periods with prices fluctuating below the fundamental value.

5. Conclusions

We have studied a simple asset pricing model following BH (1997b, 1998) where traders may choose between two predictors for future prices, a costly
Fig. 5. Projections of strange attractors on the \((x_t, x_{t-1})\)-plane.

Fig. 6. Time series of price deviations from the fundamental \(x_t\), differences in the fractions \(m_t\), and returns for (a) constant beliefs about variances, (b) time varying beliefs about variances. (b) also shows the time series of the variances \(\sigma_t^2\)
Fig. 7. Unstable manifold for case 2 ($\alpha = 0.75, \beta = 2000$): (a) the first 4 iterates of the segment $E40$, (b) the 4th iterate and the beginning and end (dotted line) of the 5th iterate of $A_0 D_0 A_1$. 

predictor based on fundamentals and a technical, trend chasing predictor. Choice of predictors is not only determined by their performance in the recent past but also by market conditions. Though the model is still very simple and stylized it possesses rich dynamics. As the intensity of choice to switch between predictors increases, different bifurcation routes to chaotic price dynamics are possible. They depend (among other things) upon the strength of the stabilizing force $\alpha$. One could interpret $\alpha$ as a parameter that determines the probability that traders do not choose the technical predictor when prices are too far away from the fundamental value, even when its recent performance was good. We have observed different periods in the market where prices switch between stable phases close to the fundamental and unstable periods with prices much higher or much lower than the fundamental value.

Introducing the parameter $\alpha$ as a stabilizing force has allowed for a two parameter bifurcation analysis focusing on codimension one bifurcation curves. Such bifurcations generally occur in higher dimensional systems also. Though the way we introduced this stabilizing force here is rather ‘ad hoc’ our analysis is useful. Similar codimension bifurcation routes may be expected in extensions of the model where the stabilizing force is modelled explicitly.

We further have introduced time varying beliefs about conditional variances of returns. The bifurcation routes to chaos differ in some details, however the global qualitative features of the price dynamics are similar to the case with constant beliefs about variances. In real markets traders would not only update their expectations about future returns but use observed data also to estimate variables such as variances of returns. Our analysis gives a justification to concentrate on the more tractable model with constant beliefs about variances, when trying to understand the behavior of financial markets.

The aim of the model is to understand stochastic properties of stock returns and trading volume observed in real financial data and the forces that account for these properties. Though statistical techniques are useful they are no substitute for a structural model in giving insight into the economic mechanism that may generate nonlinearity and observed fluctuations. Simple stylized versions of adaptive belief systems as presented in this paper also complement computer experiments like that of Arthur et al. (1997a, b) since they are able to analyze certain issues in an analytic framework. Such a scientific understanding is essential for a design of intelligent regulatory policy and also has practical value concerning risk management.

There are of course many ways to develop the model further, an extensive list is presented in BH (1997b) and Brock (1997). In a next step we will extend the model in order to catch empirical observed patterns in the dynamics of volatility and volume, such as autocorrelation functions of volatility of returns and trading volume and cross-autocorrelations between volatility measures and volume measures. A first attempt to calibrate the model to monthly IBM data was already done in BH (1997b) for a case with four different types of traders.
We have observed that the model generates time series of returns with periods of high and low volatilities. Though the time series of the actual model do not possess GARCH effects it seems promising that a further development of the model may generate such effects. Introducing more types may give price series with stochastic properties which are closer to real financial data. Also an improvement in the conditional rules for predictor choice which does not cause such an abrupt decline in prices as we have seen in some time series will be a step into this direction.

6. For further reading

Brock (1993).

Appendix

Proof of Proposition 1. Steady states $x$ are given by

$$Rx = n_2g x = \frac{1 - m^*}{2} g x,$$

where $m^*$ is the value of $m_t$ for $x_t = x_{t-2} = x^*$. It follows that $x = 0$ or

$$m^* = 1 - \frac{2R}{g} = \tanh \left[ \frac{\beta}{2} \left( \frac{g}{2a\sigma^2} x^*(g x^* - 2x^*) + x x^* - C \right) \right],$$

i.e.

$$x^* = \frac{2a\sigma^2 (\log (g/R - 1) + \beta C)}{\beta (g - 2) + 2a\sigma^2 x}.$$  (13)

Thus, besides the fundamental steady state $E_1$ there exist two non-fundamental equilibria $E_{2,3} = (\pm x^*, \pm x^*, \pm x^*)$ iff

$$\frac{\log (g/R - 1) + \beta C}{g(g - 2) + 2a\sigma^2 x} > 0 \text{ and } g > R.$$  

The Jacobian of the fundamental steady state $E_1$ has the eigenvalue $\lambda_1 = (g/2R)(1 + \tanh(\beta C/2))$, which lies in the interval $(0,1)$ iff $\beta < \beta^* = (1/C)\log[g/(g - R)]$, and a double eigenvalue $\lambda_2 = \lambda_3 = 0$.

For $g \geq 2R$, $\lambda_1 > 1$ (in that case $x^* > 0$ and therefore $x > x^*$, and $\beta^* < 0$).

In Fig. 1, $x^*$ and $\beta^*$ are the coordinates of $B$.  

Proof of Proposition 2. The characteristic polynomial of the Jacobian of the non-fundamental steady states $E_2$ and $E_3$ is given by

$$p(\lambda) = \lambda^3 - \left( 1 + \frac{g - 2a\sigma^2 x}{g} Z \right) \lambda^2 + (g - 1)Z,$$  (14)
where

\[ Z := \frac{\beta}{a^2} (g - R) x^2 = 2(g - R) \frac{\log(g/R - 1) + \beta C}{g(g - 2) + 2a\sigma^2 x}. \]  

(15)

Note that a change in \( x \) is locally equivalent to a change in the risk parameter \( a \) near all steady states.

At the pitchfork bifurcation value \( \beta = \beta^* \), \( x^* = 0 \) (\( Z = 0 \)) and \( p(\lambda) \) has a double eigenvalue 0 and an eigenvalue 1. For \( \beta \) slightly larger than \( \beta^* \) (\( Z \) slightly positive), \( p(\lambda) \) has three real, one negative and two positive eigenvalues inside the unit circle and the non-fundamental steady states are stable. Increasing \( \beta \), one of the two positive eigenvalues increases, the other one decreases. For a certain value of \( \beta \) they coincide resulting in a double real positive eigenvalue inside the unit circle, increasing \( \beta \) further, two complex conjugate eigenvalues occur. These eigenvalues might cross the unit circle at a value \( \beta_{11}(x) \), given by (19) and (20), respectively. In this case a Hopf bifurcation of the non-fundamental steady states occurs.

Let \( \lambda_1, \lambda_2, \lambda_3 \) denote the eigenvalues, where \( \lambda_{2,3} = A \pm Bi \). Since \( p(\lambda) \) has no linear term, \( \lambda_1 = -\lambda_{23}/(\lambda_2 + \lambda_3) \). Therefore,

\[
p(\lambda) = (\lambda - \lambda_1)(\lambda^2 - (\lambda_2 + \lambda_3)\lambda + \lambda_2\lambda_3)
\]

\[
= \left( \lambda + \frac{A^2 + B^2}{2A} \right)(\lambda^2 - 2A\lambda + A^2 + B^2)
\]

\[
= \lambda^3 + \left( \frac{A^2 + B^2}{2A} - 2A \right)\lambda^2 + \frac{(A^2 + B^2)^2}{2A}.
\]

(16)

**Hopf bifurcation of the non-fundamental steady states:** A Hopf bifurcation of the non-fundamental steady states occurs if \( A^2 + B^2 = 1 \). By comparing coefficients of (14) and (16) this yields

\[
1 + \frac{g - 2a\sigma^2 x}{g}Z = -\frac{1}{2A} + 2A \quad \text{and} \quad (g - 1)Z = \frac{1}{2A}.
\]

(18)

Eliminating \( A \) from these equations, we obtain that on the \((\beta, x)\)-plane the Hopf bifurcation curve is implicitly defined by

\[
\frac{(g^2 - 2a\sigma^2 x)(g - 1)}{g}Z^2 + (g - 1)Z - 1 = 0.
\]

(19)

Using (15), (19) can be rewritten as

\[
4(g^2 - 2a\sigma^2 x)(g - 1)(g - R)^2 \left( \log \left( \frac{g}{R} - 1 \right) + \beta C \right)^2
\]

\[
+ 2g(g - 1)(g - R)(g(g - 2) + 2a\sigma^2 x) \left( \log \left( \frac{g}{R} - 1 \right) + \beta C \right)
\]

\[
- g(g - 2) + 2a\sigma^2 x)^2 = 0.
\]

(20)
From (19) we further have
\[
Z = -\frac{g(g - 1) \pm \sqrt{g^2(g - 1)^2 + 4g(g - 1)(g^2 - 2a\sigma^2 \alpha)}}{2(g^2 - 2a\sigma^2 \alpha)(g - 1)},
\]
which is real for
\[
\alpha < \frac{1}{8a\sigma^2}g(5g - 1) =: \bar{\alpha}.
\]

On the \((\beta, \alpha)\)-plane the Hopf bifurcation curve \(\alpha(\beta)\) obtains its maximum at \(\bar{\alpha}\) (cf. Fig. 1).

**Flip bifurcation of the non-fundamental steady states:** A flip bifurcation of the non-fundamental steady states occurs for \(\lambda_1 = - (A^2 + B^2) / 2A = -1\). Comparing coefficients of \((14)\) and \((16)\) gives
\[
1 + \frac{g - 2a\sigma^2 \alpha}{g}Z = -1 + 2A \quad \text{and} \quad (g - 1)Z = A^2 + B^2.
\]

Eliminating \(A\) and \(B\), we obtain
\[
Z = \frac{2g}{g(g - 2) + 2a\sigma^2 \alpha}
\]
and plugging this expression into \((15)\) yields
\[
\beta = \beta_F = \frac{1}{C} \left( \frac{g}{g - R} \log \left( \frac{g}{R - 1} \right) \right).
\]

Plugging \(\beta_F\) into \((20)\) we obtain two solutions for \(\alpha\),
\[
\alpha_1 = -\frac{g(5g - 6)}{2a\sigma^2} < \alpha^* \iff g > 1,
\]
\[
\alpha_2 = \frac{g^2}{2a\sigma^2} = \alpha^{**} < \bar{\alpha} \iff g > 1.
\]

Since \(g > R \ (> 1)\) (otherwise non-fundamental steady states do not exist), the flip bifurcation line \(\beta = \beta_F\) intersects the Hopf bifurcation curve in \(C = (\beta_F, \alpha^{**})\) (see Fig. 1, recall that \(B = (\beta^*, \alpha^*)\)). Therefore, if \(\alpha \in (\alpha^*, \alpha^{**})\) the non-fundamental steady states are destabilized by a Hopf bifurcation at \(\beta = \beta_H(\alpha)\) defined by \((20)\), \(\beta_H(\alpha) < \beta_F\) in that case. If \(\alpha \in (\alpha^*, \bar{\alpha})\), \(\beta_F < \beta_H(\alpha)\), if \(\alpha > \bar{\alpha}\), \(\beta_H(\alpha) \notin \mathbb{R}\), thus the non-fundamental steady states are destabilized by a flip bifurcation at \(\beta = \beta_F\) if \(\alpha > \alpha^{**}\). □
A.1. Limits of the eigenvalues when approaching B

On the Hopf bifurcation curve (19), as \( x \to x^* \), \( g^2 - 2a\sigma^2 x \to 2g(g - 1) \), therefore for \( x \approx x^* \) (19) can approximately be written as
\[
2(g - 1)^2 Z^2 + (g - 1)Z - 1 = 0,
\]
hence \( \lim_{(g, x) \to B} Z = 1/(2(g - 1)) \) along the Hopf bifurcation curve. Thus, in the limit (14) reduces to
\[
\lambda^3 - \frac{3}{2}\lambda^2 + \frac{1}{2} = 0,
\]
which has zeros 1 (double) and \(-\frac{1}{2}\).

Approaching point B along the Hopf bifurcation curve the two complex conjugate eigenvalues on the unit circle therefore converge to 1 and the third eigenvalue to \(-\frac{1}{2}\).

On the pitchfork bifurcation line the fundamental and the non-fundamental steady states coincide. The eigenvalues of the steady state are 0 (double) and 1. On the line \( g(g - 2) + 2a\sigma^2 x = 0 \) (on the \((\beta, x)\)-plane this is a horizontal line through point \( B = (\beta^*, x^*) \), see Fig. 1) the characteristic Eq. (14) reduces to a quadratic equation with zeros \( \pm 1 \). As \( x \) approaches \( x^* \) (from above) one eigenvalue tends to infinity.

A.2. Computations of the origin points of Arnold tongues

On the Hopf bifurcation curve \( A^2 + B^2 = 1 \), using (18) we have
\[
\arg \lambda = \arctan \frac{B}{A} = \arctan \sqrt{1 - \frac{A^2}{A}}
\]
\[
= \arctan \left[ 2g(g - 1)Z \sqrt{1 - \frac{1}{4(g - 1)^2 Z^2}} \right].
\]
Hence plugging expression (15) in for \( Z \) we obtain
\[
\tan \left( 2\pi \frac{p}{q} \right) = \frac{\sqrt{16(g - 1)^2(g - R)^2(\log(g/R - 1) + \beta C)^2 - (g(g - 2) + 2a\sigma^2 x)^2}}{g(g - 2) + 2a\sigma^2 x},
\]
resp.
\[
16(g - 1)^2(g - R)^2 \left( \log \left( \frac{g}{R} - 1 \right) + \beta C \right)^2 - (g(g - 2) + 2a\sigma^2 x)^2 = (g(g - 2) + 2a\sigma^2 x)\tan^2 \left( 2\pi \frac{p}{q} \right), \tag{21}
\]
For any \( p/q \) we obtain the origin points of the Arnold tongues on the \((a, b)\)-plane, as the solutions of (20) and (21). Note that four angles \( \arg \lambda \) give the same value for \( \tan^2(\arg \lambda) \). From the second equation of (18) we conclude that the eigenvalues corresponding to the Hopf bifurcation curve have non-negative real part (when moving away from \( B \) along the Hopf bifurcation curve (20) the two complex conjugate eigenvalues move on the unit circle from \(+1\) to \( \pm i \) as \( \beta \to \infty \)). Therefore, \( p \) and \( q \) have to be chosen such that \( 2p/(p/q) \in [0, \pi/2] \) \((p/q)\) reduced) to get a \( p:q \) resonance. From (18) it also follows that \( p/q \) increases monotonically when \( b \) is increased.

**Proof of Proposition 3.** It is easy to check that the eigenvalues of the fundamental steady state \( E_1 \) are \((g/2R)(1 + \tanh(\beta C/2)) \) and \( 0^{(2)} \) (which are the same as in the case of constant beliefs about variances) and the two stable eigenvalues \( w_\sigma, w_\mu \in (0, 1) \).

The characteristic polynomial of the Jacobian \( J \) of the non-fundamental steady states is given by

\[
\det (J - \lambda I) = (w_\sigma - \lambda)(w_\mu - \lambda) \det (J_c - \lambda I),
\]

where \( J_c \) is the Jacobian for the case with constant beliefs about variances when \( \sigma^2 \) is replaced by \( \sigma_0^2 \). Therefore, in addition to the stable eigenvalues \( w_\sigma \) and \( w_\mu \), we have the same eigenvalues as in the case of constant beliefs about variances. \( \square \)

**Proof of Proposition 4.** We proceed analogous as in the proof of Lemma 4 in BH (1998). We have

\[
m_t = \begin{cases}
+1 & \text{if } \frac{g}{2d(\sigma_1^2 - \sigma_0^2)} x_t^2 (g x_t^2 - 2 x_t) + x_t^2 C,
-1 & \text{if } \frac{g}{2d(\sigma_1^2 - \sigma_0^2)} x_t^2 (g x_t^2 - 2 x_t) + x_t^2 \leq C.
\end{cases}
\]

(22)

The unstable eigenvector of the fundamental steady state is

\[
\left( \frac{R}{g} \right)^2 \cdot \left( \frac{R}{g} \right)_{1,0} \cdot \left( \frac{R - (g/R)}{w_\mu - (g/R)} \right)\).
\]

Since \( x_{t+1} = (g/2R) x_t (1 - m_t) \), \( x_{t+1} = (g/R) x_t \) if \( m_t = -1 \) and \( x_{t+1} = 0 \) if \( m_t = 1 \).
Take an initial state \( x_0 = \varepsilon(g/R)^2, x_{-1} = \varepsilon g/R, x_{-2} = \varepsilon, \varepsilon > 0 \). The expression determining whether \( m_t = -1 \) or \(+1\) is

\[
C_t := \varepsilon^2 \left( \frac{g}{R} \right)^{2t} \left[ \frac{g}{2a(\sigma_{1}^2 + \sigma_{3}^2)} (R^2 - 2g) + \alpha \left( \frac{g}{R} \right)^2 \right]
\]

and note that \( g/(2a(\sigma_{1}^2 + \sigma_{3}^2)) \leq g/2a\sigma_{3}^2 \). If \( g > R \) and the expression in brackets is positive there is some smallest \( T > 0 \) such that \( C_T > C \), so that \( m_T = +1, x_t = 0 \ \forall t \geq T + 1 \), and \( m_t = -1 \ \forall t \geq T + 3 \). Hence in this case the unstable manifold is bounded and all orbits converge to \( E \). \( \square \)

**A.3. Numerical analysis of the unstable manifold of the fundamental steady state \( E \) for large but finite \( \beta \)**

Numerical computations suggest that the situation is similar as in BH (1997a) and in Goeree and Hommes (1999). However, our system is three dimensional, therefore the analysis of the unstable manifold of \( E \) for large but finite \( \beta \) is much more delicate than for the two-dimensional models in those papers. An approximation of the unstable manifold is not obtained by connecting the segments of \( F_2^s(EA_0) \) just by straight segments. The limiting manifold will contain line segments lying on the planes \( \{ m = -1 \} \) and \( \{ m = 1 \} \) which are connected by vertical line segments. Of course, \( A_1 \) and \( A'_1 \) are connected by a vertical line segment. Note that for \( \beta = \infty \), \( m_t \) in (11) is not defined if equality holds on the right-hand side of (12). We have defined \( m_t = -1 \) in that case, however, we could have taken any other value between \(-1\) and \(1\). One could interpret the vertical line segment \( A_1A'_1 \) as image of \( A_0 \). For finite but sufficiently large \( \beta \) there is a segment \( A_0A_0 \in EA_1 \) whose image is close to the vertical segment \( A_1A'_1 \). Further analysis will be carried out numerically.

We find that, depending on \( \alpha \), two cases are possible. Figs. 3 and 7 show the first six resp. five iterates of the unstable manifold for \( g = 1.2, R = 1.01, a = 10, \sigma^2 = 0.1, C = 1 \).\(^{13}\) We analyze case 1 for the numerical example \( \alpha = 0.65 \), case 2 for \( \alpha = 0.75 \).

**Case 1:** For \( \beta \) sufficiently large the unstable manifold is arbitrary close to (see Fig. 3)

\[
F(EA_0) = EA_1,
\]

\[
F^2(EA_0) = F(EA_0) A'_1 A_2,
\]

---

\(^{13}\)The numerical computations of the unstable manifolds were carried out by plotting 6000 equally spaced points on a small unstable eigenvector according to the \( \lambda \)-lemma (see Guckenheimer and Holmes, 1986, p. 247). (To get accurate pictures, we further iterated segment \( A_1A'_1 \) and some small subsegments on \( A_1A'_1 \) taking even more, namely, 30,000 points.)
\[ F^3(EA_0) = F^2(EA_0)B_1B_2A_2A_3, \]
\[ F^4(EA_0) = F^3(EA_0)C_1C_2C_3C_4A_4, \]
\[ F^5(EA_0) = F^4(EA_0)A_3A_4E, \]
\[ F^6(EA_0) = F^5(EA_0)A_4E. \]

Except for points \( A_i \) (the points which define the segments of the unstable manifold in the case \( \beta = \infty \)), \( X' \) denotes points lying on the plane \( \{ m = -1 \} \) having the same \( x \) and \( y \)-coordinates as \( X \in \{ m = 1 \} \). \( F \) denotes the limit of \( F_\beta \) as \( \beta \to \infty \).

**Case 2**: Fig. 7(a) shows \( EA_0A'_0 \) and the first three iterates of the segment \( A_0A'_0A_1 \) (first four iterates of \( EA_0 \)), Fig. 7(b) shows the fourth iterate of the segment \( A_0A'_0A_1 \) and the first and last part (dotted line) of its fifth iterate. The first three iterates of \( EA_0 \) are as in case 1. For further iterations we obtain

\[ F^4(EA_0) = F^3(EA_0)C_1C_5C_6C_2C_7C_8C_3C_4A_4, \]
\[ F^5(EA_0) = F^4(EA_0)D_1D_2D_3D_4D_5D_6D_7A_3D_7 \]
\[ D_8D_9D_{10}A'_1A_1D'_1D'_2E. \]

For the 6th iterate we only roughly sketch the approximative way of the unstable manifold:

\[ F^5(EA_0)A_4D_1 \cdots A_0A'_0A_2 \cdots A_1A_3A_2 \cdots A_1A_0 \cdots D_1 \]
\[ \overleftarrow{A_4F^4(EA_0)}EA_4E. \]

\( F^4(EA_0) \) denotes the reverse direction of \( F^4(EA_0) \). When moving between the points (close to) \( A_2 \) and \( A'_2 \in \{ m = 1 \} \) parts of the connecting segment lie in the plane \( \{ m = -1 \} \), similar as for \( F^4(EA_0) \) and \( F^5(EA_0) \). Though this description is rather rough it shows that the 6th iterate of \( EA_0 \) contains segments which are expanded and folded along (close to) the unstable segment \( EA_0 \). This suggests that for sufficiently large \( \beta \)-values, \( F_\beta \) has as a full horseshoe over a rectangular region containing a subsegment of \( EA_0 \).

Note that in both cases the unstable manifold gets arbitrarily close to the plane \( S \) which is contained in the stable manifold. That is, the unstable manifold first moves away from the fundamental steady state but returns close to it later on (cf. Proposition 4).

In case 2 the situation seems to be similar as in BH (1997a) and suggests the occurrence of homoclinic bifurcations of period \( N (N \geq 6) \) saddle points and the associated occurrence of strange attractors (cf. BH, 1997a, Section 3.4; Viana, 1993). Computing Lyapunov characteristic exponents (LCE), which are positive for sufficient high \( \beta \)-values, gives further evidence of chaotic dynamics (see Section 4).
Case 1 appears to be similar to the case of Fig. 8(e–f) in Goeree and Hommes (1999). The fourth iterate of $EA_0$ does not contain the segment $C_7C_7C_8$ as in case 2. Indeed, when $\alpha$ is decreased, $C_7$ and $C_8$ approach each other turning case 2 into case 1 for a certain value of $\alpha$. It is not clear whether a horseshoe construction as in case 2 is possible. Numerically, all points of the segment $A_1A_1A_2$ are mapped to the steady state $E$ after four iterates. The 6th iterate might continue to move along $EA_0$. Computing LCE shows that chaos arises in that case also (see Section 4).

References


