A discrete and symmetric price adjustment process on the simplex

Jan Tuinstra*

Department of Quantitative Economics and CeNDEF, Faculty of Economics and Econometrics, University of Amsterdam, Roetersstraat 11, 1018 WB Amsterdam, The Netherlands

Accepted 30 April 1999

Abstract

We study the tâtonnement process in an exchange economy with three commodities and three agents. We argue that for the multiplicative tâtonnement process the simplex is a natural normalization rule. We analyse the influence of symmetry in preferences and endowments on the dynamics of the tâtonnement process. Depending on this symmetry there are different bifurcation routes to chaos. In particular, coexisting attractors may emerge when the symmetric equilibrium price vector becomes unstable. This coexistence of attractors is robust in the sense that it also exists for close-by asymmetric tâtonnement processes. © 2000 Elsevier Science B.V. All rights reserved.

JEL classification: D51; E30

Keywords: Tâtonnement process; Chaotic dynamics; Bifurcations with symmetry

1. Introduction

In recent years there has been a renewed interest in the discrete tâtonnement process. It has been shown that the tâtonnement process does not always

*The research for this paper was done as part of the project ‘Dynamics of nonlinear price adjustment processes’ funded by the Netherlands Organisation for Scientific Research (NWO). Helpful comments by Cars Hommes, Leo Kaas and Claus Weddepohl are gratefully acknowledged.

* Tel.: 0031-20-525-4117; fax: 0031-20-525-4349.

E-mail address: tuinstra@fee.uva.nl (J. Tuinstra)
converge to an equilibrium price vector and that it can exhibit periodic or chaotic behaviour (Bala and Majumdar, 1992; Day and Pianigiani, 1991; Goeree et al., 1998; Saari, 1985; Weddepohl, 1995). In this paper we concentrate on a two-dimensional discrete tâtonnement process. Our approach differs from previous contributions in that we choose the simplex as a price normalization rule instead of taking some commodity as numeraire. We argue that the simplex is a natural price normalization rule since it is an invariant space of the multiplicative adjustment process, just as the unit sphere is an invariant space of the continuous time linear adjustment process.

We are interested in the dynamics of the tâtonnement process for all possible economies. However, for a global analysis general equilibrium models have too many free parameters and therefore as a first step towards a systematic analysis, we focus on symmetric tâtonnement processes. We apply the literature on dynamical systems with symmetries (see Golubitsky et al., 1988) to study these symmetric tâtonnement processes. We find that different routes to strange attractors can occur. Furthermore, the type of bifurcation that occurs depends upon the symmetry of the tâtonnement process.

Since some economic models have a natural symmetry, our study of symmetric dynamical systems is important from a methodological viewpoint. Examples of symmetries arising naturally in economic models are the cyclical symmetry in overlapping generations models (see Tuinstra and Weddepohl, 1999) and the reflectional symmetry in the cobweb model with heterogeneous beliefs and linear supply and demand curves (see Brock and Hommes, 1997).

After reviewing the tâtonnement process in Section 2 we present our model and show results from numerical simulations in Section 3. In Section 4 we introduce symmetry. In Section 5 we look at the generic local bifurcations of the equilibrium price vector given a certain symmetry. Among other things we see that the so-called symmetry breaking bifurcations can occur. In Section 6 the global dynamics of a symmetric tâtonnement process is studied. In Section 7 an asymmetric tâtonnement process which is in some sense close to a symmetric process is analysed to illustrate the relevance of studying symmetric processes for processes that have no nontrivial symmetries. Section 8 summarizes. Proofs of the main results are given in the appendix.

2. The tâtonnement process

We study a price adjustment process in a competitive exchange economy. Let there be $m$ consumers and $n$ commodities. Each consumer takes prices as given and solves the optimization problem

$$\max_{x_{i1}, \ldots, x_{in}} \ U_i(x_{i1}, \ldots, x_{in}), \quad i = 1, \ldots, m,$$  \hspace{1cm} (1)
subject to his or her budget constraint

\[ \sum_{j=1}^{n} p_j x_{ij} \leq \sum_{j=1}^{n} p_j w_{ij}, \quad i = 1, \ldots, m, \quad (2) \]

where \( U_i(\cdot) \) is a strictly quasi-concave, strictly monotonic utility function of consumer \( i \), \( x_{ij} \) the amount of good \( j \) consumed by consumer \( i \), \( p_j \) the price of good \( j \) and \( w_{ij} \) the initial endowment of good \( j \) owned by consumer \( i \). Let \( x_{ij}^H = x_{ij}(p, w_i) \) be the utility maximizing demand for good \( j \) by consumer \( i \), where \( p = (p_1, \ldots, p_n) \) is the price vector and \( w_i = (w_{i1}, \ldots, w_{in}) \) the vector of initial endowments of consumer \( i \).

Aggregate excess demand for good \( j \) is

\[ z_j(p) = \sum_{i=1}^{m} x_{ij}(p, w_i) - \sum_{i=1}^{m} w_{ij}, \quad j = 1, \ldots, n. \quad (3) \]

The economy is in equilibrium at the equilibrium price vector \( p^* \) if \( z_j(p^*) = 0 \) for all \( j \). The excess demand functions satisfy Walras’ Law: \( \sum_{j=1}^{n} p_j z_j(p) = 0 \), and are homogeneous of degree zero in prices, i.e. \( z_j(\lambda p) = z_j(p), \forall \lambda > 0 \). Differentiating this last equality with respect to \( \lambda \) at \( \lambda = 1 \) gives the Euler equation: \( \sum_{k=1}^{n} p_k \partial z_j(p)/\partial p_k = 0 \).

It is commonly believed (for example expressed by the notion of the law of supply and demand) that there are economic forces that drive the economy towards equilibrium. We study the well-known tâtonnement process which is the simplest way to model these equilibrating forces and which can serve as a first approximation to more realistic adjustment processes. This process can be summarized by the following two characteristics.

- Trade occurs if and only if all markets are in equilibrium.
- If the economy is not in equilibrium at a certain price vector, prices are adjusted according to the following rule. If there is excess demand for a good its price goes up, if there is excess supply for a good its price goes down and if the market for a good is in equilibrium its price remains the same.

The tâtonnement process can be formalized as a discrete or as a continuous dynamical system

\[ p_{j,t+1} - p_{j,t} = F_j(p_{j,t}, z_j(p_t)), \quad j = 1, 2, \ldots, n, \quad (4) \]

\[ \frac{dp_j(t)}{dt} = F_j(p_j(t), z_j(p(t))), \quad j = 1, 2, \ldots, n, \quad (5) \]

with \( F_j(p_t, 0) = 0 \) and \( \partial F_j(p_t, x)/\partial x > 0 \).

\[ ^1 \text{Here} \ x' \text{ denotes the transpose of the vector (or matrix) } x. \]
The following question arises: who adjusts the prices? In the perfect competitive economy described above all agents take prices as given in determining their optimal consumption. If the economy is not in equilibrium there are profit opportunities for economic agents since they can benefit from charging a higher price for goods in excess demand. Therefore, it can be argued that perfect competition is only a good description of economic behaviour if the economy is in equilibrium (see Arrow, 1959). The traditional solution to this problem is the introduction of an auctioneer who sets prices. The tâtonnement process now works as follows: the auctioneer states an arbitrary price vector and collects information on the aggregate excess demands given this price vector. He then adjusts the price vector according to the rule stated above. This process is repeated until an equilibrium price vector is found. In this interpretation the discrete adjustment process (4) seems to be the most appropriate way to describe the tâtonnement process. Obviously the tâtonnement process with an auctioneer is a very crude approximation of the adjustment processes in reality.

The continuous tâtonnement process has been extensively studied in the literature (see Arrow and Hahn, 1971). The most important finding is that a sufficient condition for global stability is that all goods are gross substitutes, that is: \( \frac{\partial z_j}{\partial p_k}(p) > 0, \forall j, \forall k \neq j, \forall p. \) This condition also implies that the equilibrium price vector is unique.

This is a rather severe condition. There are some well-known examples (e.g. Scarf, 1960) for which this condition does not hold and for which the continuous tâtonnement process does not converge to an equilibrium price vector. Notice that the stability of the equilibrium price vector only depends on the aggregate demand functions and not on the specification of the adjustment rule \( F(\cdot). \)

For the discrete tâtonnement process the situation is even worse. Stability of the competitive equilibrium is determined by the properties of the aggregate excess demand functions as well as the specific form of the adjustment rule.

3. The model

In this section we introduce the model. In Section 3.1 we discuss the problem of finding an appropriate normalization rule and we suggest that for the multiplicative tâtonnement process normalizing prices on the simplex is the most natural normalization rule. In Section 3.2 we specify our model and in Section 3.3 we show some typical results obtained from numerical experiments.

\footnote{For small values of \( n \) this assumption can be relaxed: for \( n = 2 \) the tâtonnement process is always stable and for \( n = 3 \) a sufficient condition for stability of the tâtonnement process \( dp/dt = z(p) \) is the so-called weak law of market demand, which states that \( \sum_{i=1}^{n} (\frac{\partial z_i}{\partial p_i}) (p) < 0 \) for all \( p \) (Keenan and Rader, 1985).}
3.1. Price normalization

The tâtonnement process (4) is a discrete $n$-dimensional dynamical system. From the homogeneity of the aggregate excess demand functions we have that if $p^*$ is an equilibrium price vector then so is $z p^*$ for any $z > 0$. Instead of an equilibrium price vector we have a ray of equilibrium price vectors. None of these equilibria can be locally stable since every neighbourhood in $\mathbb{R}^n_+$ of such an equilibrium point contains other equilibrium points. The tâtonnement process then seems to be ill-defined.

The tâtonnement process can be restricted to an $n - 1$ dimensional subspace of $\mathbb{R}^n_+$ by imposing an extra condition on the prices. Such a price normalization rule $g(p) = c$, where $c$ is a constant, defines an $n - 1$ dimensional manifold in $\mathbb{R}^n_+$ and has to be chosen in such a way that the intersection of this manifold with any ray of equilibrium price vectors consists of exactly one point. This intersection point is the equilibrium price vector of the tâtonnement process restricted to this manifold. Some well-known price normalization rules are $g(p) = p_k$ for some fixed $k$, $g(p) = \sum_{j=1}^n p_j$ and $g(p) = \sum_{j=1}^n p_j^2$, all with $c = 1$, respectively, corresponding to one good being chosen as numeraire, prices on the unit simplex and prices on the unit sphere. There is no economic motivation to choose a particular price normalization rule. In special cases some normalization rule seems obvious (e.g. money as a numeraire in a monetary economy), but no normalization rule is appropriate a priori.

Having chosen a normalization rule it subsequently has to be implemented. Assume $g(p) = c$ can be rewritten as $p_k = h_k(p_1, \ldots, p_{k-1}, p_{k+1}, \ldots, p_n)$ for some $k$. The tâtonnement process restricted to $g(p) = c$ then becomes

\[
\begin{align*}
    p_{j,t+1} &= p_{j,t} + F_j p_{j,t}, z_j p_1, \ldots, p_{k-1,t}, h_k(\cdot), p_{k+1,t}, \ldots, p_n, \\
    p_{k,t} &= h_k(p_1, \ldots, p_{k-1,t}, p_{k+1,t}, \ldots, p_n).
\end{align*}
\]

(6)

So the dimension of the dynamical system is reduced by one to $n - 1$. In fact the $k$th equation of motion has been deleted. From Walras’ Law we have that if $n - 1$ markets are in equilibrium the $n$th market must also be in equilibrium, so an equilibrium price vector of (6) is also an equilibrium price vector of (4). The dynamics of the tâtonnement process however depend crucially on this reduction. Which equation is deleted obviously influences the dynamical system. The tâtonnement process with $p_1$ as numeraire may have different stability properties than the process with $p_n$ as numeraire. There is ‘goods discrimination’ in the sense that the price of one, arbitrarily chosen, good is treated differently than the prices of the other goods.

In some cases there is a nice solution to this last problem. If for some tâtonnement process $p_{t+1} = f(p_t)$ there exists a price normalization rule $g(\cdot)$ such that $g(f_1(p), f_2(p), \ldots, f_n(p)) = g(p_1, p_2, \ldots, p_n)$ then the tâtonnement process $f$ leaves the manifold defined by $g(\cdot) = c$ invariant. In this case the $n$ equations of
motion of the tâtonnement process are not independent. In fact, there is one redundant equation and any equation can be replaced by the normalization rule without affecting the dynamical system: the systems (6) and (4) are equivalent and no information is lost by imposing \( g(.) = c \).

For the continuous time case such a normalization rule exists for the linear tâtonnement process \( \frac{dp_i}{dt} = \lambda z_j(p(t)) \). We have: \( \frac{d}{dt} \sum p_i^2(t) = 2 \sum p_i(t) \frac{dp_i}{dt} = 2 \lambda \sum p_i(t) z_j(p(t)) = 0 \), due to Walras’ Law, so the linear tâtonnement process leaves the sphere invariant.

In the case of the discrete tâtonnement process a similar price normalization rule exists for the so-called multiplicative tâtonnement process

\[
p_{j,t+1} = p_{j,t}(1 + \lambda z_j(p_t)), \quad j = 1, \ldots, n,
\]

where \( \lambda \) is the speed of adjustment and is assumed to be the same for all markets.\(^3\) Summation of the prices gives (using Walras’ Law)

\[
\sum_{j=1}^{n} p_{j,t+1} = \sum_{j=1}^{n} p_{j,t}(1 + \lambda z_j(p_t)) = \sum_{j=1}^{n} p_{j,t} + \lambda \sum_{j=1}^{n} p_{j,t} z_j(p_t) = \sum_{j=1}^{n} p_{j,t}.
\]

Hence the multiplicative tâtonnement process leaves the simplex invariant.

It can be argued that (7) is a reasonable way to model price adjustments. It states that the rate of change of the price of a good is proportional to its excess demand and it is homogeneous of degree one in prices. From now on we shall use (7) to study the dynamics of price adjustments.\(^4\)

3.2. Model specifications

In order to analyse the global dynamics of our tâtonnement process we have to specify utility functions and endowments. We will study an economy with three commodities and three consumers that have Constant Elasticity of Substitution (CES) utility functions, i.e. the utility function of consumer \( i \) is

\[
U_i(x_{i1}, x_{i2}, x_{i3}) = \left( \sum_{j=1}^{3} \alpha_{ij} x_{ij}^\rho \right)^{1/\rho}, \quad \rho < 0, \quad i = 1, 2, 3.
\]

\(^3\)This is not a restrictive assumption since we can always normalize the unit of goods such that the speeds of adjustment are equal across markets.

\(^4\)For the sake of simplicity we have not imposed a nonnegativity constraint on the prices. If the speed of adjustment is not too large, say \( \lambda < \hat{\lambda} \), then for initial price vectors chosen not too close to the border of the simplex the tâtonnement process stays on the simplex. If \( \lambda > \hat{\lambda} \) then for almost all initial price vectors the price of some good becomes negative in finite time and the tâtonnement process breaks down. However, all kinds of interesting dynamics can be observed for \( \lambda < \hat{\lambda} \).
The corresponding individual demand functions are
\[ x_{ij}(p_1, p_2, p_3) = \frac{(z_{ij}/p_j)^\sigma}{\sum_{k=1}^{3} p_k (z_{ik}/p_k)^\sigma} \sum_{j=1}^{3} p_j w_{ij}, \quad i,j = 1,2,3, \] (9)
with \( \sigma = 1/(1 - \rho) \) the (constant) elasticity of substitution. The individual demand functions are homogeneous of degree zero in prices \( p \) and in preference parameters \( z_{ij}, z_{i2} \) and \( z_{i3} \) and homogeneous of degree one in initial endowments \( w_i \). Under the assumption that \( z_{ii} > 0 \) we can take, without loss of generality, \( z_{ii} = 1 \).

We assume that the initial endowments are of the following form: \( w_{ij} = 1 \) if \( i = j \) and \( w_{ij} = 0 \) if \( i \neq j \). So each consumer supplies exactly one good and is the sole supplier of that good. This restriction is made for computational simplicity and does not influence the qualitative results. The multiplicative price adjustment process now becomes
\[
\begin{align*}
    p_{1,t+1} &= p_{1t} \left(1 + \lambda \left[ \sum_{i=1}^{3} \frac{p_{1i}(z_{1i}/p_{1i})^\sigma}{\sum_{j=1}^{3} p_{j}(z_{ij}/p_{j})^\sigma} - 1 \right] \right), \\
    p_{2,t+1} &= p_{2t} \left(1 + \lambda \left[ \sum_{i=1}^{3} \frac{p_{1i}(z_{2i}/p_{2i})^\sigma}{\sum_{j=1}^{3} p_{j}(z_{ij}/p_{j})^\sigma} - 1 \right] \right), \\
    p_{3,t+1} &= p_{3t} \left(1 + \lambda \left[ \sum_{i=1}^{3} \frac{p_{1i}(z_{3i}/p_{3i})^\sigma}{\sum_{j=1}^{3} p_{j}(z_{ij}/p_{j})^\sigma} - 1 \right] \right). \quad (10)
\end{align*}
\]
This is the dynamical system under study. The parameters of the model are \( z_{ij}, \sigma \) and \( \lambda \). We will study the behaviour of the dynamical system for given preference parameters \( z_{ij} \) and substitution of elasticity \( \sigma \), as \( \lambda \) increases. We focus on \( z_{ij} \in (0,1] \) and \( \sigma \in (0,1] \).

### 3.3. Some typical numerical simulations

Before analysing our tâtonnement process in more detail we present some numerical results. We are interested in the long-run behaviour of our dynamical system: do prices settle down to an equilibrium price vector or is the discrete tâtonnement process unstable and can prices fluctuate forever? To describe the long-run behaviour of the dynamical system we use the following definition (Guckenheimer and Holmes, 1983, p. 36).

**Definition 1.** A compact set \( A \) is called an attractor of a dynamical system \( f \) if:

1. \( A \) is invariant with respect to \( f: f(A) = A \).
2. \( A \) is attracting, that is there is an open neighbourhood \( U \) of \( A \) such that the trajectory of every initial state \( p \in U \) converges to \( A \).
3. \( A \) has a dense orbit.

---

Fig. 1. Attractors of the tâtonnement process with rotational symmetry. (a) $x_{12} = x_{23} = x_{31} = 1$, $x_{13} = x_{21} = x_{32} = \frac{1}{3}$, $\sigma = \frac{1}{4}$ and $\lambda = 3.11$. (b) $x_{12} = x_{23} = x_{31} = 1$, $x_{13} = x_{21} = x_{32} = \frac{1}{3}$, $\sigma = \frac{1}{4}$ and $\lambda = 8.45$.

Fig. 2. Attractors of the tâtonnement process with reflectional symmetry. (a) $x_{12} = x_{21} = \frac{1}{3}$, $x_{13} = x_{23} = x_{31} = \frac{1}{3}$, $\sigma = \frac{1}{10}$ and $\lambda = 32$. (b) $x_{12} = x_{21} = \frac{1}{10}$, $x_{13} = x_{23} = x_{31} = x_{32} = \frac{1}{3}$, $\sigma = \frac{1}{10}$ and $\lambda = 44.2$.

The simplest example of an attractor is a locally stable fixed point. Another simple example is an attracting periodic orbit. In that case we have $f^n(p) = p$ and $f^k(p) \neq p$, $k < n$. However, more complicated quasi-periodic attractors or attractors with a fractal structure can arise. This last type of attractors is sometimes called strange or chaotic since the dynamics on these attractors may be unpredictable and exhibit sensitive dependence on initial conditions.

Some typical examples of attractors of the multiplicative tâtonnement process (10) are shown in Figs. 1–3. All these attractors were obtained by iterating (10).

---

5 To plot the pictures in two dimensions we have transformed the simplex in $\mathbb{R}^3$ into a triangle in $\mathbb{R}^2$. This transformation is also used in the proof of Proposition 5 (in the appendix).
for different values of the preference parameters \(a_{ij}\), the elasticity of substitution \(\sigma\) and the speed of adjustment \(\lambda\).

Notice that the attractors have a certain symmetry. In particular, the pictures in Fig. 1 have a rotation symmetry: rotating the pictures over an angle of \(\frac{2}{3}\pi\) results in exactly the same pictures. The pictures in Fig. 2 have a reflection symmetry: reflection of the pictures in Fig. 2 in the line with \(p_1 = p_2\) results in the same pictures. The attractors in Fig. 3 have a rotation as well as a reflection symmetry: the pictures are invariant both to a rotation over \(\frac{2}{3}\pi\) and to a reflection in the line with \(p_1 = p_2\).

The prices move erratically across these chaotic attractors. The attractors show the long-run behaviour of the prices of the tâtonnement process. From these simulations it is clear that for many parameter settings the tâtonnement process will not settle down to an equilibrium price vector but will keep on fluctuating forever. In the subsequent sections we will investigate bifurcations routes leading to these strange attractors. Symmetry will play an important role in characterizing these bifurcation routes.

4. Symmetry

The number of parameters of our model (10) is 20 (\(a_{ij}, w_{ij}, \lambda\) and \(\sigma\)). To study the global dynamics of this model in a transparent manner we have to reduce this number. A first reduction is achieved by normalizing the preference parameters, such that \(a_{ii} = 1\), and by taking endowments \(w_{ii} = 1\) and \(w_{ij} = 0\) for \(i \neq j\). This leaves us with 8 parameters. The simplex has both a rotation and a reflection symmetry. Therefore, it seems to be natural to start a systematic investigation of the global dynamics by firstly focusing on symmetric
endowments and preferences. In Section 7 we discuss properties of close-by asymmetric cases. There is a large literature on symmetry in dynamical systems (see for example Golubitsky et al., 1988). We will apply concepts and results from this literature to our multiplicative tâtonnement process.

Consider the following dynamical system on the two-dimensional simplex:

\[ p_{t+1} = f(p_t), \quad p_t \in S^2 = \left\{ p \in \mathbb{R}_+^3 \mid \sum_{j=1}^3 p_j = 3 \right\}. \]  

(11)

A linear transformation \( M : \mathbb{R}^3 \to \mathbb{R}^3 \) is called a symmetry of \( f \), if \( M \) and \( f \) commute, that is, if

\[ M \circ f = f \circ M. \]  

(12)

The simplex is invariant with respect to any permutation of prices, so the group of symmetries that is interesting to study for the tâtonnement process on the simplex is the group of permutation matrices \( D_3 = \{ M_{123}, M_{132}, M_{321}, M_{213}, M_{312}, M_{231} \} \), with

\[
M_{ijk} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} p_i \\ p_j \\ p_k \end{pmatrix}.
\]

Of course \( M_{123} = I \) is a trivial symmetry of all dynamical systems. Apart from \( I \) and \( D_3 \) there are four different subgroups of \( D_3 \), \( \{ I, M_{132} \}, \{ I, M_{321} \}, \{ I, M_{213} \} \) and \( \{ I, M_{312}, M_{231} \} \). The first three of these are the so-called \( Z_2 \) symmetries: they correspond to a permutation of two of the three prices or equivalently a reflection in one of the symmetry axes of the simplex. These symmetry axes are: \( l_1 = \{ p \in S^2 | p_2 = p_3 \}, \) \( l_2 = \{ p \in S^2 | p_1 = p_3 \} \) and \( l_3 = \{ p \in S^2 | p_1 = p_2 \} \). The last subgroup is the \( R_3 \) symmetry and corresponds to a cyclical permutation of the three prices, or equivalently a rotation over \( \frac{2\pi}{3} \).

We can now define the symmetry group of a price adjustment process \( f \) on the simplex as follows: \( \Gamma_f = \{ M \in D_3 | M \circ f = f \circ M \} \). By choosing suitable preferences and endowments we will study a price adjustment process with a \( Z_2 \) symmetry, one with a \( R_3 \) symmetry and a price adjustment process with both symmetries, which is equivalent to symmetry group \( D_3 \). So we study tâtonnement processes with three different symmetry groups: \( \Gamma_f = \{ I, M_{213} \}, \Gamma_f = \{ I, M_{312}, M_{231} \} \) and \( \Gamma_f = D_3 \). These symmetry groups correspond respectively to tâtonnement processes that are invariant with respect to a reflective, a cyclic or all possible permutations of prices.

**Definition 2.** The fixed point subspace \( \text{Fix}(M) \) of a symmetry \( M \) is the set

\[ \text{Fix}(M) = \{ p \in S^2 | Mp = p \}. \]
The fixed point subspace of the symmetry \( M \) plays an important role in the bifurcations of the equilibrium price vector in a symmetric tâtonnement process. \( \text{Fix}(M) \) is a linear subspace that consists of all points of the simplex that are invariant under symmetry \( M \). \( \text{Fix}(M) \) is a line if \( M \) is a reflectional symmetry and \( \text{Fix}(M) \) is a point if \( M \) is a rotational symmetry. For example \( \text{Fix}(M_{213}) = l_3 \) and \( \text{Fix}(M_{312}) = (1, 1, 1) \). It can be easily seen that if \( M \) is a symmetry of \( f \), that is \( M \in \Gamma_f \), then \( \text{Fix}(M) \) is invariant under \( f \), since for \( p \in \text{Fix}(M) \) we have

\[
\mathbf{f}(p) = \mathbf{f}(\mathbf{M}p) = \mathbf{Mf}(p),
\]

so \( \mathbf{f}(p) \in \text{Fix}(M) \). This means that a nonlinear dynamical system with a symmetry can have a linear invariant space. This property will prove to be very useful when we study the bifurcations of tâtonnement processes with a reflectional symmetry. This property also implies that \((1,1,1)\) is an equilibrium price vector of all tâtonnement processes that have a rotational symmetry, since \( \text{Fix}(M_{312}) = (1, 1, 1) \).

Now we want to impose symmetry on our economy with CES-utility functions. Let \( U, V : \mathbb{R}^3_+ \to \mathbb{R} \) be utility functions, \( \mathbf{p} = (p_1, p_2, p_3)' \) a vector of prices and \( \mathbf{w} = (w_1, w_2, w_3)' \), \( \mathbf{v} = (v_1, v_2, v_3)' \) vectors of initial endowments, where \( v_1 = v_2 \). Consider the two permutation matrices \( M_{231} \) and \( M_{213} \). Take \( V(\cdot) \) such that \( V(M_{213}x) = V(x) \).

- Consider an economy with three consumers with utility functions \( U_1(x) = U(x) \), \( U_2(x) = U(M_{231}x) \) and \( U_3(x) = U(M_{231}^2x) \) and vectors of initial endowments \( \mathbf{w}_1 = \mathbf{w}, \mathbf{w}_2 = M_{231}\mathbf{w} \) and \( \mathbf{w}_3 = (M_{231})^2\mathbf{w} \), respectively. Then the corresponding multiplicative tâtonnement process has rotation symmetry \( M_{231} \).
- Consider an economy with three consumers with utility functions \( U_1(x) = U(x), \ U_2(x) = U(M_{213}x), \) and \( U_3(x) = V(x) \) and vectors of initial endowments \( \mathbf{w}_1 = \mathbf{w}, \mathbf{w}_2 = M_{213}\mathbf{w} \) and \( \mathbf{w}_3 = \mathbf{v} \), respectively. Then the corresponding multiplicative tâtonnement process has reflectional symmetry \( M_{213} \).

This result gives us a way to impose symmetry on our price adjustment process with CES utility functions. Let \( \mathbf{A} = (a_{ij}) \) be the matrix of preferences. We then have

\[
\text{Corollary 4. For an economy with three commodities and three agents with CES utility functions and initial endowments \( w_{ii} = 1 \) and \( w_{ij} = 0 \) for \( i \neq j \) we have that the following types of preference matrices}
\]

\[
\mathbf{A}^R_3 = \begin{pmatrix} 1 & \alpha & \beta \\ \beta & 1 & \alpha \\ \alpha & \beta & 1 \end{pmatrix}, \quad \mathbf{A}^Z_3 = \begin{pmatrix} 1 & \alpha & \beta \\ \alpha & 1 & \beta \\ \gamma & \gamma & 1 \end{pmatrix}, \quad \mathbf{A}^P_3 = \begin{pmatrix} 1 & \alpha & \alpha \\ \alpha & 1 & \alpha \\ \alpha & \alpha & 1 \end{pmatrix},
\]

\[
(13)
\]
induce respectively a rotational, a reflectional and both kinds of symmetries of the multiplicative price adjustment process.

The attractors shown in Section 3.3 were obtained by using the matrices (13). The attractors in Figs. 1–3 obviously exhibit the same symmetry as the corresponding tâtonnement processes. Rotating the pictures in Fig. 1 over $\frac{2}{3}\pi$ results in exactly the same pictures and reflection of the attractors in Fig. 2 in the line with $p_1 = p_2$ leaves the attractors invariant. For the pictures in Fig. 3 any permutation of the prices can be executed without changing the picture. The symmetry of the tâtonnement process seems to determine the symmetry of the resulting attractor.

If $f$ has symmetry $M$ and $p_0$ converges to an attractor $A$ then obviously $Mp_0$ converges to an attractor $MA$. If they converge to the same attractor, that is if $MA = A$, then this attractor can be said to have symmetry $M$. So we can define the symmetrygroup of an attractor of a dynamical system with symmetrygroup $\Gamma_f$ as $\Sigma_A = \{M \in \Gamma_f | MA = A\}$. For the attractors of Section 3.3 we have $\Sigma_A = \Gamma_f$. This might suggest that the attractor always has the same symmetrygroup as the dynamical system, but in general that need not be the case. We can have $MA \neq A$ and then we must have coexistence of attractors. These coexisting attractors are related by symmetry and are called conjugate attractors. In the sequel we shall encounter symmetry breaking bifurcations, where a single attractor breaks up into multiple coexisting conjugate attractors with a smaller symmetrygroup, and symmetry increasing bifurcations where coexisting conjugate attractors merge into one single (symmetric) attractor.

5. Local bifurcation analysis

In this section we study which local bifurcations at the equilibrium price vector can occur in our tâtonnement process. In Section 5.1 we show that the symmetry of the tâtonnement process determines the form of the Jacobian matrix. In Sections 5.2–5.4 we study the implications of this result for the generic⁶ bifurcations that occur for the tâtonnement processes with different symmetries.

5.1. Symmetry and the Jacobian matrix

To study the local bifurcations of the equilibrium price vector that can occur in our tâtonnement process we have to look at the eigenvalues $\mu_i$ of the Jacobian

---

⁶In this paper generic means with respect to the set of dynamical systems with a given symmetry group.
matrix $J = ((\partial f_i/\partial p_j)(p))$ evaluated at the equilibrium price vector $p^*$. These eigenvalues depend on the parameters of the tâtonnement process, so we have $\mu_i = \mu_i(\lambda, \sigma, A)$ where $A$ is the matrix with preference parameters. The equilibrium price vector $p^*$ is locally stable if all eigenvalues lie inside the unit circle and $p^*$ is unstable when at least one eigenvalue lies outside the unit circle. If an eigenvalue crosses the unit circle, as a parameter varies, a bifurcation occurs. At such a bifurcation the behaviour of the tâtonnement process changes. There are three different ways in which an eigenvalue can cross the unit circle. At the bifurcation value an eigenvalue can be real and equal to +1, real and equal to −1 or two eigenvalues can be complex conjugates with absolute value 1. These three different cases correspond to different kinds of bifurcations.

First we show that symmetry induces a special form of the Jacobian matrix evaluated in the (symmetric) equilibrium price vector.

**Proposition 5.** The Jacobian of the two-dimensional tâtonnement process evaluated at a symmetric equilibrium has eigenvalues that are

- **real** if the tâtonnement process has a $Z_2$ symmetry. In this case one eigenvector lies in the fixed point subspace of the $Z_2$ symmetry and the other eigenvector lies perpendicular to this fixed point subspace,
- **complex conjugates** if the tâtonnement process has an $R_3$ symmetry,
- **real and equal to each other** if the tâtonnement process has both symmetries.

From this proposition it follows that the symmetry of the tâtonnement process determines which type of primary bifurcation occurs as the symmetric equilibrium price vector loses stability. We now derive the eigenvalues of the Jacobian evaluated at the symmetric equilibrium for each symmetry. The Jacobian of our tâtonnement process (7) evaluated in the equilibrium price vector $p^*$ is

$$J^* = I + \lambda \begin{pmatrix} p^*_1z_{11}(p^*) & p^*_1z_{12}(p^*) & p^*_1z_{13}(p^*) \\ p^*_2z_{21}(p^*) & p^*_2z_{22}(p^*) & p^*_2z_{23}(p^*) \\ p^*_3z_{31}(p^*) & p^*_3z_{32}(p^*) & p^*_3z_{33}(p^*) \end{pmatrix},$$

(14)

where $z_i(p) \equiv \partial z_i(p)/\partial p_j$. Notice that the matrix $(p^*_i z_{ij}(p^*))_{i,j = 1,2,3}$ has linear dependent rows ($\sum_{i=1}^{3} p^*_i z_{ij}(p^*) = -z_i(p^*) = 0$ from Walras’ Law) which implies that one eigenvalue of $J^*$ is equal to 1 for all possible parameter values. This unit eigenvalue corresponds to the fact that every simplex is invariant under the multiplicative tâtonnement process. Since we are only interested in the motion of the dynamical system on the simplex we can safely ignore this eigenvalue. The other two eigenvalues are

$$\mu_{1,2} = 1 + \frac{1}{2}\lambda \{-p^*_1(z_{11} - z_{13}) + p^*_2(z_{22} - z_{23}) \pm \sqrt{D}\},$$

$$D = [p^*_1(z_{11} - z_{13}) - p^*_2(z_{22} - z_{23})]^2 + 4p^*_1p^*_2(z_{11} - z_{13})(z_{21} - z_{23}).$$

(15)
Table 1
Eigenvalues of (7) given the symmetry group

<table>
<thead>
<tr>
<th>Symmetry Group</th>
<th>Eigenvalue Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_3 = {I, M_{312}}$</td>
<td>$1 + \frac{1}{\sqrt{3}} \left( 3z_{11}(p^<em>) + (z_{13}(p^</em>) - z_{13}(p^*)) \right)$</td>
</tr>
<tr>
<td>$Z_5 = {I, M_{213}}$</td>
<td>$1 + \frac{1}{\sqrt{3}} \left( z_{11}(p^<em>) - z_{13}(p^</em>) + (z_{13}(p^<em>) - z_{13}(p^</em>)) \right)$</td>
</tr>
<tr>
<td>$D_3 = {I, M_{312}, M_{213}}$</td>
<td>$1 + \frac{1}{\sqrt{3}} z_{11}(p^*)$</td>
</tr>
</tbody>
</table>

Under symmetry the expressions for these eigenvalues simplify considerably. These expressions are summarized in Table 1.

Notice that if $\Gamma_f = R_3$ or $\Gamma_f = D_3$ and $z_{11}(p^*) > 0$ both eigenvalues lay outside the unit circle for all $\lambda > 0$. Hence, in that case the equilibrium price vector is unstable for all positive values of $\lambda$.

In the following subsections we analyse which bifurcations occur for each of the symmetry groups, as the speed of adjustment $\lambda$ increases.

5.2. Rotational symmetry and the Hopf bifurcation

In the previous subsection we found that the eigenvalues of the Jacobian of the tâtonnement process with rotational symmetry (and no other symmetry) evaluated at the symmetric equilibrium price vector $p^* = (1, 1, 1)$ are complex conjugates. The equilibrium price vector loses stability if these eigenvalues cross the unit circle. Let $\lambda^{bif}$ denote the value of $\lambda$ such that $|\mu_1(\lambda^{bif})| = |\mu_2(\lambda^{bif})| = 1$. We then have the following.

**Proposition 6.** Assume that the tâtonnement process with symmetry group $\Gamma_f = \{I, M_{312}\}$ has a stable symmetric equilibrium price vector for $\lambda > 0$ small enough and that $z_{11}(p^*) < z_{13}(p^*)$. Furthermore assume that $(\mu_4(\lambda^{bif}))^r \neq 1$ for $r = 1, 2, 3, 4$. Then a Hopf bifurcation occurs at

$$\lambda^{bif} = -\frac{4z_{11}(p^*)}{3[z_{11}(p^*)]^2 + [z_{12}(p^*) - z_{13}(p^*)]^2}. \quad (17)$$

For $\lambda$ close to $\lambda^{bif}$ there exists an invariant closed curve.

Numerical observations suggest the following: for $\lambda < \lambda^{bif}$ the equilibrium price vector $p^* = (1, 1, 1)$ is locally stable. For $\lambda > \lambda^{bif}$ the equilibrium price vector is unstable and there is an attracting invariant closed curve. This closed curve has the same symmetry group as the tâtonnement process. Fig. 4a shows an example of such an attracting invariant closed curve obtained after a Hopf bifurcation.
Fig. 4. Primary bifurcations for the tâtonnement process with different symmetry groups. (a) Invariant closed curve created in Hopf bifurcation for the tâtonnement process with cyclical symmetry, $\alpha = 1$, $\beta = \frac{1}{2}$, $\sigma = \frac{1}{2}$ and $\lambda = 4.5$. (b) Period two orbit off the symmetry axis created in period doubling bifurcation for the tâtonnement process with reflectional symmetry, $\alpha = \frac{3}{4}$, $\beta = \frac{1}{2}$, $\sigma = \frac{1}{10}$ and $\lambda = 25$. (c) Period two orbit on symmetry axis created in period doubling bifurcation for the tâtonnement process with cyclical symmetry, $\alpha = \frac{1}{4}$, $\beta = \frac{1}{2}$, $\sigma = \frac{1}{10}$ and $\lambda = 25$. (d) Six period two orbits created in symmetry breaking period doubling bifurcation in tâtonnement process with cyclical and reflectional symmetry, $\alpha = 1$, $\sigma = \frac{1}{100}$ and $\lambda = 220$. The period two orbit indicated by $i$ ($i = 1, 2, 3$) is the period two orbit on the symmetry axis $l_3$, the period two orbit indicated by $i^*$ ($i^* = 1, 2, 3$) is the period two orbit off, but symmetric with respect to, $l_3$.

5.3. Reflection symmetry and the period doubling bifurcation

Consider the tâtonnement process with symmetry group $\Gamma_f = \{I, M_{213}\}$. The fixed point subspace of this $\mathbf{Z}_2$ symmetry is the symmetry axis $l_3$. From Proposition 5 we found that the eigenvalues of the Jacobian of this dynamical
system evaluated in a point in the fixed point subspace are real and have
eigenvectors that lay perpendicular to the fixed point subspace and in the fixed
point subspace, respectively.

The equilibrium price vector can only lose stability when the largest (in
absolute value) eigenvalue goes through \( \pm 1 \). If an eigenvalue goes through \(+ 1\)
a saddle-node, pitchfork or transcritical bifurcation occurs (see e.g. Guckenheimer
and Holmes, 1983). All these bifurcations in some way deal with the multiplicity
of equilibria. Since the equilibrium price vectors of the tâtonnement process
correspond to the zeros of the system of aggregate excess demand functions, the
location and multiplicity of these equilibrium price vectors is independent of the
speed of adjustment \( \lambda \). Therefore we know that these bifurcations cannot occur in
our model as \( \lambda \) varies. When one of the eigenvalues goes through \( -1 \) a flip-
or period doubling bifurcation occurs: at this bifurcation a period two orbit
emerges. If this period doubling bifurcation occurs at \( \lambda = \lambda_{\text{bif}} \) then we have the
following scenario. For \( \lambda < \lambda_{\text{bif}} \) the equilibrium price vector is stable and for
\( \lambda > \lambda_{\text{bif}} \) the equilibrium price vector is unstable. For \( \lambda \) close to \( \lambda_{\text{bif}} \) a period two orbit exists. Numerical observations suggest that this period two orbit exists for
\( \lambda > \lambda_{\text{bif}} \) and that it is stable.

**Proposition 7.** Assume that the equilibrium price vector of the tâtonnement process
with symmetry group \( \Gamma_f = \{I, M_{213}\} \) is stable for \( \lambda > 0 \) small enough and that
\( z_{12}(p^*) \neq z_{13}(p^*) \). Then we have

- the equilibrium loses stability through a period doubling bifurcation at

\[
\lambda_{\text{bif}} = \min \left\{ -\frac{2}{p_1^*(z_{11}(p^*) - z_{12}(p^*))}, -\frac{2}{p_1^*(z_{11}(p^*) + z_{12}(p^*) - 2z_{13}(p^*))} \right\},
\]

(18)

- the resulting period two orbit has the same symmetry as the tâtonnement

process. The period two orbit lies in \( l_3 \) if \( z_{12}(p^*) < z_{13}(p^*) \) and it lies off but

symmetric with respect to \( l_3 \) when \( z_{12}(p^*) > z_{13}(p^*) \).\(^7\)

According to Proposition 7 there are two different types of period two orbits
with \( \mathbb{Z}_2 \) symmetry. Examples of these two period two orbits are shown in Fig. 4b
and c. Since the three prices have to sum up to 3 in every period an increase of
one price has to be accompanied by a decrease of another price. Consider the
period two orbit \( \{ p_1^*, p_2^* \} \) in Fig. 4b that lies off \( l_3 \). This period two orbit has the
property that the market for the third good is in equilibrium in every period but

\(^7\) These two types of period doubling bifurcations correspond to pitchfork bifurcations of \( f^2 \). One
of these pitchfork bifurcations is symmetry preserving (\( \mu_2 = -1 \)) and the other one is symmetry
breaking (\( \mu_1 = -1 \)).
the first two prices fluctuate between the same values: when \( p_1 \) is high, \( p_2 \) is low and vice versa. In this case \( p_2 \) has to adjust to the fluctuations in \( p_1 \). This happens when \( z_{12}(p^*) > z_{13}(p^*) \).

The period two orbit in Fig. 4c that lies in \( l_3 \) has the property that prices of the first two goods are equal to each other in every period: in one period they are both high (above their equilibrium value) and in the next period they are both low (below their equilibrium value). In this case \( p_3 \) has to adjust to neutralise the fluctuations in \( p_1 \) (and \( p_2 \)). This happens when \( z_{13}(p^*) > z_{12}(p^*) \). So the burden of a change in \( p_1 \) is completely carried by the price of that good that is the closest substitute to good 1 (at the equilibrium price vector).

5.4. \( D_3 \) symmetry and the equivariant branching lemma

In the previous sections we saw that a rotational symmetry leads to a Hopf bifurcation and that a reflectional symmetry leads to a period doubling bifurcation of the equilibrium price vector when it loses stability. We now want to study the case where the adjustment process has both a rotational and a reflectional symmetry. We can then apply the equivariant branching lemma for period doubling (Chossat and Golubitsky, 1988, see the appendix) to arrive at the following result.

**Proposition 8.** Assume that the symmetric equilibrium price vector \( p^* = (1, 1, 1) \) of the tâtonnement process with rotational and reflectional symmetry group \( \Gamma_f = D_3 = \{I, M_{312}, M_{213}\} \) is stable for \( \lambda > 0 \) small enough. \( p^* \) undergoes a period doubling bifurcation as \( \lambda \) goes through

\[
\lambda_{\text{bif}} = - \frac{4}{3z_{11}(p^*)}.
\]

At this value of \( \lambda \) six period two orbits emerge, three stable and three unstable. For every fixed point subspace \( l_i \), \( i = 1, 2, 3 \) there is a period two orbit on \( l_i \) and a period two orbit off \( l_i \) but with a reflectional symmetry with respect to \( l_i \).

For \( \lambda > \lambda_{\text{bif}} \) six period two orbits exist, three of which are stable. Fig. 4d shows the six period two orbits for a tâtonnement process with \( D_3 \) symmetry \((x = 1, \sigma = \frac{1}{10}, \lambda = 220)\). In this case the three period two orbits off the symmetry axes are stable. Up till now we have seen that when the symmetric equilibrium price vector loses stability the resulting attractor has the same symmetry group as the symmetric equilibrium price vector. For the period two orbits emerging in the case where the tâtonnement process has both symmetries this no longer holds. These period two orbits only have a reflection symmetry. In
fact rotating each of these period two orbits over $\frac{2}{3} \pi$ gives one of the other period two orbits. Thus the symmetry of the three coexisting period two orbits is smaller than the symmetry of the equilibrium price vector. Therefore the bifurcation described in Proposition 8 is called a symmetry breaking bifurcation.

6. Global dynamics

In Section 5.4 we have seen that when the symmetric equilibrium price vector of a tâtonnement process with both a reflectional and a rotational symmetry loses stability three attracting and coexisting period two orbits emerge. Each of these new attractors only has a $Z_2$ symmetry. The symmetry group of the attractor of the tâtonnement process decreases from $\{I, M_{213}, M_{312}\}$ to either $\{I, M_{213}\}$, $\{I, M_{321}\}$ or $\{I, M_{132}\}$. However, the attractors in Fig. 3 which come from tâtonnement processes with $D_3$ symmetry are fully symmetric. So as $\lambda$ increases at a certain point the symmetry group of the attractor has to increase again. This happens at a symmetry increasing bifurcation. Such bifurcations seem to be generic in tâtonnement processes with $D_3$ symmetry.

Consider one of the stable period two orbits that emerge after the symmetry breaking period doubling bifurcation. This period two orbit undergoes a bifurcation route to chaos, resulting in a $Z_2$-symmetric strange attractor. By symmetry there are two other, conjugate $Z_2$-symmetric strange attractors. As the speed of adjustment increases these attractors grow in magnitude. At a certain point these attractors with $Z_2$ symmetry merge into one attractor which inherits all symmetries of the conjugate attractors and therefore has $D_3$ symmetry again.

Fig. 5 illustrates such a bifurcation scenario. After the symmetry breaking period doubling bifurcation there are three stable period two orbits (one of them is shown in Fig. 5a). The period two orbit undergoes a Hopf bifurcation and an invariant set consisting of two closed curves emerges. More important for the global dynamics is the emergence of a stable period six orbit in the neighborhood of each stable period two orbit. This period six orbit emerges through a saddle-node bifurcation of $f^6$. Each of these stable period six orbits undergoes a Hopf bifurcation resulting in an attracting set consisting of six closed curves (Fig. 5b). From these six closed curves a two piece strange attractor is created (Fig. 5c). There are three of these strange attractors. If the speed of adjustment grows a little more a symmetry increasing bifurcation occurs and these three attractors merge into a large attractor with $D_3$ symmetry (Fig. 5d).

This kind of bifurcation scenario (from an equilibrium price vector with $D_3$ symmetry to three attractors with $Z_2$ symmetry and then again to an attractor with $D_3$ symmetry) seems to occur for all tâtonnement processes with $D_3$ symmetry.

---

8 This particular bifurcation scenario corresponds to scenario 3 in Chossat and Golubitsky (1988).
Fig. 5. Bifurcation scenario for tâtonnement process with $D_3$ symmetry, $\alpha = 1, \sigma = \frac{1}{10}$. (a) $\lambda = 220$, (b) $\lambda = 258$, (c) $\lambda = 260$, (d) $\lambda = 261$.

$D_3$ symmetry. However, Tuinstra (1999) gives examples where these symmetry breaking and increasing bifurcation routes also occur for systems with only a $Z_2$ or $R_3$ symmetry (see Tuinstra, 1999).

7. An asymmetric price adjustment process

Thus far we concentrated on global dynamics of symmetric tâtonnement processes. Depending on this symmetry, different kinds of bifurcation routes to strange attractors can occur. In particular, a bifurcation from a stable steady state to six coexisting period two orbits occurs in the tâtonnement process with
$D_3$ symmetry. Then, as the equilibrium price vector loses stability, three coexisting attracting period two orbits emerge. Obviously, a price adjustment process with a $D_3$ symmetry is very special. The question thus arises whether the analysis of a special symmetric tâtonnement process is useful in order to understand the global dynamics of general asymmetric tâtonnement processes. In this section we show that nearby asymmetric tâtonnement processes exhibit bifurcation routes to strange attractors very similar to the special symmetric bifurcation routes. Therefore, symmetric tâtonnement processes can serve as an ‘organizing centre’ in the analysis of the global dynamics in general tâtonnement processes.

We study a numerical example with the following parameter values

\[
A = \begin{pmatrix}
1 & 0.95 & 0.9 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}, \quad \sigma = \frac{1}{100}
\]  

and initial endowments as before.

Notice that the multiplicative price adjustment process with these preferences has no nontrivial symmetry, but that it is close to a price adjustment process with $D_3$ symmetry.

The equilibrium price vector is $p^* = (1.0177656, 1.0002052, 0.9820292)$ and the eigenvalues of the Jacobian evaluated at this equilibrium price vector are $\mu_1 = 1 - 0.00982501\lambda$ and $\mu_2 = 1 - \frac{1}{100}\lambda$, respectively. A period doubling bifurcation occurs at $\lambda^{\text{bif}} = 200$ resulting in a stable period two orbit. Only one period two orbit emerges. However, as $\lambda$ increases to $\lambda^{\text{sn1}} \approx 228.2$ the second iterate of the price adjustment process, $f^2$, undergoes a saddle-node bifurcation and two additional period two orbits emerge, one stable and one unstable. As $\lambda$ increases even more to $\lambda^{\text{sn2}} \approx 237.0$ $f^2$ undergoes a second saddle-node bifurcation again resulting in another stable and unstable period two orbit. Just as in the price adjustment process with $D_3$ symmetry three stable period two orbits coexist in the asymmetric tâtonnement process. In this case however, these period two orbits are not created through a single bifurcation at a single value of the bifurcation parameter $\lambda$, but through a series of three independent bifurcations at different values of $\lambda$. Fig. 6 shows two bifurcation diagrams that were created with the software package LOCBiF (Khibnik et al., 1992). Fig. 6a shows the three stable period two orbits created in the symmetry breaking period doubling bifurcation for the tâtonnement process with $D_3$ symmetry ($a_{ij} = 1$, $\sigma = \frac{1}{100}$). Fig. 6b shows three stable period two orbits and three unstable period two orbits created in a period doubling and two subsequent saddle-node bifurcations for the asymmetric tâtonnement process described above.

Each of the three coexisting stable period two orbits undergoes a bifurcation route to a strange attractor, with bifurcations occurring for different values of the speed of adjustment $\lambda$. Therefore for certain values of $\lambda$ there are three different coexisting attractors. Fig. 7a–c shows the three different coexisting
attractors for $\lambda = 259$. As $\lambda$ increases more the three coexisting attractors merge, just as in the symmetric case, to one large attractor. This attractor is shown in Fig. 7d for $\lambda = 265$.

This example illustrates that the bifurcation scenario of the price adjustment process with a $D_3$ symmetry is useful in understanding the behaviour of a price adjustment process that has no nontrivial symmetry but that is close to a price adjustment process with $D_3$ symmetry.

8. Summary and conclusions

Price normalization plays an important role in the dynamics of the tâtonnement process. The tâtonnement process can be stable under one normalization rule and unstable under another. We have argued that for the multiplicative price adjustment process, using the simplex as a normalization rule is natural since the multiplicative price adjustment process leaves the simplex invariant.

In studying the dynamical behaviour of our tâtonnement process we have focused on the cases where some nontrivial symmetry is present. The symmetry of the tâtonnement process influences the bifurcation scenario as the speed of adjustment increases. In particular, we may have symmetry breaking
bifurcations where the symmetric equilibrium price vector becomes unstable and three coexisting stable period two orbits emerge.

In this last case the symmetry breaking bifurcation is followed by a sequence of bifurcations leading to three coexisting conjugate strange attractors. Eventually a symmetry increasing bifurcation occurs where the three conjugate attractors merge into one unique attractor, which is fully symmetric again. We also encountered symmetry breaking and increasing bifurcations when the tâtonnement process has only a cyclical or reflectional symmetry. Obviously the symmetry of the tâtonnement process plays an important role in the dynamics of that tâtonnement process. Studying symmetric tâtonnement processes has been a first step, and may serve as an organizing centre in understanding the possible bifurcation routes in the multiplicative tâtonnement process on the simplex. The
example in Section 7 shows that the bifurcation scenario in a symmetric tâtonnement process reveals information about possible bifurcation scenario’s in general asymmetric tâtonnement processes.

Appendix

In this appendix proofs of the main results are given.

Proof of Proposition 3. Let the utility maximizing consumption bundle be given by
\[ x = d^U(p, w) = \arg \max_{x \in \mathbb{R}^3} \{ U(x) | p'x \leq p'w \} \]
Consider the problem of maximizing \( U(Mx) \) subject to the budget constraint \( p'x \leq p'(Mw) \), where \( M \in \{ M_{231}, M_{213} \} \). Since \( M^{-1} = M' \), this problem is equivalent with maximizing \( U(Mx) \) subject to \( (Mp)'Mx \leq (Mp)'w \). The solution to this last problem is easily seen to be
\[ x = M'd(Mp, w). \]

- First consider \( M = M_{231} \). Notice that \( M_{231}^2 = M_{231} \) and \( M_{231}^3 = I \). The individual demand functions become \( x_{k + 1} = (M_{231})^k d^U(M_{231}^k p, w) \), with \( k = 0, 1, 2 \). The aggregate excess demand functions \( z(p) = \sum_{i=1}^{3} x_i - \sum_{i=1}^{3} w_i \) can then be written as
\[ z(p, w) = \begin{pmatrix}
  (d_1^U(p, w) + d_2^U(M_{231}^2 p, w) + d_3^U(M_{231}^3 p, w) - w_1 - w_2 - w_3) \\
  (d_1^U(M_{231}^2 p, w) + d_2^U(p, w) + d_3^U(M_{231}^3 p, w) - w_1 - w_2 - w_3) \\
  (d_1^U(M_{231}^2 p, w) + d_2^U(M_{231}^3 p, w) + d_3^U(p, w) - w_1 - w_2 - w_3)
\end{pmatrix}. \]
It can be easily checked that we have \( z(M_{231} p, w) = M_{231} z(p, w) \). This means that \( M_{231} \) is a symmetry of the aggregate excess demand functions.

- Now consider \( M = M_{213} \). The demand functions are respectively \( x_1 = d^U(p, w) \), \( x_2 = M_{213} d^U(M_{213} p, w) \) and \( x_3 = d^V(p, v) \) with \( d_2^U(p, v) = d_1^V(M_{213} p, v) \) and \( d_3^V(M_{213} p, v) = d_3^V(p, v) \). The aggregate excess demand functions become
\[ z(p, w) = \begin{pmatrix}
  (d_1^U(p, w) + d_2^U(M_{213} p, w) + d_1^V(p, v) - w_1 - w_2 - v_1) \\
  (d_1^U(M_{213} p, w) + d_2^U(p, v) + d_1^V(M_{213} p, v) - w_1 - w_2 - v_1) \\
  (d_2^U(p, v) + d_3^V(M_{213} p, w) + d_3^V(p, v) - 2w_3 - v_3)
\end{pmatrix}. \]
We have \( M_{213} z(p, w) = z(M_{213} p, w) \).
Now if the aggregate demand functions have symmetry $M_{231}$ then for the multiplicative price adjustment process we have
\[
\begin{pmatrix}
  f_1(M_{231}p) \\
  f_2(M_{231}p) \\
  f_3(M_{231}p)
\end{pmatrix} = \begin{pmatrix}
  (1 + \lambda z_1(M_{231}p)) \\
  (1 + \lambda z_2(M_{231}p)) \\
  (1 + \lambda z_3(M_{231}p))
\end{pmatrix}
\]
\[
= \begin{pmatrix}
  p_2(1 + \lambda z_2(p)) \\
  p_3(1 + \lambda z_3(p)) \\
  p_1(1 + \lambda z_1(p))
\end{pmatrix} = M_{231}f(p)
\]
and if the aggregate demand functions have symmetry $M_{213}$ then for the multiplicative price adjustment process we have
\[
\begin{pmatrix}
  f_1(M_{213}p) \\
  f_2(M_{213}p) \\
  f_3(M_{213}p)
\end{pmatrix} = \begin{pmatrix}
  (1 + \lambda z_1(M_{213}p)) \\
  (1 + \lambda z_2(M_{213}p)) \\
  (1 + \lambda z_3(M_{213}p))
\end{pmatrix}
\]
\[
= \begin{pmatrix}
  p_2(1 + \lambda z_2(p)) \\
  p_1(1 + \lambda z_1(p)) \\
  p_3(1 + \lambda z_3(p))
\end{pmatrix} = M_{213}f(p)
\]

**Proof of Proposition 5.** Without loss of generality we can apply a linear transformation to the prices to transform the simplex in $\mathbb{R}^3$ to a triangle in the plane. Let $q = \mathcal{T}p$ with
\[
\mathcal{T} = \begin{pmatrix}
  \frac{1}{2}\sqrt{3} & -\frac{1}{2}\sqrt{3} & 0 \\
  -\frac{1}{2} & -\frac{1}{2} & 1 \\
  1 & 1 & 1
\end{pmatrix}.
\]
The price adjustment process in terms of these new variables becomes $q_{t+1} = \mathcal{T} f \circ \mathcal{T}^{-1}(q_t) = h(q_t)$. Since we have $q_{3t} = \sum_{i=1}^3 p_{it} = 3$, for all $t$, this is equivalent to
\[
\begin{pmatrix}
  q_{1,t+1} \\
  q_{2,t+1}
\end{pmatrix} = \begin{pmatrix}
  h_1(q_{1t}, q_{2t}, 3) \\
  h_2(q_{1t}, q_{2t}, 3)
\end{pmatrix}.
\]
Notice that $\mathcal{T}$ transforms the centre of the simplex to the origin and the symmetry axis $l_3$ to the $q_2$ axis. We have $J_h = \mathcal{T} J_f \mathcal{T}^{-1}$ and the eigenvalues of $J_h$ are equal to the eigenvalues of $J_f$. The $Z_2$ and $R_3$ symmetries can be represented respectively by
\[
M_z = \begin{pmatrix}
  -1 & 0 \\
  0 & 1
\end{pmatrix}, \quad M_r = \begin{pmatrix}
  -\frac{1}{2} & \frac{1}{2}\sqrt{3} \\
  -\frac{1}{2}\sqrt{3} & -\frac{1}{2}
\end{pmatrix}.
\]
$M_z$ corresponds to a reflection in the $q_2$ axis and $M_r$ corresponds to a rotation over $\frac{2}{3} \pi$. With this transformation of variables it is easy to establish our result. Let $M$ be a symmetry of $h$. Then by definition we have $Mh(q) = h(Mq)$. Differentiating both sides with respect to $q$ gives

$$M \frac{\partial h}{\partial q}(q) = \frac{\partial h}{\partial q}(Mq)M.$$ 

Now if $q^*$ is symmetric (that is $Mq^* = q^*$) we have

$$MJ_{h}^* = J_{h}^* M.$$ 

So the Jacobian evaluated at the symmetric equilibrium commutes with the symmetries of the dynamical system. An easy calculation shows that the only $2 \times 2$ matrices commuting with $M_z$ and with $M_r$ are $J_z$ and $J_r$, respectively with

$$J_z = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad J_r = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$ 

The eigenvalues of $J_z$ are $a$ and $b$ with eigenvectors $v_a = (1, 0)'$ and $v_b = (0, 1)'$, respectively, the eigenvalues of $J_r$ are $a \pm bi$ and therefore complex as long as $b \neq 0$. Note that if the dynamical system has both symmetries we have

$$J_{D_i} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = aI.$$ 

The statements of the proposition follow. □

**Proof of Proposition 7.** The eigenvalues are

$$\mu_1 = 1 + \lambda p_1^i(z_{11}(p^*) - z_{12}(p^*)), \quad \mu_2 = 1 + \lambda p_1^i(z_{11}(p^*) + z_{12}(p^*) - 2z_{13}(p^*))$$

and the corresponding eigenvectors $v_1$ and $v_2$ lay perpendicular to $l_3$ and in $l_3$, respectively. If an eigenvalue goes through $-1$ a period doubling bifurcation occurs in the dynamical system restricted to the centre manifold (see Guckenheimer and Holmes, 1983, p. 158). The centre manifold is locally $Z_2$-symmetric (Kuznetsov, 1995, Theorem 7.6). When $\mu_1 = -1$ the centre manifold lies tangent to $v_1$ and therefore perpendicular to the fixed point subspace. When $\mu_2 = -1$ the centre manifold coincides with the fixed point subspace. We have $\mu_1 < \mu_2$ if $z_{12}(p^*) > z_{13}(p^*)$ and $\mu_1 > \mu_2$ if $z_{12}(p^*) < z_{13}(p^*)$. □

To prove Proposition 8 we use the following result (Chossat and Golubitsky, 1988, Theorem 4.1).
Theorem 9 (Equivariant Branching Lemma for period doubling). Consider a dynamical system \( h: \mathbb{R}^n \to \mathbb{R}^n \) with a symmetry group \( \Gamma_h \). Assume that \( \Gamma_h \) acts absolutely irreducibly on \( \mathbb{R}^n \), that \( J_h^k = -I_n \) and that \( \dim \text{Fix}(M) = 1 \), for a subgroup \( M \subset \Gamma_h \oplus \{ \pm I_n \} \). Then generically, there exists a branch of period two points for \( h \) bifurcating at the fixed point and tangent to \( \text{Fix}(M) \) at the fixed point. If \( M \subset \Gamma_h \), then the branch lies in \( \text{Fix}(M) \).

Proof (Sketch). The symmetry group \( \Gamma_h \) is said to be acting absolutely irreducible on \( \mathbb{R}^n \) if the only matrices commuting with all symmetries of \( \Gamma_h \) are multiples of the identity matrix. Then at a bifurcation all eigenvalues are equal to \( +1 \) or all eigenvalues are equal to \( -1 \). For a good understanding of the bifurcation that occurs when all eigenvalues are equal to \( -1 \), it is useful to discuss the case where all eigenvalues are \( +1 \) first. Recall that the fixed point subspace of a symmetry is an invariant space of the dynamical system. Now if \( h(.) \) has a symmetry with a one-dimensional fixed point subspace then the search for fixed points can be confined to this fixed point subspace. This gives the Equivariant Branching Lemma which states that for every subgroup of \( \Gamma_h \) that has a one-dimensional fixed point subspace there exists a branch of fixed points of \( h \) bifurcating at the origin. This branch lies in this fixed point subspace (for a discussion of this theorem see Golubitsky et al. (1988, Chapter XIII)).

If all eigenvalues are equal to \( -1 \) the analysis is more complicated. The normal form does not only commute with the symmetry group \( \Gamma_h \) but also with the linear part of the dynamical system which is \( -I_n \) (Chossat and Golubitsky, 1988). Therefore the normal form commutes with \( \Gamma_h \oplus \{ \pm I_n \} \). Now we can apply the same reasoning as above, only we have to distinguish between subgroups with a one-dimensional fixed point subspace that are in \( \Gamma_h \) and ones that are in \( \Gamma_h \oplus \{ \pm I_n \} \) but not in \( \Gamma_h \). Therefore, there exists a branch of period two points bifurcating at the fixed point tangent to \( \text{Fix}(M) \), for each \( M \in \Gamma_h \oplus \{ \pm I_n \} \) with \( \dim \text{Fix}(M) = 1 \).

Proof of Proposition 8. From Proposition 5 it follows that the symmetrygroup \( D_3 \) acts absolutely irreducible on \( \mathbb{R}^3_+ \). Furthermore, \( D_3 \oplus \{ \pm I_3 \} \) contains six subgroups that have a one-dimensional fixed point subspace. Three of these are also in \( D_3 \), namely the three \( Z_2 \) symmetries with fixed point subspaces \( l_1 \), \( l_2 \) and \( l_3 \), respectively. The three other fixed point subspaces \( l_{1\pi}, l_{2\pi} \) and \( l_{3\pi} \) can be obtained by rotating \( l_1 \), \( l_2 \) and \( l_3 \) over \( \frac{\pi}{2} \). Application of Theorem 9 then gives the result. \( \square \)

References