Optimal consumption/investment policies with undiversifiable income risk and liquidity constraints

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Abstract

This paper examines the continuous time optimal consumption and portfolio choice of an investor having an initial wealth endowment and an uncertain stream of income from non-traded assets. The income stream is not spanned by traded assets and the investor is not allowed to borrow against future income, so the financial market is incomplete. We solve the corresponding stochastic control problem numerically with the Markov chain approximation method, prove convergence of the method, and study the optimal policies. In particular, we find that the implicit value the agent attaches to an uncertain income stream typically is much smaller in this incomplete market than it is in the otherwise identical complete market. Our results suggest that this is mainly due to the presence of liquidity constraints. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

The consumption/investment choice of a price-taking investor is a classical problem of financial economics. In two pioneering papers, Merton (1969, 1971) introduced stochastic control techniques to analyze the continuous-time version of the problem for an investor with an additively time-separable utility function. In particular, Merton studied the case where the investor has access to a complete financial market in which risky asset prices follow geometric Brownian motions and the investor’s utility function for consumption is of the constant relative risk aversion type. He was able to solve analytically the Hamilton–Jacobi–Bellman (HJB) equation associated with the problem and hence to obtain closed-form expressions for the optimal control policies in feedback form, both for a finite horizon and an infinite horizon.

One of many interesting generalizations of Merton’s setting appears when the investor besides having an initial endowment of wealth also receives a stream of income throughout her planning horizon. Merton (1971, Section 7) stated that the optimal policies when the agent has a deterministic stream of income are as if the agent has no income stream but instead adds the capitalized lifetime income flow discounted at the risk-free rate to her initial wealth. However, it is easy to show, see e.g. He and Pagès (1993, Example 1), that under the policies derived this way the wealth process may go below zero. Due to moral hazard and adverse selection problems, it may be impossible for the investor to borrow against future income, so that the investor can only choose her consumption/investment policy among those that keep her financial wealth non-negative.

He and Pagès (1993) study a model where the income rate is spanned, so that the only source of incompleteness is that liquid wealth has to stay non-negative. Using the martingale techniques of Cox and Huang (1989) they find that the presence of liquidity constraints has a smoothing effect on the optimal consumption across time. If the investor expects her income to rise, she will increase her consumption at a smaller rate than if she was not subjected to liquidity constraints. In a similar set-up, El Karoui and Jeanblanc-Picqué (1996) demonstrate that the optimal trading strategy is to invest part of the wealth in the strategy which is optimal in the corresponding unconstrained case, and the remainder in an American put option written on the optimal wealth process in the unconstrained case. They also derive a formula linking the optimal consumption rate to current wealth and current income, and they show that for zero wealth the optimal consumption rate is a smaller fraction of current income in the liquidity constrained case than in the unconstrained case.

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1 Cox and Huang (1989) demonstrated a similar result for the no-income case.
Maintaining the liquidity constraints of He and Pagès, but dropping the spanning assumption, Duffie and Zariphopoulou (1993) study an infinite horizon model with a single risky asset whose price follows a geometric Brownian motion and a non-negative income rate given by an Itô process driven by a Brownian motion imperfectly correlated with the risky asset price. In this general setting we cannot be sure that the value function, also known as the indirect utility function, is a solution of the associated HJB equation. In fact the value function may not even be smooth. Duffie and Zariphopoulou are able to show that the value function is the unique constrained viscosity solution in the class of concave functions of the HJB equation.

The model of Duffie and Zariphopoulou is specialized in Duffie et al. (1997), henceforth abbreviated DFSZ. For power utility of consumption they reduce the control problem from a two-variable (wealth and income) to a one-variable (wealth divided by income) problem and show that the HJB equation for the reduced problem has a unique smooth solution. The optimal policies and the value function of the original problem can easily be restored from the solution to the reduced problem. With analytical derivations DFSZ are able to find some characteristics of the solution, but they cannot solve the problem completely. In an almost identical model Koo (1998) expresses the optimal policies in terms of the current liquid wealth and a measure of the agent’s implicit value of the future uncertain income stream and is able to analytically derive some general properties of these policies. In Section 2 below, we shall review the main findings of DFSZ and Koo, and we introduce an intuitively more appealing measure of the implicit value of future income and relate it to Koo’s measure. For other continuous-time models with stochastic income, see Andersson et al. (1995), Cuoco (1997), Detemple and Serrat (1997), and Svensson and Werner (1993).

The main contributions of this paper are to demonstrate how the reduced problem of DFSZ can be solved by a relatively simple converging numerical scheme and to study the properties of optimal policies in the set-up of DFSZ in detail. The numerical method adopted is the Markov chain approximation approach which basically approximates the continuous time, continuous state stochastic control problem with a discrete time, discrete state stochastic control problem that is easily solved numerically. The Markov chain approximation approach is described in Kushner (1990) and more detailed in Kushner and Dupuis (1992). See also Fleming and Soner (1993, Chap. IX). The method has previously been applied to consumption/portfolio problems by Fitzpatrick and Fleming (1991) and Hindy et al. (1997).

We contrast the numerically computed value function and optimal controls to the complete market case where the income rate is spanned by traded assets and the investor is not liquidity constrained. In particular, we find that the implicit value the investor attaches to the uncertain income stream is much smaller in the non-spanned, liquidity constrained case than in the complete
market case, even for high ratios of initial wealth to initial income. We find that this implicit value of income is very insensitive to the correlation between changes in the risky asset price and changes in the income rate. Since a perfectly positive correlation corresponds to spanning, this suggests that the large difference between the complete markets case and the non-spanned, liquidity constrained case is to be attributed to the liquidity constraints. We study the sensitivity of both the optimal policies and the implicit value of the income stream with respect to various parameters. Among other things we find that, \textit{ceteris paribus}, liquidity constraints as modeled in this paper are most restrictive for agents with a low financial wealth relative to income, an income rate positively correlated with changes in the risky asset price, and a high time preference for consumption.

While this paper apparently contains the first quantitative study of the properties of the optimal consumption and portfolio policies in a continuous-time set-up with undiversifiable income risk and liquidity constraints, related discrete-time models have been considered by other authors. Koo (1995) also has a non-spanned income process and a liquidity constraint, but in a discrete-time set-up the liquidity constraint explicitly implies bounds on the risky investment in each period, which is not the case with continuous trading. Furthermore, Koo imposes even tighter bounds on the investment in the risky asset than those implied by his liquidity constraint. Hence, it is difficult to evaluate the implications of the liquidity constraint from his numerical results. Koo only considers the case where the shocks to the income rate process and the shocks to the risky asset price are uncorrelated, while we allow for a non-zero correlation and provide a detailed comparative statics analysis. For other discrete-time models of optimal consumption and investment choice with income, see, e.g., Cocco et al. (1998), Deaton (1991), and Heaton and Lucas (1997).

The outline of the rest of the paper is as follows. The problem is formalized in Section 2 which also reviews the analytical findings of DFSZ and discusses two implicit measures of the value of an uncertain non-spanned stream of income. In Section 3, we implement the Markov chain approximation method and prove its convergence. Numerical results are presented and discussed in Section 4. In Section 5, we study the sensitivity of the results with respect to selected parameters. Finally, Section 6 briefly summarizes the paper.

2. The problem

2.1. Statement of the problem

Let \( W = (W_1, W_2) \) be a standard two-dimensional Brownian motion on some probability space \((\Omega, \mathcal{F}, \mathbb{P})\), and let \( \mathcal{F} = \{ \mathcal{F}_t; t \geq 0 \} \) be the filtration generated by
Consider an investor who wants to maximize her expected life-time utility from consumption \( E \left[ \int_0^T e^{-\beta t} u(c(t)) \, dt \right] \). The investor can invest in a risky asset with a price process \( S(\cdot) \) given by

\[
dS(t) = S(t)(b \, dt + \sigma \, dW_1(t)), \quad S(0) > 0,
\]
and in a riskless asset with a constant continuously compounded rate of return \( r \). The investor has an initial wealth endowment \( x \geq 0 \) and receives income from non-traded assets at a rate given by the process \( Y(\cdot) \), where

\[
dY(t) = Y(t)(\mu \, dt + \delta \rho \, dW_1(t) + \delta \sqrt{1 - \rho^2} \, dW_2(t)), \quad Y(0) = y > 0.
\]

Here \( b, \sigma, \mu, \) and \( \delta \) are positive constants, and \( \rho \in (-1,1) \) is the correlation between changes in the risky asset price \( S \) and changes in the income rate \( Y \). Hence, the income rate process is not spanned by traded assets, i.e. the investor faces undiversifiable income risk.

A consumption process is defined to be an \( \mathcal{F} \)-progressively measurable process \( c: \Omega \times \mathbb{R}_+ \to \mathbb{R}_+ \) with \( E[\int_0^T c(t) \, dt] < \infty \) for all \( T \geq 0 \) and \( E[\int_0^\infty e^{-\beta t} u(c(t)) \, dt] < \infty \). \( c(t) \) is the rate of consumption at time \( t \). The set of consumption processes is denoted by \( \mathcal{C} \). A portfolio process is an \( \mathcal{F} \)-progressively measurable process \( \pi: \Omega \times \mathbb{R}_+ \to \mathbb{R} \) satisfying \( \int_0^T \pi(t)^2 \, dt < \infty \) (a.s.) for all \( T \geq 0 \). \( \pi(t) \) is the dollar amount invested at time \( t \) in the risky asset. The set of portfolio processes is denoted by \( \mathcal{P} \).

Define \( X(t) \) as the time \( t \) value of the agent’s portfolio of financial assets. Hence, \( X(t) \) is the liquid wealth of the investor at time \( t \) and does not take into account the value of her future uncertain income stream. Given a consumption process \( c \in \mathcal{C} \) and a portfolio process \( \pi \in \mathcal{P} \), \( X(\cdot) = X^{c,\pi}(\cdot) \) evolves as

\[
dx(t) = (rX(t) + \pi(t)(b - r) - c(t) + Y(t)) \, dt + \pi(t)\sigma \, dW_1(t), \quad X(0) = x.
\]

The investor can choose \( c \) and \( \pi \) from the set \( \mathcal{A}(x,y) \) of admissible controls, where

\[
\mathcal{A}(x,y) = \{(c,\pi) \in \mathcal{C} \times \mathcal{P}: X^{c,\pi}(t) \geq 0 \text{ (a.s.), } \forall t \geq 0\}.
\]

The admissible control policies have the property that the liquid wealth is non-negative (with probability one) at all points in time. The investor is not allowed to borrow funds against her future uncertain income.

Define the value function of the investor’s problem by

\[
\nu(x,y) = \sup_{(c,\pi) \in \mathcal{A}(x,y)} E \left[ \int_0^\infty e^{-\beta t} u(c(t)) \, dt \right].
\]
The HJB equation\(^3\) associated with this control problem is
\[
\beta v(x, y) = \frac{1}{2} \frac{\partial^2 v}{\partial y^2} (x, y) \sigma^2 y^2 + \frac{\partial v}{\partial x} (x, y) [rx + y] + \frac{\partial v}{\partial y} (x, y) \mu y + \sup_{\tilde{c} \in \mathbb{R}} \left\{ u(\tilde{c}) - \frac{\partial v}{\partial y} (x, y) \tilde{c} \right\}
+ \sup_{x \in \mathbb{R}} \left\{ \frac{1}{2} \frac{\partial^2 v}{\partial x^2} (x, y) \pi^2 \sigma^2 + \frac{\partial^2 v}{\partial x \partial y} (x, y) \pi \sigma \delta p y + \frac{\partial v}{\partial x} (x, y) \pi (b - r) \right\}.
\]

(3)

2.2. Reduction of the problem

Now assume that the investor has a CRRA (constant relative risk aversion) utility function
\[ u(c) = c^\gamma, \quad 0 < \gamma < 1, \]
where \(1 - \gamma\) is the coefficient of relative risk aversion. The subsequent steps meant to reduce the dimension of the problem from two to one follow DFSZ. For any positive constant \(k\), it follows from the linearity of the wealth dynamics in Eq. (1) that \((c, \pi) \in \mathcal{A}(x, y)\) if and only if \((kc, k\pi) \in \mathcal{A}(kx, ky)\) and hence that \(v(kx, ky) = k^\gamma v(x, y)\), i.e., \(v\) is homogeneous of degree \(\gamma\). As demonstrated below, this allows a reduction in the dimension of the problem from two to one.\(^4\)

Define \(F: \mathbb{R}_+ \to \mathbb{R}_+\) by \(F(z) = v(z, 1)\). Then \(v\) can be recovered from \(F\) since, for \(y \neq 0\), \(v(x, y) = y^\gamma v(x/y, 1) = y^\gamma F(x/y)\). For \(y = 0\), the value function \(v(x, 0)\) is known from Merton’s work. Define
\[
A = \frac{\beta - r\gamma}{1 - \gamma} - \frac{\gamma(b - r)^2}{2(1 - \gamma)^2 \sigma^2}.
\]

\(^3\)It is not clear a priori that the value function \(v\) is twice differentiable and hence that the HJB equation (3) has a solution in the classical sense. In fact, the HJB equation is of the degenerate elliptic type, so one cannot expect \(v\) to be a smooth solution to Eq. (3); see, e.g., the discussion in Fleming and Soner (1993, Section IV.5).

\(^4\)The reduction idea was suggested by Davis and Norman (1990). A reduction is also possible in the case of logarithmic utility, \(u(c) = \log c\), and for the finite horizon case, a reduction of the problem’s dimension is possible, e.g., when the utility function for terminal wealth is identically equal to zero or if it is identical to the utility function of consumption. The homogeneity property is also sustained if the amount invested in the risky asset is constrained to a convex cone, a specification which contains many portfolio constraints of interest; see, e.g., Cuoco (1997). There can also be multiple risky assets as long as their drift and noise terms are constant. Of course, this will increase the number of controls in the reduced problem, but not the dimension of the state variable. Indeed, the extension to multiple risky assets is a real extension of the model in the sense that portfolio separation does not obtain.
Under the assumption $A > 0$, Merton showed that $v(x, 0) = A^y x^y$ with optimal policies $c(t) = C(X(t), 0)$ and $\pi(t) = \Pi(X(t), 0)$, where

$$ C(x, 0) = Ax, \quad \Pi(x, 0) = \frac{b - r}{\sigma^2(1 - \gamma)} x. \tag{4} $$

Exploiting $v(x, y) = y^\gamma F(x/y)$, we get from Eq. (3) that

$$ \hat{\beta} F(z) = \frac{1}{2} \delta^2 z^2 F''(z) + k_2 z F'(z) + \sup_{\zeta \geq -1} \left\{ - \zeta F'(z) + (1 + \zeta) \right\} $$

$$ + \sup_{\psi \in \mathbb{R}} \left\{ \frac{1}{2} \sigma^2 \psi^2 - \sigma \delta \rho \psi \right\} F''(z) + k_1 \psi F'(z), \tag{5} $$

where

$$ \hat{\beta} = \beta - \mu_\gamma + \frac{1}{2} \delta^2 \gamma (1 - \gamma), $$

$$ k_1 = b - r - (1 - \gamma) \sigma \delta \rho, \quad k_2 = \delta^2 (1 - \gamma) + r - \mu, $$

and the variables $\zeta = \bar{c}/y - 1$ and $\psi = \bar{\pi}/y$ have been introduced. The maximizers of the two sup-terms are

$$ \zeta^*(z) = (F'(z)/\gamma)^1/(\gamma - 1) - 1, \quad \psi^*(z) = \frac{\delta \rho z}{\sigma} - \frac{k_1 F'(z)}{\sigma^2 F''(z)}, $$

so for $x, y > 0$ the candidate optimal consumption and investment policies in feedback form are $c(t) = C(X(t), Y(t))$, $\pi(t) = \Pi(X(t), Y(t))$, where

$$ C(x, y) = y \left( \frac{F'(x/y)}{\gamma} \right)^1/(\gamma - 1), \quad \Pi(x, y) = \frac{\delta \rho}{\sigma} x - \frac{k_1 y F'(x/y)}{\sigma^2 F''(x/y)}. \tag{6} $$

By direct computations it can be verified that

$$ \sup_{\psi \in \mathbb{R}} \left\{ \frac{1}{2} \sigma^2 \psi^2 - \sigma \delta \rho \psi \right\} F''(z) + k_1 \psi F'(z) $$

$$ = - \frac{1}{2} k_1^2 F'(z)^2 + \frac{1}{2} \delta^2 \rho^2 z^2 F''(z) + \frac{k_1 \delta \rho}{\sigma} F'(z) $$

$$ = \sup_{\phi \in \mathbb{R}} \left\{ \frac{1}{2} \sigma^2 \phi^2 F''(z) + k_1 \phi F'(z) \right\} + \frac{\rho k_1 \delta \rho}{\sigma} F'(z) - \frac{1}{2} \delta^2 \rho^2 z^2 F''(z), $$

where the maximizers $\psi^*$ and $\phi^*$ of the two sup-terms are related by the equation $\psi^* = \delta \rho z/\sigma + \phi^*$, and hence Eq. (5) may be rewritten as

$$ \hat{\beta} F(z) = \frac{1}{2} \delta^2 (1 - \rho^2) z^2 F''(z) + k_2 z F'(z) + \sup_{\zeta \geq -1} \left\{ - \zeta F'(z) + (1 + \zeta) \right\} $$

$$ + \sup_{\phi \in \mathbb{R}} \left\{ \frac{1}{2} \sigma^2 \phi^2 F''(z) + k_1 \phi F'(z) \right\}, \tag{7} $$

where $k = k_2 + \rho k_1 \delta/\sigma$. 
Eq. (7) is actually the HJB equation associated with the control problem

\[ F(z) = \sup_{(\zeta, \phi) \in \mathcal{A}(z)} \mathbb{E} \left[ \int_0^\infty e^{-\hat{\beta}t}(1 + \zeta(t))^\gamma dt \right], \quad (8) \]

where

\[
dZ(t) = (kZ(t) + k_1 \phi(t) - \zeta(t)) dt + \sigma \phi(t) dW_1(t) + \delta Z(t) \sqrt{1 - \rho^2} dW_2(t), \quad Z(0) = z, \quad (9)
\]

\[
\hat{A}(z) = \{(\zeta, \phi) \in \mathcal{C}_{-1} \times \mathcal{P}: Z(t) \geq 0 \text{ (a.s.), } \forall t \geq 0\}.
\]

Here \( \mathcal{C}_{-1} \) is the set of progressively measurable processes \( \zeta \) with \( \mathbb{E} \left[ \int_0^\infty \zeta(s) ds \right] < \infty \) and \( \zeta(t) \geq -1 \) for all \( t \geq 0 \), which is as the set \( \mathcal{C} \) except that the non-negativity constraint is replaced with the greater than or equal to \(-1\) constraint. Note that the HJB equation (7) can be written as

\[
\hat{\beta} F(z) = \sup_{\zeta \geq -1, \phi \in \mathbb{R}} \{(1 + \zeta)^\gamma + \mathcal{L}_{\zeta, \phi} F(z)\}, \quad (10)
\]

where the operator \( \mathcal{L}_{\zeta, \phi} \) is given by

\[
\mathcal{L}_{\zeta, \phi} F(z) = \frac{1}{2} (\delta^2 (1 - \rho^2) z^2 + \sigma^2 \phi^2) F''(z) + (kz + k_1 \phi - \zeta) F'(z).
\]

2.3. Review of analytical results

Applying viscosity solution techniques DFSZ are able to show that, under the parameter restrictions \( \hat{\beta} > 0, A > 0, \) and \( r > \mu \), the reduced HJB equation (7) has a unique \( C(\mathbb{R}_+) \cap C^2([0, \infty)) \) solution \( F \) in the class of concave functions; cf. Theorem 1 of DFSZ. Furthermore, the unique optimal feedback policy \((C, \Pi)\) for the original problem (2) is given by Eq. (6) for \( x, y > 0 \), by Eq. (4) for \( y = 0 \), and by

\[
C(0, y) = \left( \frac{F(0)}{y} \right)^{1/(\gamma - 1)} y, \quad \Pi(0, y) = 0,
\]

for \( x = 0 \) and \( y > 0 \), and DFSZ provide some results on the behavior of \( F(\cdot) \) near zero; cf. their Proposition 1. DFSZ also show that as the ratio of wealth to income goes to infinity, the value function, respectively the optimal policies, are asymptotically equal to the value function, respectively the optimal policies, in Merton’s problem.

Koo (1998) shows that, for \( x > 0 \) and \( y > 0 \), the homogeneity property of the value function implies that the following two mappings \( B(\cdot) \) and \( A(\cdot) \) are well-defined continuous functions of \( z = x/y \):

\[
B(x/y) = \frac{\partial v}{\partial y} (x, y), \quad A(x/y) = \left( \frac{v(x, y)}{(x + B(x/y)y)^{\gamma}} \right)^{1/(\gamma - 1)}.
\]
Indeed,
\begin{align*}
B(z) &= \gamma F(z) - z, \\
A(z) &= \left(\frac{\gamma}{F'(z)}\right)^{\gamma/(1-\gamma)} / F(z).
\end{align*}

The value function can be written as
\[ v(x,y) = A(z)^{\gamma-1}(x + B(z)y)^\gamma, \quad x > 0, \; y > 0, \]
and hence
\[ F(z) = A(z)^{\gamma-1}(z + B(z))^\gamma. \]}

In the complete market where there are no liquidity constraints and the income rate is spanned, the certainty wealth equivalent of lifetime income is
\[ E[\int_0^{\infty} p(t)Y(t)dt] \]
where
\[ p(t) = \exp\left\{-rt - \frac{1}{2} \left(\frac{b-r}{\sigma}\right)^2 t - \frac{b-r}{\sigma} W_1(t)\right\} \]
is the unique state-price density. Define the constant \( \lambda = r - \mu + \delta(b-r)/\sigma \). Then a simple computation yields that, if \( \lambda > 0 \), the certainty wealth equivalent of lifetime income is equal to \( y/\lambda \). If the constants \( A \) and \( \lambda \) are both positive, we therefore have that, in the complete market, the value function is given by
\[ v_{\text{com}}(x,y) = A^{\gamma-1}\left(\frac{x + y}{\lambda}\right)^\gamma \]
with optimal consumption policy
\[ C_{\text{com}}(x,y) = A\left(\frac{x + y}{\lambda}\right). \]

The optimal risky investment is\(^5\)
\[ \Pi_{\text{com}}(x,y) = \frac{b-r}{\sigma^2(1-\gamma)}\left(\frac{x + y}{\lambda}\right) - \frac{\delta y}{\sigma\lambda}. \]

Thus, in the complete market, \( B(z) = 1/\lambda \) and \( A(z) = A \) for all \( z > 0 \).

Koo shows that, if \( \beta > 0, \; A > 0, \; r > \mu, \) and \( |\rho| \neq 1 \), then, for \( x > 0 \) and \( y > 0 \), optimal consumption is given by
\[ C(x,y) = A(z)(x + B(z)y), \]}

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\(^5\)In the case of a certain income stream where \( \delta = 0 \), we see that the value function and the optimal policies are exactly as suggested by Merton (1971, Section 7), cf. our introduction.
and the optimal amount invested in the risky asset is given by

\[ \Pi(x, y) = \frac{b - r}{\sigma^2} \frac{x + B(z)y}{1 - \gamma + B'(z)} + \frac{\delta \rho}{\sigma} \left( x - \frac{1 - \gamma}{1 - \gamma + B'(z)}(x + B(z)y) \right), \]

where \( z = x/y \); cf. his Theorem 4.1. If, furthermore, \( \lambda > 0 \), then \( B(z) \) is strictly increasing and

\[ \lim_{z \to \infty} B(z) = 1, \quad \lim_{z \to \infty} A(z) = A. \]

In particular, the ratio of optimal consumption to the accounting total wealth \( x + y/\lambda \) is strictly increasing in \( z \).

Based on Eq. (14), Koo interprets the term \( B(z)y \) as the implicit value the agent associates with her future uncertain income stream in the presence of liquidity constraints and undiversifiable income risk. However, \( B(z) \) is by definition a marginal rate of substitution between income and liquid wealth. An alternative, and perhaps more natural, measure of the value of the entire income stream is the wealth equivalent

\[ \mathcal{W}(x, y) = \inf\{ q \geq 0 : v(x + q, 0) \geq v(x, y)\}, \]

i.e. the least increase in initial wealth the investor would accept in exchange for the entire stream of income. Since \( v(x + q, 0) = A^{\gamma - 1}(x + q)^\gamma \), we get

\[ \mathcal{W}(x, y) = A^{(1 - \gamma)/\gamma} v(x, y)^{1/\gamma} - x = A^{(1 - \gamma)/\gamma} y F(x/y)^{1/\gamma} - x = B^*(z)y, \]

where

\[ B^*(z) = A^{(1 - \gamma)/\gamma} F(z)^{1/\gamma} - z \]

and, as before, \( z = x/y \). Notice that from Eq. (12) we have

\[ B(z) = A(z)^{1 - \gamma/\gamma} F(z)^{1/\gamma} - z. \]

Since \( \lim_{z \to \infty} A(z) = A \), we see that for large \( z \) the two income multipliers \( B(z) \) and \( B^*(z) \) will be approximately equal.

3. The numerical method

3.1. An approximating Markov chain

The reduced control problem (8) is solved with the Markov chain approximation approach. Similar approximations have been used to solve other consumption/investment problems by Fitzpatrick and Fleming (1991) and Hindy et al. (1997). We approximate the controlled state variable process \( Z = (Z(t))_{t \in \mathbb{R}} \), by a controlled discrete-time Markov chain \( \xi^h = (\xi^h_n)_{n \in \mathbb{Z}} \), on the discrete state space
$\mathcal{H}^h = \{0, h, 2h, \ldots, Ih\}$. Here, $\tilde{z} \equiv Ih$ is an artificially imposed upper boundary. A control is a sequence $(\zeta^h, \varphi^h) = (\zeta^h_n, \varphi^h_m)_{m \in \mathbb{Z}}$, where $\zeta^h_n = \zeta^h(z^h_n)$ and $\varphi^h_n = \varphi^h(z^h_n)$. The controls are bounded by the requirements

$$-1 \leq \zeta^h(z) \leq K_\zeta z \quad \text{and} \quad |\varphi^h(z)| \leq K_\varphi z,$$

where $K_\zeta$ and $K_\varphi$ are positive constants. The stochastic evolution of the controlled Markov chain $\zeta^h$ is given by the transition probabilities$^6$

$$p^h(z, z-h | \zeta, \varphi) = \frac{\lambda (\sigma^2 \varphi^2 + \delta^2 (1 - \rho^2)z^2) + h(k^- z + (k_1 \varphi)^- + \zeta^+)}{Q^h(z)},$$  \hspace{1cm} (17a)

$$p^h(z, z+h | \zeta, \varphi) = \frac{\lambda (\sigma^2 \varphi^2 + \delta^2 (1 - \rho^2)z^2) + h(k^+ z + (k_1 \varphi)^+ + \zeta^-)}{Q^h(z)},$$  \hspace{1cm} (17b)

$$p^h(z, z | \zeta, \varphi) = 1 - p^h(z, z-h | \zeta, \varphi) - p^h(z, z+h | \zeta, \varphi)$$  \hspace{1cm} (17c)

for $z \in \{h, 2h, \ldots, (1-1)h\}$, where

$$Q^h(z) = \sigma^2 K_{\varphi z^2} + \delta^2 (1 - \rho^2)z^2 + h(|k| z + |k_1| K_\varphi z + \max\{1, K_\zeta z\}).$$

Here, $p^h(z, z' | \zeta, \varphi)$ denotes the probability of the state changing from $z$ to $z'$ in one “time step” when the control $(\zeta, \varphi)$ is currently applied. At the upper boundary we can take

$$p^h(\tilde{z}, \tilde{z}-h | \zeta, \varphi) = \frac{\lambda (\sigma^2 \varphi^2 + \delta^2 (1 - \rho^2)\tilde{z}^2) + h(k^- \tilde{z} + (k_1 \varphi)^- + \zeta^+)}{Q^h(\tilde{z})},$$  \hspace{1cm} (17d)

$$p^h(\tilde{z}, \tilde{z} | \zeta, \varphi) = 1 - p^h(\tilde{z}, \tilde{z}-h | \zeta, \varphi).$$  \hspace{1cm} (17e)

Since $Z$ must stay non-negative, we conclude from Eq. (9) that $\varphi$ at $Z=0$ must be zero and subsequently that $\zeta$ has to be non-positive at $Z=0$. Therefore, we take

$$p^h(0, h | \zeta, \varphi) = \frac{h \zeta^-}{Q^h(0)} = \zeta^-,$$  \hspace{1cm} (17f)

$$p^h(0, 0 | \zeta, \varphi) = 1 - p^h(0, h | \zeta, \varphi) = 1 - \zeta^-.$$  \hspace{1cm} (17g)

All other transition probabilities are zero.

The control $(\zeta^h, \varphi^h) = (\zeta^h_n, \varphi^h_m)_{m \in \mathbb{Z}}$, is called an admissible control for $\zeta^h$ if Eq. (16) is satisfied for all $z \in \mathcal{H}^h$ and $\zeta^h$ is a Markov chain when it is controlled by $(\zeta^h, \varphi^h)$. The set of admissible controls for $\zeta^h$ given $\zeta^h_0 = z$ is denoted $\mathcal{A}^h(z)$.

$^6$The plus and minus superscripts indicate the positive and negative part, respectively, i.e. $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$. 
Define the interpolation interval function $\Delta t^h(z) = h^2 / Q^h(z)$ and let $\Delta t^h = \Delta t^h(z_n)$ and $t_n^h = \sum_{n=0}^{N-1} \Delta t^h_n$. Define

$$F^h(z) = \sup_{(\xi^h, \phi^h) \in \mathcal{U}(z)} \left\{ \sum_{n=0}^{\infty} e^{-\beta \Delta t^h} (1 + \xi^h_n)^\gamma \Delta t^h_n \xi^h_0 = z \right\}, \quad z \in \mathcal{R}^h.$$  

The dynamic programming equation for the discrete-time Markov chain control problem (18) is

$$F^h(z) = \sup_{-1 \leq \zeta \leq K_\zeta, |\phi| \leq K_\phi} \left\{ \Delta t^h(z)(1 + \zeta)^\gamma + e^{-\beta \Delta t^h(z)} \sum_{z' \in \mathcal{R}^h} p^h(z, z' | \zeta, \phi) F^h(z') \right\}. \tag{19}$$

We note that the Markov chain $\xi^h$ is locally consistent with the process $Z(\cdot)$, since

$$E[\zeta^h_{n+1} - \zeta^h_n | \zeta^h_n = \zeta, \phi^h_n = \phi] = \Delta t^h(z)(kz + k_1 \phi - \zeta), \tag{20}$$

$$\text{Var}[\zeta^h_{n+1} - \zeta^h_n | \zeta^h_n = \zeta, \phi^h_n = \phi] = \Delta t^h(z)(\sigma^2 \phi^2 + \delta^2 (1 - \rho^2) z^2)$$

$$+ o(h \Delta t^h(z)), \tag{21}$$

for every $z \in \mathcal{R}^h$, $-1 \leq \zeta \leq K_\zeta z$, and $|\phi| \leq K_\phi z$. Eq. (20) says that the expected change in the Markov chain $\xi^h$ divided by the length of the time step is equal to the drift of the process $Z(\cdot)$, cf. Eq. (9). Similarly, Eq. (21) says that the variance of the change in $\xi^h$ divided by the length of the time step is approximately equal to the squared volatility of $Z(\cdot)$.

### 3.2. Convergence of the numerical method

We shall argue that the discrete-time value function $F^h(\cdot)$ converges to the continuous-time value function $F(\cdot)$ as $h \to 0$, $K_\zeta, K_\phi \to \infty$, and $\bar{z} = Ih \to \infty$. The convergence of the Markov chain approximation approach has been proved for various stochastic control problems by, e.g., Kushner and Dupuis (1992, Chap. 14), Fleming and Soner (1993, Chap. IX), Fitzpatrick and Fleming (1991), and Hindy et al. (1997) using viscosity solution techniques. While these proofs only require that the value function of the continuous-time control problem is a viscosity solution of the associated HJB equation, we know from DFSZ that the value function for the problem (8) is actually a classical solution of the HJB equation (7). Therefore, convergence will follow immediately from the stability and consistency of the Markov chain approximation approach which we shall prove below.

Define the operator $\mathcal{F}^h$ on the space $\mathcal{B}$ of real functions on $\mathcal{R}^h$ by

$$\mathcal{F}^h F(z) = e^{-\beta \Delta t^h(z)} \sum_{z' \in \mathcal{R}^h} p^h(z, z' | \zeta, \phi) F(z').$$
Obviously, 
\[
|\mathcal{F}^h F_1(z) - \mathcal{F}^h F_2(z)| \leq e^{-\beta \Delta t} \sum_{z' \in \mathcal{X}} p^h(z, z' | \zeta, \phi) |F_1(z') - F_2(z')|
\]
\[
\leq e^{-\beta \Delta t} \max_{z' \in \mathcal{X}} |F_1(z') - F_2(z')|,
\]
where \(\Delta h = \min_{x \in \mathcal{X}} \Delta t^h(z) > 0\). Hence, \(\mathcal{F}^h\) is a strict contraction in terms of the operator norm \(\|\mathcal{F}\| = \sup_{F} \|\mathcal{F} F\|_{\infty} / \| F\|_{\infty}\) where \(\| \cdot \|_{\infty}\) is the sup-norm on \(\mathcal{B}\). This implies the stability of the method.

Note that with the transition probabilities given by Eq. (17) we have
\[
\sum_{z' \in \mathcal{X}} p^h(z, z' | \zeta, \phi) F^h(z') = \Delta t^h(z) \mathcal{L}^h_{\zeta, \phi} F^h(z) + F^h(z),
\]
where the operator \(\mathcal{L}^h_{\zeta, \phi}\) is given by
\[
\mathcal{L}^h_{\zeta, \phi} F^h(z) = \frac{1}{2} (\delta^2 (1 - \rho^2) z^2 + \sigma^2 \phi^2) D^2 F^h(z)
\]
\[
+ (k^+ z + (k_1 \phi)^+ + \zeta^+) D^+ F^h(z) - (k^- z + (k_1 \phi)^- + \zeta^-) D^- F^h(z),
\]
\[
D^+ F^h(z) = \frac{F^h(z + h) - F^h(z)}{h}, \quad D^- F^h(z) = \frac{F^h(z) - F^h(z - h)}{h},
\]
\[
D^2 F^h(z) = \frac{F^h(z + h) - 2F^h(z) + F^h(z - h)}{h^2}.
\]

Hence, the discrete dynamic programming Eq. (19) can be rewritten as
\[
\frac{1 - e^{-\beta \Delta t}}{\Delta t^h(z)} F^h(z) = \max_{-1 \leq \zeta \leq K_{\zeta, \phi} | \phi | \leq K_{\phi, z}} \{ (1 + \zeta)^+ e^{-\beta \Delta t} \mathcal{L}^h_{\zeta, \phi} F^h(z) \}. \quad (22)
\]
Comparing Eq. (22) with the HJB equation (10) we see that consistency of the method follows from the facts that \((1 - e^{-\beta \Delta t}) / \Delta t^h(z) \to \hat{\beta}\), \(e^{-\beta \Delta t} \to 1\), and \(\mathcal{L}^h_{\zeta, \phi} \to \mathcal{L}^\infty_{\zeta, \phi}\) as \(h \to 0\) and \(K_{\zeta, \phi} \to \infty\). Note the link between the consistency of the method and the local consistency property of the approximating Markov chain.

The optimal control policies can be expressed in terms of the value function and its derivatives (for the continuous-time, continuous-state problem) or its differences (for the approximating problem). Given the convergence of the value function and the convergence of finite differences to derivatives it follows that the optimal policies for the approximating Markov control problem converges to the optimal control policies of the continuous-time, continuous-state problem. For details the reader is referred to Hindy et al. (1997, Theorem 5) and Fitzpatrick and Fleming (1991, Section 4).

As noted by, e.g., Fleming and Soner (1993, Chap. IX) and Hindy et al. (1997), the specification of the Markov chain at the artificial upper bound is not important for the convergence of the method, but will, of course, affect the quality of the numerical results. Also note the importance of the reduction of the problem form a
two-variable to a one-variable problem. Due to the non-trivial control-dependence of the variance–covariance matrix of the original two-dimensional state-variable \((X,Y)\) it is not possible to approximate it by a locally consistent Markov chain.

3.3. Solving the approximating Markov chain control problem

The DPE (19) is solved with the policy iteration algorithm. Given an arbitrary admissible control policy \(\zeta^h_0, \phi^h_0\), i.e. given \((\zeta^h_0(z), \phi^h_0(z))\) for all \(z \in \mathcal{S}^h\), perform a policy evaluation by solving the linear system of equations

\[
F^h_0(z) = \Delta^h(z)(1 + \zeta^h_0(z))^p + e^{-\beta \Delta^h(z)} \sum_{z' \in \mathcal{S}^h} p^h(z,z' | \zeta^h_0(z), \phi^h_0(z))F^h_0(z'), \quad z \in \mathcal{S}^h.
\]

Next, a policy improvement is computed by

\[
(\zeta^h_1(z), \phi^h_1(z)) = \arg \max_{-1 \leq \zeta \leq K_z, 1 \leq K_z} \left\{ \Delta^h(z)(1 + \zeta)^p + e^{-\beta \Delta^h(z)} \sum_{z' \in \mathcal{S}^h} p^h(z,z' | \zeta, \phi)F^h_0(z') \right\}.
\]

Then a new guess \(F^h_1\) on the value function is computed on the basis of \((\zeta^h_1, \phi^h_1)\) similarly to the computation of \(F^h_0\), etc. Note that each policy iteration step always generates an improvement so that \(F^h_m\) converges to \(F^h\) from below. A simple criterion for stopping the algorithm is to stop the first time \(\sup_{z \in \mathcal{S}^h} |F^h_m(z) - F^h_{m-1}(z)| < \varepsilon\) for some tolerance \(\varepsilon > 0\).

Since the transitions of the approximating Markov chain are to only ‘nearest neighbors’, the equation system Eq. (23) has a tri-diagonal matrix structure which makes it very fast to solve. For interior states \(z\), the \(j\)th policy improvement step amounts to computing

\[
\zeta^h_j(z) = \arg \max_{-1 \leq \zeta \leq K_z} \left\{ (1 + \zeta)^p + e^{-\beta \Delta^h(z)}(-\zeta + D^- F^h_{j-1}(z) + \zeta D^+ F^h_{j-1}(z)) \right\},
\]

\[
\phi^h_j(z) = \arg \max_{|\psi| \leq K_z} \left\{ \frac{1}{2} \sigma^2 \psi^2 D^2 F^h_{j-1}(z) - (k_1 \psi)^2 D^- F^h_{j-1}(z) + (k_1 \psi)^+ D^+ F^h_{j-1}(z) \right\}.
\]

At the upper bound \(\bar{z}\) the corresponding equations are

\[
\zeta^h_j(\bar{z}) = \arg \max_{-1 \leq \zeta \leq K_{\bar{z}}} \left\{ (1 + \zeta)^p - e^{-\beta \Delta^h(\bar{z})} D^- F^h_{j-1}(\bar{z}) \right\},
\]

\[
\phi^h_j(\bar{z}) = \arg \max_{|\psi| \leq K_{\bar{z}}} \left\{ - D^- F^h_{j-1}(\bar{z}) \left( \frac{1}{2} \sigma^2 \psi^2 + h(k_1 \psi) \right) \right\}
\]

implying \(\phi^h_j(\bar{z}) = 0\) as \(D^- F^h_{j-1}(\bar{z}) \geq 0\).
Convergence of the policy space algorithm will, other things equal, be faster, the faster the probability mass ‘spreads’. A locally consistent Markov chain with $p^h(z, z | \zeta, \phi) = 0$ is obtained by replacing the denominator $Q^h(z)$ of $p^h(z, z + h | \zeta, \phi)$ with the control-dependent denominator

$$Q^h(z, \zeta, \phi) = \sigma^2 \phi^2 + \delta^4(1 - \rho^2)z^2 + h(|k| z + |k_1 \phi| + |\zeta|).$$

This procedure does not require bounding of the controls, but the policy improvement step is now significantly more complicated, since the controls then enter both the numerator and the denominator of the probabilities. Therefore, we prefer the scheme with bounded controls. \(^7\)

4. Numerical results

In this section we present and discuss results from an implementation of the numerical method outlined in Section 3. The basic economic parameter values are taken to be

$$\gamma = 0.5, \quad r = 0.1, \quad b = 0.15, \quad \mu = 0.05,$$

$$\beta = 0.2, \quad \sigma = 0.3, \quad \delta = 0.1, \quad \rho = 0$$

unless otherwise indicated. The auxiliary parameters are then

$$\hat{\beta} = 0.17625, \quad k_1 = 0.05, \quad k_2 = 0.055, \quad k = 0.055, \quad A \approx 0.27222.$$ 

$K_\zeta$ and $K_\phi$ must be determined experimentally and are, of course, dependent on the other parameters. They are chosen as small as possible under the condition that their values do not affect the optimal controls. Unnecessarily high values slow down the method.

4.1. Properties of the numerical method

The Markov chain approximation approach does not provide any measures of the precision of the numerical results. The reduced problem has a structure similar to Merton’s no-income consumption/investment problem for which the

\(^7\)Note that $Q^h(z) = \sup [Q^h(z, \zeta, \phi): -1 \leq \zeta \leq K_\zeta, |\phi| \leq K_\phi]$. Fitzpatrick and Fleming (1991) also impose bounds on the controls to avoid control dependent denominators. Their scheme corresponds to replacing $Q^h(z, \zeta, \phi)$ with a constant $Q^h$ given by $Q^h = \sup [Q^h(z, \zeta, \phi): z \in \mathfrak{P}, -1 \leq \zeta \leq K, -K \leq \phi \leq K]$ for some constant $K$. Of course, this will increase the probability of staying at a state which tends to slow down convergence. Since the advantages of having a constant $Q^h$ and, hence, a constant $\Delta u^h$, are very limited, the $Q^h(z)$ probability scheme is adopted here.
solution is as shown in Section 2.2. An indication of the numerical properties of our proposed scheme can, therefore, be obtained by studying the performance of the Markov chain approximation approach on Merton’s problem. Such an analysis is contained in Munk (1998). The numerically computed value function was found to be very precise over the entire range of the wealth level, which is the only state variable of that problem. The numerically computed optimal controls were rather imprecise near the artificially imposed upper bound on the state variable, but otherwise also very precise. For our problem there is also no reason to believe that the numerically computed optimal controls at the upper bound are anywhere near the true optimal controls. The error introduced this way will propagate to lower values of the state variable. The range of states in which the numerically computed controls can be considered reliable is highly dependent on the value of the contraction parameter of the DPE, which in our case is the parameter $\beta$. The higher the contraction parameter, the wider the reliable range.

For practical applications the particular values of the controls might not be that important, as long as they induce a utility level very close to the true, but unknown value function. But since we want to study the economic properties of the optimal control policies, we cannot ignore this feature of the numerical scheme. Therefore, the numerical results discussed below do not rely on values near the artificial upper bound on $z$.

The numerical values presented below are computed with a grid refinement of $I = 20\,000$ (the corresponding $h$-values depend, of course, on $\tilde{z}$). The policy iteration algorithm is implemented with tolerance $\varepsilon = 10^{-5}$. With these specifications running the computer program on an HP9000/E35 128MB computer takes about 5–20 s depending on $\tilde{z}$ and the parameter values.

Our numerical results confirm the analytical findings of DFSZ concerning the behavior of $F(\cdot)$ near zero and that the value function and optimal policies of the unreduced problem asymptotically are equal to the value function and optimal

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8 A standard way to assess the speed and smoothness of the convergence of a numerical method is to look at the so-called experimental order of convergence, which can be estimated by examining the variations in the numerically computed quantity when the grid refinement $h$ (or, equivalently, $I$) is varied. With the default parameter values in Eq. (24) the experimental order of convergence is typically very unstable as the grid refinement $h$ is varied. For higher values of the contraction parameter $\beta$ the convergence is much smoother and the convergence order is roughly one for both the value function and the controls. (For high levels of $z$ relative to $\tilde{z}$ the value function converges nicely, but for both controls, most pronouncedly for $q$, the computed experimental order indicates a lack of convergence.) With such a smooth convergence the Richardson extrapolation technique can be successfully superimposed, but since our focus is on lower and economically more appealing $\beta$-values we cannot benefit from extrapolation methods.

9 The main determinant of the running time is the number of iterations the policy iteration algorithm requires for convergence. With the stated value of $\varepsilon$ the policy space iteration algorithm typically converges in 5–10 iterations. A smaller $\varepsilon$ will result in additional time-consuming and essentially futile iterations.
policies of Merton’s no-income problem as the ratio of wealth to income goes to infinity.

4.2. The computed value function and optimal controls

Next, we present the computed value function and optimal controls for non-extreme values of the wealth/income ratio. Results discussed in this subsection are obtained from an implementation with 100. Fig. 1 shows the value function and Figs. 2 and 3 depict the optimal controls. All three surfaces are smooth and increasing in both coordinate directions as one would expect them to be.

Recall that in Merton’s no-income problem, both the optimal consumption rate and the optimal risky investment are constant fractions of wealth. With undiversifiable income risk and liquidity constraints these two fractions depend on wealth and income. For relatively high levels of wealth the fractions of consumption and risky investment to wealth are nearly constant across income rates, but they are rapidly increasing with the initial income rate for small levels of wealth. When measured relative to the initial income rate both controls are rather insensitive to the level of wealth as long as income is large relative to wealth, but for small levels of the income rate both the consumption/income ratio and the investment/income ratio increase rapidly as wealth increases.
Fig. 2. The optimal consumption rate, $C(x, y)$.

Fig. 3. The amount optimally invested in the risky asset, $\Pi(x, y)$. 
In this subsection we consider the implicit value that the investor associates with her stochastic income stream given the initial income rate. As discussed in Section 2.3, we can measure this value by \( B(z)y \), where \( B(z) \) is given by Eq. (11), or alternatively by \( B^\ast(z)y \), where \( B^\ast(z) \) is given by Eq. (15). Fig. 4 shows the two multipliers \( B(z) \) and \( B^\ast(z) \) of the implicit value of the income stream as a function of \( z \). The multipliers are strictly increasing in \( z \), so the value associated with a given uncertain stream of income increases with wealth. Intuitively this is because the importance of the liquidity constraint, i.e. the non-negative wealth constraint, decreases as wealth increases. Recall that for the complete markets case the multiplier is \( 1/\lambda \). With the parameters in Eq. (24), \( 1/\lambda = 15 \), so the implicit value attached to the income stream is much smaller in the presence of liquidity constraints and non-spanning. Since the magnitude of \( B(z) \) and \( B^\ast(z) \) is relatively insensitive to the value of the correlation parameter \( \rho \), as will be discussed in the next section, the large difference between the complete and the incomplete market income valuation must be imputed mainly to the liquidity constraints.
5. Parameter sensitivity

In this subsection we examine the sensitivity of results to the income process parameters $\rho$, $\delta$, and $\mu$, and to the time preference rate $\beta$. We shall not present a similar analysis for the remaining parameters, since the comparative statics with respect to those parameters are qualitatively the same as for the well-known complete market problem.

The correlation parameter, $\rho$. Fig. 5 shows the ratio of optimal risky investment to wealth, $\Pi(x,y)/x$, as a function of $\rho$ for four different values of $z = x/y$.\(^{10}\) This ratio, the portfolio weight of the risky asset, is obviously a decreasing function of $\rho$ for all four values of $z$. Intuitively this is because a negative correlation between changes in the income rate and changes in the risky asset price provides insurance against negative wealth: If the risky asset price and therefore the value of the investment decreases, the income rate will tend to increase and vice versa. For high values of $z$ the portfolio weight is nearly constant, whereas for low $z$ it is steeply decreasing in $\rho$. The intuition for this property is straightforward: When the financial wealth is small relative to the income rate it is very important to hedge against future changes in the income

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\(^{10}\)Since $\varphi(z) = \Pi(x,y)/y - \delta \rho z / \sigma$ we have $\Pi(x,y)/x = \varphi(z)/z + \delta \rho / \sigma$ where $z = x/y$. 

Fig. 6. The dependence of the relative optimal consumption rate \( C(x, y)/x \) on the correlation parameter \( \rho \). The results are from an implementation with \( z = 50 \).

The ratio is increasing in \( \rho \), but nearly constant for high wealth/income ratios. The value function is decreasing in \( \rho \), although only slightly, due to the fact that a lower \( \rho \) implies that the risky investment provides a better implicit hedge of the income risk.

Fig. 7 displays the dependence of the income multiplier \( B^s(z) \) on \( \rho \). For low wealth/income ratios the income multiplier is only slightly decreasing in \( \rho \), intuitively because with low wealth the liquidity constraint is more important.

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11 Since \( \zeta(z) = C(x, y)/y - 1 \), we have \( C(x, y)/x = (1 + \zeta(z))/z \).

12 Any linearly increasing transformation of the utility function of consumption will give the same optimal policies, but may change the resulting value function drastically. Therefore, one should generally avoid assessing the magnitude of changes in the value function. Instead, one can translate the value function into a measure which is independent of the specific representation of preferences. One such measure is the constant consumption equivalent \( \tilde{C}(x, y) \) defined as

\[
\tilde{C}(x, y) = \inf \left\{ c \geq 0 : \int_0^\infty e^{-\rho t} u(c) \, dt = v(x, y) \right\}.
\]

Whenever we address changes in the value function in the text, it is to be understood as changes in the constant consumption equivalent.
than the ability to hedge income risk in the future. Note that the implicit value of the uncertain income stream is substantially lower than in the complete markets case. For higher wealth/income ratios the liquidity constraint is less of a problem and the hedging possibilities are more important. Therefore, the implicit value of the income stream decreases significantly with $\rho$. The lower the correlation is, the better is the implicit income risk hedging properties of a positive investment in the risky asset and, hence, the more valuable is any given income process to the investor.

In Figs. 5–7 we have included the computed optimal policies and income multiplier for the cases $\rho = \pm 1$, although the analytical derivations leading to the numerical approach presume that $|\rho| < 1$. The case of $\rho = 1$ corresponds to a model where the income rate process is perfectly positively correlated with the risky asset price, so there is no undiversifiable income risk. For this particular case El Karoui and Jeanblanc-Picqué (1996, Theorem 4.3) have derived a closed-form expression of the current wealth/income ratio $z = x/y$ as a function of the ratio of optimal consumption to current income, $C(x,y)/y$. Table 1 shows the ratio of optimal consumption to wealth, $C(x,y)/x$, for $\rho = 1$ computed both with our numerical method and the El Karoui and Jeanblanc-Picqué relation for representative values of $z$. The numerically computed consumption/wealth ratios are within a few percent of the ratios computed with the closed-form relation. This indicates that our numerical method provides reliable
Table 1
The optimal consumption rate relative to current wealth, \( C(x,y)/x \), for various values of the wealth/income ratio, \( z = x/y \), in the case of perfectly positive correlation, \( \rho = 1 \)

<table>
<thead>
<tr>
<th>( z = x/y )</th>
<th>0.2</th>
<th>1</th>
<th>5</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Our numerical method</td>
<td>5.962</td>
<td>1.675</td>
<td>0.641</td>
<td>0.389</td>
</tr>
<tr>
<td>El Karoui and Jeanblanc</td>
<td>5.670</td>
<td>1.609</td>
<td>0.625</td>
<td>0.384</td>
</tr>
</tbody>
</table>

Fig. 8. The dependence of the relative optimal consumption rate \( C(x,y)/x \) on the income volatility parameter \( \delta \). The results are from an implementation with \( \bar{z} = 50 \).

results also for the case where \( \rho = 1 \), which again allows us to separate the effects of liquidity constraints from the effects of undiversifiable income risk.

The income rate volatility, \( \delta \). Higher income risk induces a risk-averse agent to increase total savings and decrease consumption in order to protect the wealth against possible declines in future income. Consistent with that argumentation Fig. 8 shows that the ratio of optimal consumption to initial wealth is a decreasing function of \( \delta \). The decrease is most pronounced for small values of the wealth/income fraction where the precautionary motive is strongest.

It is less clear how the allocation of total savings on the riskless and the risky asset varies with \( \delta \). Obviously, a certain income stream acts as a substitute for the riskless asset, but, as \( \delta \) increases, the income stream becomes more like a risky asset. Therefore, the fraction of wealth invested in the risky asset should decrease with \( \delta \), ceteris paribus. For the complete market case, Eq. (13) implies
Fig. 9. The dependence of the relative optimal risky investment $\Pi(x, y)/x$ on the income volatility parameter $\delta$. The results are from an implementation with $\tilde{z} = 50$.

that the fraction optimally invested in the risky asset is a linearly decreasing function of $\delta$. The higher the wealth/income ratio, the flatter the line. While Fig. 9 confirms the complete market intuition for moderate and high levels of wealth relative to income, it also shows that for a very low wealth/income ratio the fraction invested in the risky asset actually increases with $\delta$.\textsuperscript{13}

Fig. 10 shows that the income multiplier $B^*(z)$ is a concavely decreasing function of $\delta$, so that the implicit value of an uncertain income stream decreases with the income rate volatility. In the complete market case, the income multiplier $B(z) = 1/\lambda$ is a convex, decreasing function of $\delta$ for the set of values chosen here for the other parameters. The value function is also a decreasing, concave function of $\delta$. However, the changes of both the controls and the value function are relatively small when $\delta$ is varied from 0 up to about 0.2. Hence, for investors with an income which is not extremely volatile, the precise value of $\delta$ is not that important.

The income rate drift, $\mu$. The optimal consumption rate is an increasing function of $\mu$ for all levels of the wealth/income ratio, as can be seen from Fig. 11. The higher the wealth/income ratio, the flatter the curve. For low wealth/income ratios, a higher $\mu$ implies that a significantly higher consumption rate is possible.

\textsuperscript{13}Koo (1995) finds a similar relation for his discrete-time model which we briefly described in the Introduction.
Fig. 10. The dependence of the implicit value of income multiplier $B^*(z)$ on the income volatility parameter $\delta$. The results are from an implementation with $\bar{z} = 50$.

Fig. 11. The dependence of the relative optimal consumption rate $C(x, y)/x$ on the income drift parameter $\mu$. The results are from an implementation with $\bar{z} = 50$.

without leading the investor into bankruptcy. The dependence of the proportion of wealth invested in the risky asset on the income rate drift is less intuitive. Fig. 12 unveils that the risky asset weight is a decreasing function of $\mu$ for small values of the wealth/income ratios, but a slightly increasing function of $\mu$ for
large wealth/income ratios. As expected, the value function increases convexly with $\mu$. The income multipliers $B(z)$ and $B^*(z)$ increase significantly, also in a convex manner, with $\mu$.

The time preference rate, $\beta$. The value of the time preference rate, $\beta$, has a great impact on the optimal policies and the implicit value of the income stream. With a high time preference rate, the investor is very eager to transform some of the future income to current consumption. Due to the liquidity constraints, this is not possible, and therefore the implicit value of the income stream is much smaller than in the unconstrained case. Liquidity constraints are thus more restrictive for investors with a high time preference rate. Fig. 13 depicts the value of the income multiplier $B^*(z)$ for different values of $\beta$. For relatively high values of $\beta$, the income multiplier is much smaller than in the complete markets case, even for very high values of the wealth/income ratio $z$.

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14 In his related discrete-time model, Koo (1995) presents some results for two different expected growth rates of the income stream. His results are consistent with our findings.
6. Concluding remarks

In this paper we have studied the properties of the optimal consumption and investment policies of a liquidity constrained investor with an uncertain, non-spanned income process in a continuous-time set-up. The underlying optimization problem was solved numerically with the Markov chain approximation approach and the numerical solution was shown to converge to the true solution of the problem. We have found that the implicit value the investor associates with the entire income process is much smaller in the presence of liquidity constraints and undiversifiable income risk than without such imperfections. Our results suggest that this is mainly due to the presence of liquidity constraints.

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