Oligopoly equilibria in nonrenewable resource markets

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Abstract

Most Nash–Cournot oligopoly models of nonrenewable resources apply open-loop equilibrium concepts and are based on physical resource depletion. This paper studies feedback equilibria and economic depletion. Assuming affine-quadratic functional forms, the existence, uniqueness, and explicit solutions for the equilibria are derived for duopoly and n-player oligopoly with multiple resource stocks. For the cases of nonquadratic criteria, we develop a numeric solution scheme for the Nash feedback equilibrium. This scheme is an application of a discrete time, discrete state controlled Markov chain approximation method originally developed for solving deterministic and stochastic dynamic optimization problems. In our Nash–Cournot equilibrium, the degree of concentration in supply declines over time whereas the previous models with physical depletion and open-loop equilibrium concepts predict that a Nash–Cournot resource market will develop in the direction of monopoly supply. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Many natural resource industries (e.g. oil, diamonds, bauxite, uranium, mercury, copper) cannot be described using pure monopoly or competitive models. Determining oligopoly equilibrium for resource markets leads to differential game models, which at the most general level belong to capital accumulation games (Reynolds, 1987, 1991; Dockner et al., 1996). Several recent studies have developed a particular subclass of these models where two or more players extract a common property resource stock (van der Ploeg, 1987; Benhabib and Radner, 1992; Clemhout and Wan, 1994; Haurie et al., 1994; Dutta, 1995; Dockner et al., 1996). This subclass has turned out to be tractable since the common property assumption normally leads to one state variable specifications. Compared to this development, the empirically well-justified case of well-defined property rights and two or some finite number of producers extracting their own stocks (e.g. oil) but supplying to the same market is still open in several respects. There is a growing interest in understanding the oligopolistic description of resource markets due to attempts to analyze the effects of international carbon dioxide control (or the Kioto agreement) on noncompetitive oil markets (e.g. Wirl and Dockner, 1995; Tahvonen, 1995). However, these studies have had to assume that oil is supplied by a monopoly.

Existing oligopoly models for nonrenewable resource markets and the ‘oil’igopoly theory (e.g. Loury, 1986; Polansky, 1992) rest on the Nash open-loop equilibrium concept and physical depletion of natural resources. This paper develops a model for Nash–Cournot feedback equilibria and for economic depletion, both in the case of duopoly and n-player oligopoly. We solve analytically the affine-quadratic specifications and extensions with more complex functional forms by the Markov chain approximation method designed for solving deterministic or stochastic dynamic optimization problems.

Oligopoly models of nonrenewable resources have an interesting history which goes back to the informal discussion included in Hotelling (1931). Later, Salant (1976) viewed oil markets in a cartel-competitive fringe setup where the cartel may take the competitive sales path as given and face a sequence of excess demand curves and the resource constraint (cf. Dasgupta and Heal, 1979). A more consistent extension of the static Nash–Cournot model is found in Lewis and Schmalensee (1980). They replace the static choice of output level by the choice of an output time path under the constraint of initial reserves. With equal, constant unit costs, the producers with larger reserves produce for

The analysis by Louy (1986) continues the use of a similar model and the open-loop equilibrium concept. According to the ‘oil’igopoly theory, the resource rent and the agent’s initial reserves are inversely related. The agent’s market share is increasing if it exceeds the average market share. Typically the degree of concentration in supply increases over time and before physical depletion the resource is finally supplied by a monopoly. The theory is further developed and empirically tested in an inspiring study by Polansky (1992). ‘Oil’igopoly theory predicts that producers with larger stocks are more conservative, i.e. they produce larger amounts but use smaller fractions of their reserves than do producers with small reserves. These types of predictions are interesting because they can be tested with existing data. Polansky finds that the predictions are consistent with oil market data. For a large-scale computer version of this model, see Salant (1982).

The ‘oil’igopoly theory is problematic in two respects. First, the theory rests on the Nash open-loop equilibrium concept. Thus producers observe only initial reserves and do not use information on the levels of reserves after the initial moment of the game (Basar and Olsder, 1995, p. 231). Second, the main results are based on the ‘cake eating’ description of nonrenewable resources, which neglects that reserves are heterogeneous and that their complete physical depletion may be implausible.1 We focus on economic depletion and thus study an unsolved problem that may give a more valid empirical description of resource markets. In addition, economic depletion leads to a formulation in which the feedback equilibria can either be solved analytically or in the case of complex functional forms by using numerical approximation methods.

In the Nash feedback equilibrium the players continuously observe the reserves and condition extraction on this information (Basar and Olsder, 1994, p. 317). We restrict our study to these equilibria which are prefect in Markov strategies. Since our solutions constitute threats in trigger strategy equilibria (see e.g. Mehlmann, 1994) we offer a useful basis for extending the analysis toward this direction in the future studies.

In the case of affine-quadratic functional forms, we prove the existence and uniqueness of the Nash feedback equilibrium and obtain the explicit form for both duopoly and n-player oligopoly. Such solutions and their properties have

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1 Toman and Walls (1995) present a general critique of the ‘cake eating’ approach. In the case of oil, the approach is criticized by Adelman (1993) and in the case of uranium by Neff (1984, p. 74–79). Heal’s (1976) is among the first studies on economic resource depletion. In their well-known text, Dasgupta and Heal (1979, p. 192) mention the difference between physical and economic resource depletion but choose to study physical depletion only because of its simplicity and intuitiveness. Geologists have found it difficult to accept the idea that a nonrenewable resource could be used up physically (see e.g. Goeller and Weinberg, 1976).
been absent in nonrenewable resource literature. Within capital accumulation games, our analytical solution may be compared to Reynolds (1987, 1991), with the difference that in deriving our solution we do not need parameter restrictions like zero rate of discounting. Compared to the nonuniqueness problem of the Nash feedback equilibrium in Reynolds (1987, 1991), an interesting feature of the nonrenewable resource game is that similar reasons for multiple equilibria seem to be ruled out.

Several authors have found the analysis based on affine-quadratic specifications restrictive (e.g. Reynolds, 1987; Clemhout and Wan, 1994). However, the nonzero-sum differential games literature offers quite a few examples of the analysis of more general problems, even using numerical approximation. For this purpose, we apply a numerical method for the Nash feedback equilibrium and extend the Markov chain approximation approach originally designed for solving deterministic or stochastic optimal control or dynamic programming problems (Kushner and Dupuis, 1992). In this method, the continuous time optimal control problem of each producer, given the feedback supply strategy of the other producer, is approximated by a discrete time, discrete-state-controlled Markov chain. In dynamic programming, a classical iterative method, called approximation in the policy space, is globally convergent under mild conditions, and as the approximation parameter approaches zero the value function solution converges to the value function of the continuous time problem. However, in extending the method to a Nash feedback equilibrium problem, one can no longer guarantee its global convergence. In all four of our numerical examples, the convergence of the method was very rapid, and in the affine-quadratic case it leads to the same solution (within high accuracy) as the analytic solution.\footnote{Note that in this methodology we are following the general suggestions by Judd (1997).} In the case of fish resources and monitoring cooperative equilibria similar approach has been developed by Haurie et al. (1994).

With affine-quadratic functional forms, our model yields a couple of reverse hypotheses compared to the ‘oil’igopoly theory. This is mainly explained by our replacement of the ‘cake eating’ property by economic depletion. The crucial differences between these two approaches are not fully noted in studies on competitive or monopoly equilibria (see Sweeney, 1993) nor by Hansen et al. (1985). In the ‘oil’igopoly model, small producers deplete their stocks before large producers do, implying that the degree of concentration in supply increases over time (Lewis and Schmalensee, 1980, Proposition 4). In our model, small producers enter the market when the price rise makes it profitable to extract the higher cost deposits, and thus markets become more competitive. In addition, the Nash–Cournot feedback equilibrium always approaches a stable turnpike, where the market shares of symmetric producers are equal. Such a stable turnpike is, in general, absent in the ‘oil’igopoly model (Loury, 1986,
The unit extraction cost function can be derived from the problem of a country using several individual and heterogeneous deposits in optimal order (see Solow and Wan, 1976). The hypothesis that costs increase with decreasing reserves receives support in empirical studies (Chermak and Patrick, 1995; Epple and Londregan, 1993; Polansky, 1992).

Theorem 4). With nonlinear demand and nonquadratic costs, our equilibria may temporarily resemble the features of the ‘oil’igopoly theory, given that the extraction costs are initially independent of the size of reserves. However, when economic depletion is approached, the features revealed by the affine-quadratic specification always dominate. Thus the basic predictions of the ‘oil’igopoly theory are conditional on the assumptions regarding physical depletion, extraction costs and functional forms. Especially the prediction that resource markets always finally end with a period of monopoly supply is conditional on the assumption of physical resource depletion. Whether these hypotheses are conditional on Nash–Cournot equilibrium depends e.g. on the properties of the Stackelberg feedback equilibrium, which is more or less unsolved.

The paper is organized as follows. Section 2 derives the duopoly and Section 3 the n-player oligopoly. Section 4 presents the numerical approximation method and results. Section 5 concludes the paper.

2. Nash feedback equilibrium in duopolistic resource markets

Assume that the players continuously observe the (physical) state variables and that at each moment they condition their extraction on their own and other players’ reserve levels. This leads to the Nash feedback equilibrium which is subgame perfect in Markov strategies (Fudenberg and Tirole, 1993, p. 520; Basar and Olsder, 1995, p. 325).

Demand is linear, \( p = \bar{p} - q_1 - q_2 \), where \( p \) is price, \( \bar{p} \) the choke price and \( q_1, q_2 \) the extraction of agents 1 and 2, respectively. Unit costs equal \( c_{0i} - c_i X_i \), \( i = 1, 2 \), where \( c_{0i} \) (\( > 0 \)) are parameters and \( X_i \) is the physical stock of producer \( i \). Assume that \( c_{0i} > \bar{p} \), i.e. the cost of extracting the ‘last’ physical units exceeds the choke price. Define \( \bar{X}_i \) by \( c_{0i} - c_i \bar{X}_i = \bar{p} \) and let \( x_i \equiv X_i - \bar{X}_i \). Thus, \( x_i \) is the economical fraction of the physical stock of producer \( i \). Unit extraction costs now equal \( \bar{p} - c_i x_i \). This specification, together with state equations \( x_i = -q_i \), satisfies affine-quadratic functional form. We can postulate that extraction levels are linear functions of the state, i.e. \( q_1 = \alpha_1 x_1 + \alpha_2 x_2 \) and \( q_2 = \mu_1 x_1 + \mu_2 x_2 \), where \( \alpha_i \) and \( \mu_i, i = 1, 2 \), are unknown parameters. The problem of producer 1 is to

\[
\max_{q_1 \geq 0} J_1 = \int_0^\infty [(\bar{p} - q_1 - \mu_1 x_1 - \mu_2 x_2)q_1 - q_1(\bar{p} - c_1 x_1)]e^{-\delta_1 t} dt \quad (1)
\]

3This unit extraction cost function can be derived from the problem of a country using several individual and heterogeneous deposits in optimal order (see Solow and Wan, 1976). The hypothesis that costs increase with decreasing reserves receives support in empirical studies (Chermak and Patrick, 1995; Epple and Londregan, 1993; Polansky, 1992).
One then obtains the costate for $x_2$. The Hamiltonian\(^4\) is \(\bar{p} - q_1 - \mu_1 x_1 - \mu_2 x_2\) and for interior solutions, the Modified Hamiltonian Dynamic System (MHDS) takes the form

\[
\begin{align*}
\dot{x}_1 &= - q_1(x_1, x_2, \phi_1), \\
\dot{x}_2 &= - \mu_1 x_1 - \mu_2 x_2,
\end{align*}
\]  

\[
\begin{align*}
\phi_1 &= - q_1(x_1, x_2, \phi_1)(c_1 - \mu_1) + \psi_1 \mu_1 + \delta_1 \phi_1, \\
\psi_1 &= q_1(x_1, x_2, \phi_1)(c_1 - \mu_1) - (c_1 - \mu_1) + \delta_1 + \mu_2,
\end{align*}
\]

where \(q_1(x_1, x_2, \phi_1) = \frac{1}{2}(c_1 x_1 - \mu_1 x_1 - \mu_2 x_2 - \phi_1)\). Write next the Jacobian matrix for the MHDS as

\[
J = \begin{bmatrix}
\frac{1}{2} (\mu_1 - c_1) & \mu_2/2 & \frac{1}{2} & 0 \\
-\mu_1 & -\mu_2 & 0 & 0 \\
\frac{1}{2} (\mu_1 - c_1)(c_1 - \mu_1) & \mu_2(c_1 - \mu_1)/2 & \frac{1}{2} (c_1 - \mu_1) + \delta_1 & \mu_1 \\
\frac{1}{2} (c_1 - \mu_1)\mu_2 & -\mu_2/2 & -\mu_2/2 & \delta_1 + \mu_2
\end{bmatrix}.
\]

One then obtains

\[
\Delta = \frac{1}{2} c_1 \delta_1 \mu_2(\delta_1 + \mu_2),
\]

\[
\Omega = \frac{1}{2} \left[ 2\mu_2(\mu_1 - \mu_2) - c_1 \delta_1 - \delta_1(2\mu_2 - \mu_1) \right],
\]

where \(\Delta\) is the determinant of the Jacobian and

\[
\Omega = \begin{vmatrix}
\partial \dot{x}_1/\partial x_1 & \partial \dot{x}_1/\partial \phi_1 \\
\partial \phi_1/\partial x_1 & \partial \phi_1/\partial \phi_1
\end{vmatrix} + \begin{vmatrix}
\partial \dot{x}_2/\partial x_2 & \partial \dot{\psi}_2/\partial \psi_1 \\
\partial \psi_1/\partial x_2 & \partial \psi_1/\partial \psi_1
\end{vmatrix} + 2 \begin{vmatrix}
\partial \dot{x}_1/\partial x_2 & \partial \dot{\psi}_1/\partial \psi_1 \\
\partial \phi_1/\partial x_2 & \partial \phi_1/\partial \psi_1
\end{vmatrix}
\]

(see Dockner, 1985; or Kemp et al., 1993). Given that a linear equilibrium exists, it must be (saddle point) stable, i.e. two of the characteristic roots must have negative real parts. Complex roots imply oscillations, which violate the restrictions \(q_i \geq 0\) and thus cannot exist in an interior equilibrium. The negative

\[\text{4} \text{The following solution procedure yields us exactly the same solution than the use of Hamilton–Jacobi–Bellmann equations but is more compact for presentation.}\]
roots are \( r_{1,2} = \frac{1}{2} \delta_1 - \sqrt{\delta_1^2/4 - \Omega/2 \pm \frac{1}{2} (\Omega^2 - 4A)^{1/2}} \). The solution can be written as

\[
\begin{align*}
x_1(t) &= A_1 e^{r_1 t} + A_2 e^{r_2 t}, \quad \text{(8)} \\
x_2(t) &= - \mu_1 A_1 e^{r_1 t}/(\mu_2 + r_1) - \mu_1 A_2 e^{r_2 t}/(\mu_2 + r_2), \quad \text{(9)} \\
q_1 &= x_4(\mu_2 + r_1 + r_2) - x_2(\mu_2 + r_1)/(\mu_2 + r_2)/\mu_1, \quad \text{(10)}
\end{align*}
\]

where

\[
\begin{align*}
A_1 &= (\mu_2 + r_1)[\mu_1 x_{10} + x_{20}(\mu_2 + r_2)]/\mu_1 (r_1 - r_2), \\
A_2 &= - (\mu_2 + r_2)[\mu_1 x_{10} + x_{20}(\mu_2 + r_1)]/\mu_1 (r_1 - r_2).
\end{align*}
\]

In Eqs. (8)–(10), we assume \( \mu_1 \neq 0, \mu_2 + r_2 \neq 0, \) and \( r_1 \neq r_2 \). We next formulate an analogous problem for resource owner 2. Recall that \( q_1 = \alpha_1 x_1 + \alpha_2 x_2 \).

Carrying out the procedure above yields (cf. Eq. (10))

\[
q_2 = - x_1(\alpha_1 + u_1)/(\alpha_1 + u_2)/x_2 - x_2(\alpha_1 + u_1 + u_2), \quad \text{(11)}
\]

where \( u_i \) are the characteristic roots for the producer 2 problem. Again, we require \( x_2 \neq 0, u_1 \neq u_2, \alpha_1 + u_2 \neq 0 \). The linear equilibrium must satisfy the system

\[
\begin{align*}
\mu_2 + r_1 + r_2 + \alpha_1 &= 0, \\
(\mu_2 + r_1)(\mu_2 + r_2)/\mu_1 + \alpha_2 &= 0, \\
\alpha_1 + u_1 + u_2 + \mu_2 &= 0, \\
(\alpha_1 + u_1)(\alpha_1 + u_2)/\alpha_2 + \mu_1 &= 0.
\end{align*}
\]

If the resource owners are symmetric and \( \delta_1 = \delta_2 = \delta \) and \( c_1 = c_2 = c \), it must hold that \( \alpha_1 = \mu_2 \) and \( \alpha_2 = \mu_1 \). Thus, the above system simplifies to

\[
\begin{align*}
2\mu_2 + r_1 + r_2 &= 0, \quad \text{(12)} \\
(\mu_2 + r_1)(\mu_2 + r_2) + \mu_1^2 &= 0. \quad \text{(13)}
\end{align*}
\]

Because \( r_1 < r_2 < 0 \), Eq. (12) implies that \( \mu_2 > 0 \). If \( \mu_1 > 0 \), it can be shown that there exists only a degenerate combination of initial states, so that the solution for the system \( \dot{x}_1 = - \mu_2 x_1 - \mu_1 x_2, \dot{x}_2 = - \mu_1 x_1 - \mu_2 x_2 \) does not reach either \( x_1 = 0 \) or \( x_2 = 0 \) in finite time. When \( x_1 = 0 \), for example, \( q_1 = \mu_1 x_2 > 0, q_2 = \mu_2 x_2 > 0 \) and \( p = \bar{p} - q_1 - q_2 < \bar{p} \). For agent 1, the extraction costs equal \( \bar{p} \), implying that he can increase his profit by not extracting the last unit in finite time. Such paths cannot form the equilibrium solution. Thus, if the linear equilibrium exists more generally than in knife edge cases, it must hold that \( \mu_1 \leq 0 \). It is possible to show that \( \mu_1 = 0, \mu_2 = \delta/2c[(1 + 2\delta/c)^{1/2} - 1] \) implies \( \mu_2 = - r_2 = - r_1 \), and solves (12) and (13). Such a case was excluded when we deduced Eqs. (12) and (13). However, this candidate contradicts the
necessary conditions of problem (1)–(3) for $\forall x_2 > 0$. Thus, in the (saddle point) stable equilibrium, it is necessary that $\mu_1 < 0$, $\mu_2 > 0$.

**Proposition 1.** Given that $\mu_1 < 0$ and $\mu_2 > 0$, it follows that $r_1$, $r_2$ are real and a unique solution for (12), (13) exists, and equals $\mu_1 = -a_1(1 + s^2)^{-1/2}$, $\mu_2 = (a_2 s + \frac{1}{2}a_1)(1 + s^2)^{-1/2} - \frac{1}{2}\delta$, where $s = a_0/3[p_0 + 2p_1\cos(\frac{\pi}{2}\arccos(p_2/p_3))]$, $p_0 = 3\gamma + 5$, $p_1 = \sqrt{64 + 60\gamma + 9\gamma^2}$, $p_2 = 404 + 666\gamma + 270\gamma^2 + 27\gamma^3$, $a_0 = 1/\sqrt{7}$, $a_1 = \sqrt{\delta(2c + \delta)/7}$, $a_2 = \frac{1}{2}\sqrt{\delta(2c + \delta)}$ and $\gamma = \delta/c$. For proof, see Appendix A.

The proof shows that $\mu_1 + \mu_2 > 0$ and that the solutions to Eqs. (12) and (13) must lie on an arc of an ellipse. A parametric representation of this arc leads to an equation which is shown to have a unique solution in the region $\mu_1 < 0$, $\mu_1 + \mu_2 > 0$.

Reynolds (1987) proves the existence of an equilibrium and obtains its explicit form for a symmetric Nash–Cournot capital accumulation game by applying Bellman equations and restricting the model parameters. As he notes, his model may have equilibria with nonlinear subgame perfect Nash strategies. This is due to the lack of a ‘natural boundary condition’; i.e. the assumption of feedback strategies is not restrictive enough to yield a unique steady state and unique feedback strategies (see Karp, 1995; Clemhout and Wan, 1994). However, in a nonrenewable resource game, the origin forms a natural boundary condition, implying that the existence of equilibria with nonlinear feedback strategies is highly unlikely.

### 2.1. Economic properties of the feedback equilibrium

When $\mu_1 < 0$ and $\mu_2 > 0$, the equilibrium is (saddle point) stable (Fig. 1). The slopes of the paths are equal to $q_2/q_1$. Thus, a lower slope implies a larger market share for agent 1. Between the isoclines $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$, there is a linear turnpike with equal stock and extraction levels for both players. In the proof of Proposition 1, we show that $\mu_1 + \mu_2 > 0$; i.e. extraction is more sensitive to changes in agents’ own stock than to changes in the rival’s stock. Thus, an increase in $x_i$ always increases total supply. An agent’s extraction increases with his own stock ($\mu_2 > 0$) because a greater stock implies lower extraction costs. Because $\mu_1 < 0$, extraction decreases with rival’s stock level. An increase in the rival’s stock decreases his extraction costs and increases his supply, causing a decrease in price. Thus, the other agent decreases supply and sells in the future at a higher price.

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5 In Fig. 1 $\delta = 1/20, c = 1/2$. Thus $\mu_1 \approx -0.029781, \mu_2 \approx 0.0974679, r_1 \approx -0.1127249$, and $r_2 \approx -0.067865$. 
In Fig. 1, a high stock level held by the rival implies that the extraction of the agent with the lower stock is finally zero. When the state exists below the $x_2 = 0$ isocline, we postulate that $q_1 > 0$ and $q_2 = 0$. For agent 1, this equilibrium must satisfy the necessary conditions of problem (1)-(3) with $q_2 = \dot{x}_2 = 0$. This yields $q_1 = \frac{1}{2}(c_1x_1 - q_1), \phi_1 = -c_1q_1 + \delta \phi_1, \psi_1 = \delta \psi_1$ and $\dot{x}_2 = 0$. At the switching moment to a regime where both agents produce, the Hamiltonian must be continuous, implying continuity of extraction, costates, and resource price. The other conditions needed to determine the equilibrium are the initial level $x_{10}$ and the continuity requirement that $x_{10} = -\mu_2 x_{20}/\mu_1$ at the beginning of simultaneous production.

Below the $x_2 = 0$ isocline, $\dot{q}_1 = -\frac{1}{2}\delta \phi_1 < 0$ and price is increasing. After the solution has intersected the $x_2 = 0$ isocline, the extraction of agent 2 initially increases. However, by $\mu_1 + \mu_2 > 0$, we obtain $\dot{q}_1 + \dot{q}_2 = -(x_1 + x_2)(\mu_1 + \mu_2) < 0$; i.e. total extraction is always decreasing and thus price increases monotonically. The same holds in the ‘oil’igopoly theory (Loury, 1986, Lemma 3).

We next examine the inefficiency properties of the duopoly equilibrium. Note first that it cannot be efficient to produce from both stocks when unit extraction costs differ. Thus, in contrast to duopoly, the efficient trajectory reaches the turnpike vertically or horizontally, depending on whether $c_1x_1 < c_2x_2$ or $c_1x_1 > c_2x_2$ and then follows the turnpike toward the origin. In the symmetric
case \((\delta_1 = \delta_2, c_1 = c_2)\), the sloped of the efficient and duopoly turnpikes are equal.

**Lemma 1.** Along the turnpike of the Nash feedback equilibrium the rate of extraction represents a smaller fraction of existing resources than along the turnpike of the efficient equilibrium. (Proof: Appendix B).

Drawing the surface \(w - (\mu_1 + \mu_2)\) as a function of \(\delta\) and \(c\) reveals that the degree of inefficiency of the duopoly equilibrium increases with the levels of \(\delta\) and \(c\) (see Appendix B).

In the ‘oil’ oligopoly theory, producers’ necessary conditions satisfy the ‘generalized Hotelling arbitrate condition’ \(\dot{m} = \delta[m - c(x)]\), where \(m\) is marginal revenue (Polansky, 1992). In the open-loop equilibrium, producers take rivals’ extraction time paths as given, and these enter as nonautonomous components in revenues without changing the form of the conditions. Here the producers take the feedback control rule as given, and \(\dot{m}_1 = \delta[m_1 - (\hat{p} - cx_1)] + \mu_1(q_1 + \psi_1)\). Along the turnpike, \(q_1 + \psi_1 > 0\) or otherwise \(\psi_1 \to -\infty\) as \(t \to \infty\) (Eq. (7)), implying that \(\dot{m}_1 < \delta[m_1 - (\hat{p} - cx_1)]\). Below but near the \(x_1 = 0\) isocline, \(q_1 \simeq 0\) and \(\dot{m}_1 > \delta[m_1 - (\hat{p} - cx_1)]\). Thus, the Hotelling rule fails in the feedback duopoly equilibrium. For interpretation, write Eq. (6) as

\[
\varphi(t) = \int_t^\infty [q_1(\tau)c - q_1(\tau)\mu_1 - \psi(\tau)\mu_1]e^{\delta(t - \tau)} d\tau.
\]

Thus, rent includes the marginal present value of extraction costs and two components related to the rival’s preemptive behavior. First, as agent 1’s stock level increases, his rival’s extraction decreases and this agent 1’s marginal present value of revenue increases. This effect must be ‘corrected’ by \((-\psi_1\mu_1 < 0)\) because an upward shift in the rival’s stock path has a secondary effect on agent 1’s extraction profile. Note from Eq. (6) that when \(q_i(0) \simeq 0\), the rent of agent \(i\) first increases but later converges toward zero.

When \(x_{10} < x_{20}\), it follows that \(A_1 > 0\) and that in the \(x_1 < x_2\) state space the solution converges toward the turnpike from below. This implies that such a path is strictly concave.

**Proposition 2.** (A) The agent with the higher stock has a higher but diminishing market share. (B) Given any \(t \in (0, \infty)\), the agent with the higher initial stock has extracted a larger proportion of his initial stock than the agent with the smaller initial stock. (C) If the initial stock of agent 1 is higher than that of agent 2, then agent 1’s production/reserves ratio is higher than that of agent 2 \(\forall t \in [0, \infty)\) (Proof: Appendix C).
We next explain why all these outcomes are the reverse of those in the ‘oil’igopoly theory. Loury (1986, Lemma 6) shows that a producer’s market share is increasing when his market share is higher than the average (cf. Proposition 2A). This follows because the agents with smaller initial reserves typically deplete their stocks faster and then exit from the market. Just before the physical depletion of the resource endowment, there may exist only one producer. In the case of equal costs and initial reserves, the industry evolves along a path such as our turnpike (Lewis and Schmalensee, 1980). The crucial difference is that, excluding a special case (Loury, 1980, Theorem 2) in the ‘oil’igopoly theory, this path is not stable, so that all solutions would converge toward it as $t \to \infty$. This is demonstrated in Fig. 2, which depicts the ‘oil’igopoly equilibria with linear demand and zero extraction costs ($\overline{p} = 100, \delta = 0.1$). All equilibria, except the degenerate one where the players have equal stock levels, finally lead to a pure monopoly. In general, the differences in the concavity/convexity properties of the paths in Figs. 1 and 2 generate major differences in the economic outcomes of these models.

Loury (1986) and Lewis and Schmalensee (1980, Proposition 4) have shown that, with equal costs, the degree of concentration in supply increases over time. This refers to the number of firms, their (reserve) size, and their market shares. These features can be seen immediately for Fig. 2. In our duopoly, the number of (active) producers cannot decrease, their market shares converge toward $\frac{1}{2}$, and the difference in their sizes decreases toward zero. Thus, the degree of concentration in supply decreases over time. An increasing price induces new small producers whose extraction costs were initially too high for profitable production.
Polansky (1992, Proposition 4) shows that, when unit extraction costs depend on stocks, producers with higher reserves extract more but have a smaller ratio of cumulative production to initial reserves than do producers with smaller reserves (cf. Proposition 2B). This difference follows from Polansky’s cost function, \( c(1 - x_i/x_0)q_i \), where \( c(\cdot) \) satisfies \( c' > 0, c'' \geq 0 \). Thus, unit costs increase with the ratio of extracted stock to initial reserves. Therefore, all agents initially have equal unit extraction costs, independent of the size of their reserves.\(^6\) The rationale for this formulation is not explained. Note that Polansky’s formulation would yield an asymmetric outcome for our model.

In Proposition 2, Polansky (1992) shows that, with constant and equal unit costs, the producers with higher reserves have lower production/reserves ratios than do producers with small reserves. In his model, producers with higher reserves have an incentive to save resources for sale when smaller producers have depleted their stocks. In contrast, our model suggests (Proposition 2C) that small producers (with high unit extraction costs) have an incentive to wait until the larger producers have depleted their lowest cost reserves. Thus, the outcome is again the reverse.\(^7\)

We now turn to the effect of discounting on resource supply. For this purpose, we depict \( \mu_1 \) and \( \mu_2 \) as functions of the model parameters \( \delta \) and \( c \) (Fig. 3a and b). Fig. 3a and b shows that the absolute values of \( \mu_1 \) and \( \mu_2 \) increase with \( \delta \) and \( c \). As a comparison of Fig. 3a and b suggests, \( \mu_2 \) is more sensitive to changes in \( \delta \) (and \( c \)) than \( \mu_1 \). This is shown more exactly in Fig. 3c, where the positive sign of \( \partial(\mu_1 + \mu_2)/\partial \delta \) is independent of \( \delta \) and \( c \).

Along the turnpike, \( x_1 = x_2 \equiv x \) and \( \partial q_i/\partial \delta = (\partial \mu_2/\partial \delta + \partial \mu_1/\partial \delta)x > 0 \); i.e. the higher the value of \( \delta \), the higher the level of \( q_i \), \( i = 1, 2 \). When \( x_1 \neq x_2 \) but \( q_1 > 0 \) and \( q_2 > 0 \), we obtain \( \partial q_1 + q_2)/\partial \delta = (x_1 + x_2)(\partial \mu_1 + \mu_2)/\partial \delta > 0 \). Thus, at a given level of reserves, total supply increases with rate of discount. These properties are in line with the results for competitive and monopoly equilibria (Sweeney, 1993).

Because \( \partial q_i/\partial \delta = x_i(\partial \mu_1/\partial \delta) + x_i(\partial \mu_2/\partial \delta), \partial \mu_1/\partial \delta < 0, \) and \( \partial \mu_2/\partial \delta > 0 \); an increase in \( \delta \) causes a larger increase in extraction for the agent with larger reserves. An increases in \( \delta \) causes a positive primary effect on extraction \( \partial \mu_2/\partial \delta x_i > 0 \) because future profits will become less important. The secondary effect is negative \( \partial \mu_1/\partial \delta x_i < 0 \) due to the positive effect in the rival’s supply.

\(^6\)In the symmetric case, our formulation implies that the greater the agent’s stock the lower his unit extraction costs compared with other producers. This is in line with several empirical studies (Eppl and Londregan, 1993).

\(^7\)For asymmetric cases, we have found (numerically) that, if agent 1 has lower extraction costs \((c_1 > c_2)\), the slope of the turnpike exceeds 1, implying that agent 1 speeds up his extraction and has a lower long-run market share than agent 2. If \( \delta_1 < \delta_2 \) the slope of the turnpike is less than 1 and agent 1 slows down his extraction and has a higher long-run market share. In asymmetrical cases, the slopes of the duopoly turnpikes do not equal the slopes of the efficient turnpikes.
Thus, if the agent’s stock is small enough compared to the rival’s stock, an increase in $\delta$ implies a large increase in the rival’s supply, causing a large negative effect, while the primary positive effect may be small.

In Fig. 1, the solid isoclines are computed assuming $\delta = 0.05$, $c = 0.5$ and the dotted lines assuming $\delta = 10$, $c = 0.5$. A higher $\delta$ implies that the slope of $\dot{x}_1 = 0$ decreases, the slope of $\dot{x}_2 = 0$ increases, and that the agent with the ‘low’ stock reduces his extraction as $\delta$ increases. Fig. 3d shows that this outcome holds in general. It depicts $\partial(-\mu_1/\mu_2)/\partial\delta$ as function of the model parameters. This derivative is positive, implying that increases in the rate of discount increase (decrease) the slope of the $\dot{x}_2 = 0$ ($\dot{x}_1 = 0$) isocline independently of the levels of $S$. Salo, O. Tahvonen / Journal of Economic Dynamics & Control 25 (2001) 671–702
δ and c. This comparative dynamic result is a unique feature of oligopoly equilibrium, and its presence cannot be deduced by studying the competitive or monopoly equilibria, nor is a similar result reported in ‘oil’igopoly theory.

We have compared the feedback and the open-loop Nash–Cournot equilibrium. The equilibria never coincide. The latter implies less conservative resource use, a different region for interior solutions and, in the asymmetric case, different slopes for the turnpikes. We omit the details of this comparison.

3. Perfect oligopoly equilibrium in resource markets

When $n \geq 2$ and resource owners are symmetric ($\delta_i = \delta, c_i = c, i = 1 \ldots n$), we can write for $i = 1$

$$
\sum_{j \neq 1} q_j = \mu_3(x_1 + x_3 + x_4 + \ldots + x_n) + \mu_2 x_2 \\
+ \mu_1(x_1 + x_2 + x_4, \ldots + x_n) + \mu_2 x_3 \\
+ \mu_1(x_1 + x_2 + x_3 + x_5, \ldots + x_n) + \mu_2 x_4 + \ldots, \\
+ \mu_1(x_1 + x_2 + x_3 + \ldots + x_{n-1}) + \mu_2 x_n
$$

$$
= (n - 1)\mu_1 x_1 + \mu_1 x_2(n - 2) + \mu_1 x_3(n - 2) + \ldots, \\
+ \mu_1 x_n(n - 2) + \mu_2 x_2 + \mu_2 x_3 + \ldots + \mu_2 x_n
$$

$$
= (n - 1)\mu_1 x_1 + [\mu_1(n - 2) + \mu_2] \sum_{j \neq 1} x_j.
$$

Let us denote $\sum_{j \neq i} x_j = z$. We obtain the problem for resource owner $i = 1, \ldots, n$ in the form

$$
\max_{q_i \geq 0} J_i = \int_0^\infty \left\{ (\ddot{p} - (n - 1)\mu_1 x_i - [\mu_1(n - 2) + \mu_2] z - q_i)q_i \\
- q_i(\ddot{p} - c x_i) e^{-\delta t} dt \right\}
$$

s.t. $\dot{x}_i = -q_i$, $x_i(0) = x_{i0} > 0$, $\lim_{t \to \infty} x_i \geq 0 \forall i \in 1, \ldots, n,$

$$
\dot{z} = -(n - 1)\mu_1 x_i - [\mu_1(n - 2) + \mu_2] z, \quad z(0) = \sum_{j \neq i} x_{j0}. 
$$

For any $i = 1, \ldots, n$, the Hamiltonian is

$$
H = (\ddot{p} - (n - 1)\mu_1 x_i - [\mu_1(n - 2) + \mu_2] z - q_i)q_i \\
- q_i(\ddot{p} - c x_i) - \phi q_i - \psi(\mu_1 x_i + [\mu_1(n - 2) + \mu_2] z),
$$

where $\phi, \psi$ are constants.
where $\psi$ is now the costate for $z$. We obtain the following MHDS:

$$
x_i = -q_i(x_i, z, \phi),$$

$$\dot{z} = -(n-1)\mu_1 x_i - [\mu_1(n-2) + \mu_2]z,$$

$$\dot{\phi} = q_i(x_i, z, \phi)[(n-1)\mu_1 - c] + \psi(n-1)\mu_1 + \delta \phi,$$

$$\dot{\psi} = [\mu_1(n-2) + \mu_2]q_i(x_i, z, \phi) + \psi(\delta + [\mu_1(n-2) + \mu_2]),$$

where $q_i(x_i, z, \phi) = \frac{1}{2}(-n-1)\mu_1 x_i - [\mu_1(n-2) + \mu_2]z + c x_i - \phi$. Following Section 2, we compute

$$
A = \frac{1}{2} \{c \delta + \mu_1(n-2) + \mu_2\} \{[\mu_1(n-2) + \mu_2]\},
$$

$$
\Omega = -\frac{1}{2} \{c \delta + \delta [\mu_1(n-3) + 2\mu_2] - 2[\mu_1^2(n-2) + \mu_1 \mu_2(3-n) - \mu_2^2]\},
$$

$$
r_{1,2} = \frac{1}{2} \delta - \left[\delta^2/4 - \frac{1}{2} \Omega \right]^{1/2} \Omega \pm \frac{1}{2} \Omega^{1/2}. \quad (19)
$$

Again, the negative roots must be real because complex roots imply oscillations, which violates $q_i \geq 0$. The solution can be written as

$$
x_i = V_1 e^{r_1 t} + V_2 e^{r_2 t}, \quad (17)
$$

$$
z = -(n-1)\mu_1 V_1 e^{r_1 t}/[r_1 + [\mu_1(n-2) + \mu_2]]
$$

$$
- (n-1)\mu_1 V_2 e^{r_2 t}/[r_2 + [\mu_1(n-2) + \mu_2]], \quad (18)
$$

where, by using the initial values for $x_i$ and $z$, one obtains

$$
V_1 = \{\mu_1^2(n-2)[n(x_{i0} + z_0) - x_{i0} - 2z_0]
$$

$$+ \mu_1 \mu_2 [n(x_{i0} + 2z_0) - x_{i0} - 4z_0]
$$

$$+ n[r_1(x_{i0} + z_0) + r_2 z_0] - r_1(x_{i0} + 2z_0) - 2r_2 z_0\}
$$

$$
+ z_0(\mu_2 + r_1)(\mu_2 + r_2)/[\mu_1(n-1)(r_1 - r_2)],
$$

$$
V_2 = \{\mu_1(n-2) + \mu_2 + r_2\} \{\mu_1[n(x_{i0} + z_0) - x_{i0} - 2z_0]
$$

$$+ z_0(\mu_2 + r_1)\}(/[\mu_1(n-1)(r_2 - r_1)].
$$

Above we assume $\mu_1 \neq 0, r_1 \neq r_2$ and $\mu_2 \neq r_1$. By the fact that $q_i = -x_i$, it follows that

$$
q_i = x_i[\mu_1(2-n) - \mu_2 - r_1 - r_2] + z[\mu_1^2(n-2)^2
$$

$$+ \mu_1(n-2)(2\mu_2 + r_1 + r_2) + (\mu_2 + r_1)(\mu_2 + r_2)\}(/[\mu_1(1-n)]. \quad (19)
$$

By symmetry, the following equations determine $\mu_1$ and $\mu_2$:

$$
\mu_1(2-n) - 2\mu_2 - r_1 - r_2 = 0, \quad (20)
$$

$$
(\mu_2 + r_1)(\mu_2 + r_2) - \mu_1^2(1-n) = 0, \quad (21)
$$
where (21) has been simplified by (20). When \( n = 2 \), Eqs. (20) and (21) reduce to (12) and (13).

We first determine the signs of \( \mu_1 \) and \( \mu_2 \). Saddle point stability requires \( r_1, r_2 < 0 \). The case where \( \mu_1 < 0, \mu_2 < 0 \) violates Eq. (20) and the case \( \mu_1 > 0, \mu_2 < 0 \) violates Eq. (21). Thus, \( \mu_2 > 0 \). As in Section 2, the case where \( \mu_1 > 0, \mu_2 > 0 \) implies the existence of a unique level of \( z_0 \) for each \( x_{i0} \) in order for a stable equilibrium to be possible. Thus, if stable equilibria exist more generally than in degenerate cases, it must hold that \( \mu_1 \leq 0 \). In the duopoly case, \( \mu_1 = 0 \) and \( \mu_2 = \frac{1}{2}\sqrt{\delta(\sqrt{2c + \delta} - \sqrt{\delta})} \) solves (20), (21). However, this solves problem (14)–(16) only if \( z_0 = 0 \). Thus, in the interior equilibrium, \( \mu_1 < 0 \) and \( \mu_2 > 0 \).

Proposition 3. Eqs. (20) and (21) possess a unique solution, \( \mu_1 < 0, \mu_2 > 0 \), so that \( r_1 \) and \( r_2 \) are real and negative. This solution must lie in the cone \( \mu_1 < 0, \mu_2 + (n - 1)\mu_1 > 0 \) and is given by \( \mu_1 = -a_1/\sqrt{1 + s^2}, \mu_2 = [a_2s + \frac{1}{2}a_1(n - 1)]/\sqrt{1 + s^2} - \frac{1}{2}\delta \), where \( s = a_0/3[p_0 + 2p_1\cos(\frac{1}{2}\arccos(p_2/p_3))] \), \( p_0 = -1 + 3n + \gamma n, p_1 = (16 + 4\gamma + \gamma^2 + 24n + 16\gamma n + 2\gamma^2 n + 6\gamma n^2 + \gamma^2 n^2)^{1/2}, p_2 = 44 + 30\gamma + 6\gamma^2 + \gamma^3 + (180 + 102\gamma + 30\gamma^2 + 3\gamma^3)n + (108\gamma + 33\gamma^2 + 3\gamma^3)n^2 + (9\gamma^2 + \gamma^3)n^3 \), \( a_0 = 1/\sqrt{(3n + 1)(n - 1)}, a_1 = a_0\sqrt{\delta(2c + \delta)}, a_2 = \frac{1}{2}\sqrt{\delta(2c + \delta)} \) and \( \gamma = \delta/c \) (Proof: Appendix A).

These results can be compared to Reynolds (1991), who proves the existence of an equilibrium for a capital accumulation oligopoly and an explicit solution with zero discounting. We note that our equilibrium for nonrenewable resource oligopoly exists only with strictly positive discounting.

3.1. Properties of the oligopoly feedback equilibrium

The signs of \( \mu_1 < 0 \) and \( \mu_2 > 0 \) have the same interpretation as in the duopoly case. From (20), (21), one obtains \( \mu_1(n - 1) + \mu_2 = -r_2 \), implying \( \mu_1(n - 2) + \mu_2 > 0 \). We can depict the system

\[
\dot{x}_i = -\mu_1 z - \mu_2 x_i,
\]

\[
\dot{z} = -(n - 1)\mu_1 x_i - [\mu_1(n - 2) + \mu_2]z
\]

in \( x_i - z \) state space (Fig. 4).\(^8\) The slope of \( \dot{x}_i = 0 \) is greater than the slope of \( \dot{z} = 0 \) by \( \mu_1 < 0, \mu_2 > 0, \mu_1(n - 2) + \mu_2 > 0 \) and Eqs. (20) and (21). In Fig. 4, the interior solution exists between the isoclines. In these equilibria, the restrictions \( q_i \geq 0 \ \forall i \) are not binding. When the initial state \( (x_{i0}, z_0) \) is above, but close

\(^8\) When \( \delta = 1/20, c = 1/2 \) and \( n = 20 \), then \( \mu_1 \approx -0.004518, \mu_2 \approx 0.102834, r_1 \approx -0.0169871 \) and \( r_2 \approx -0.107352 \).
The region of interior solutions expands if the extraction costs depend on the rate of extraction and take the form
\[ q(c_a + c_b q - c_c x), \]
where \( c_a, c_b \) and \( c_c \) are positive parameters.

The explicit time paths for \( x_i \) and \( q_i \), \( i = 1, \ldots, n \), can be obtained by using the solutions for \( k_1 \) and \( k_2 \) and solving Eqs. (17) and (18) for each producer.

In general, along an interior equilibrium there may be several agents with increasing or decreasing extraction. Total extraction and its time derivative can be given as

\[
\sum_{i=1}^{n} q_i = \sum_{i=1}^{n} x_i [\mu_1(n - 1) + \mu_2] > 0,
\]

\[
\sum_{i=1}^{n} \dot{q}_i = -\sum_{i=1}^{n} q_i [\mu_1(n - 1) + \mu_2] < 0.
\]

The explicit time paths for \( x_i \) and \( q_i, i = 1, \ldots, n \), can be obtained by using the solutions for \( \mu_1 \) and \( \mu_2 \) and solving Eqs. (17) and (18) for each producer.
The signs follow from the fact that \( \mu_1(n - 1) + \mu_2 = -r_2 > 0 \) (Eqs. (20) and (21)). Thus, total extraction is decreasing and price is increasing along the oligopoly equilibrium (cf. Loury, 1986, Lemma 4). We next generalize some of the results derived in the duopoly case.

**Proposition 4.** If \( x_i = z/(n - 1) \) or \( x_i > z/(n - 1) \) or \( x_i < z/(n - 1) \), then the market share of agent \( i \) is constant, decreasing or increasing, respectively (Proof: Appendix D).

Proposition 4 means that if the stock of agent \( i \) is greater (smaller) than the average stock level, his market share is decreasing (increasing). Thus, when \( t \rightarrow \infty \), the market shares of all producers converge toward \( 1/n \). Recall that in the ‘oil’igopoly model with equal costs the number of firms decreases and before the physical depletion there may exist a pure monopoly (Loury, 1986, Lemma 6; Lewis and Schmalensee, 1980, Proposition 4). We next generalize the other results of Proposition 2:

**Proposition 5.** If \( x_i > z/(n - 1) \), then (A) \( q_i > \sum_{j \neq i} q_j/(n - 1) \), (B) \( q_i/x_i > \sum_{j \neq i} q_j/z \) and (C) \( x_i/x_{10} \leq z/z_0 \). (Proof: Appendix E).

Thus, if the stock of agent \( i \) is greater than the average stock level, then agents \( i \)'s extraction exceeds the average extraction, his production/reserves ratio exceeds the average production/reserves ratio, and he has extracted a larger proportion of his initial reserves than the average producer. Again, these results are the reverse of those in the ‘oil’igopoly theory.

We next compare the oligopoly equilibrium to perfect collusion and competitive equilibria when the number of producers changes. In all cases, the slope of the turnpike equals \( n - 1 \). A simple way of comparing is to compute the turnpike production/reserves ratios as functions of \( n \). Given \( n \) price takers, it is possible to show that the production reserves/ratio along the turnpike equals \( \frac{1}{4}[\delta - (\delta^2 + 4\delta c/n)^{1/2}] \). Accordingly, in collusion, the ratio equals \( \frac{1}{4}[\delta - (\delta^2 + 2\delta c/n)^{1/2}] \). These ratios decrease with \( n \) because by increasing \( n \), ceteris paribus, we increase the amount of total reserves and potential supply, implying that when \( n \rightarrow \infty \), the production/reserves ratio must converge toward zero. In oligopoly, this ratio equals \( -r_2 \). A numerical comparison is given in Fig. 5. Along the oligopoly turnpike, the production/reserves ratio is between the ratios in the competitive and collusion equilibria and as \( n \) increases, the ratio converges toward the ratio for competitive equilibrium. This result is similar to that in the ‘oil’igopoly theory (Loury, 1986, Corollary 2) and in Reynolds (1991), who presents an analogous result for a prototype capital accumulation game.

For duopoly, we showed that if the stock level of the agent is low enough, an increase in the rate of discount decreases his extraction. In Fig. 3, it is assumed
that \( \delta = \frac{1}{10} \), but the dotted lines denote the isoclines when \( \delta = 1 \), showing that, as in the duopoly case, by increasing \( \delta \) we shift the isoclines closer to each other and that the initial extraction of an agent or group of agents with small reserves may decrease when the rate of discount increases.

4. Duopoly equilibria with nonlinear demand and unit cost functions

In order to gain some insight into the robustness of the results discussed in Sections 2 and 3, we next attempt to generalize from the linear demand and cost functions. Unfortunately it appears difficult, if not impossible, to present analytical solutions to Nash feedback equilibria in cases of nonlinear demand and unit costs. This is because we are out of the affine-quadratic dynamic game setting. Therefore, we resort to numerical methods in three nonlinear cases all of which share the common feature that the choke price \( \tilde{p} \) equals the extraction costs of the last economic unit of both producers.

Assume two producers with initial stocks \((x_{10}, x_{20})\) in some rectangle \(G = \{(x_1, x_2)|0 \leq x_1 \leq X, 0 \leq x_2 \leq X\}\), but otherwise identical. Let \( p(q_1, q_2) = p(q_1 + q_2) \) be the demand (instead of \( \tilde{p} - q_1 - q_2 \)) and \( c(x_i), i = 1, 2 \) the unit cost (instead of \( \tilde{p} - cx_i \)). Function \( p \) and \( c \) are assumed continuous and bounded. Let \( q_j(x_1, x_2) \) be producer \( j \)'s (feasible) supply strategy. Then the problem of
producer $i$ is to

$$\max_{q_i \geq 0} J^i = \int_0^\infty \left[ p(q_i, q_j) - c(x_i) \right] q_i e^{-\delta t} \, dt,$$

s.t. \( \dot{x}_i = -q_i, x_i(0) = x_{i0}, \lim_{t \to \infty} x_i(t) \geq 0 \), \( x_{j0}, \lim_{t \to \infty} x_j(t) \geq 0 \),

$$\dot{x}_j = -q_j(x_1, x_2), x_2(0) = x_{20}. \quad (3')$$

Let \( V'(x_{10}, x_{20}) \) be the corresponding optimal value of \( J^i \) in problem \((1')\)–\((3')\) for producer \( i \). Clearly \( V'(0, x) = V'(x, 0) = 0 \) and \( q_i(0, x) = q_2(x, 0) = 0 \) for all \( x \geq 0 \). If these value functions are differentiable on \( G \), then \( V^1 \) and \( V^2 \) satisfy the following Hamilton–Jacobi–Bellman (HJB) equations on \( G \) (see Basar and Olsder, 1995; or Fleming and Soner, 1993)

$$\delta V^1(x_1, x_2) = \max_{q_i \geq 0} \left\{ p[q_i, q_2(x_1, x_2)]q_1 - c(x_1)q_1 - q_1 \frac{\partial V^1(x_1, x_2)}{\partial x_1} \right\}$$

$$- q_2(x_1, x_2) \frac{\partial V^1(x_1, x_2)}{\partial x_2}, \quad (22)$$

$$\delta V^2(x_1, x_2) = \max_{q_2 \geq 0} \left\{ p[q_1(x_1, x_2), q_2]q_2 - c(x_2)q_2 \right\}$$

$$- q_1(x_1, x_2) \frac{\partial V^2(x_1, x_2)}{\partial x_1} - q_2 \frac{\partial V^2(x_1, x_2)}{\partial x_2}, \quad (23)$$

with boundary conditions

$$V'(0, x) = V'(x, 0) = 0, x \geq 0. \quad (24)$$

If there are supply strategies \( q_i(x_1, x_2), i = 1, 2 \) so that the corresponding value functions \( V^1 \) and \( V^2 \) are differentiable on \( G \), and if in addition

$$q_1(x_1, x_2) = \arg\max_{q_1 \geq 0} \left\{ p[q_1, q_2(x_1, x_2)]q_1 - c(x_1)q_1 \right\}$$

$$- q_1 \frac{\partial V^1(x_1, x_2)}{\partial x_1} - q_2(x_1, x_2) \frac{\partial V^1(x_1, x_2)}{\partial x_2}, \quad (25)$$

$$q_2(x_1, x_2) = \arg\max_{q_2 \geq 0} \left\{ p[q_1(x_1, x_2), q_2]q_2 - c(x_2)q_2 \right\}$$

$$- q_1(x_1, x_2) \frac{\partial V^2(x_1, x_2)}{\partial x_1} - q_2 \frac{\partial V^2(x_1, x_2)}{\partial x_2}, \quad (26)$$

on \( G \), then \( q_i(x_1, x_2), i = 1, 2 \) constitute a Nash feedback equilibrium solution for duopoly problem \((1')\)–\((3')\), \( i = 1, 2 \).

Finding a numerical solution for a Nash feedback equilibrium may be a demanding task, as one does not know beforehand the nature of the equilibrium, feedback policies. In the quadratic-linear case of Section 2, the supply strategies of a Nash feedback duopoly equilibrium are continuous and the value functions are both continuous and differentiable and therefore satisfy HJB
equations (22)–(24). We would expect that a slight nonlinear modification of problem would not modify the Nash feedback policies very much.

4.1. Markov chain approximation and iteration in policy function space

An appealing possibility for finding a numerical solution is offered by the Markov chain approximation method designed for solving deterministic or stochastic optimal control problems (for a thorough exposition, see Kushner and Dupuis, 1992). In this method the continuous time optimal control problem is approximated by a discrete time, discrete-state-controlled Markov chain problem. This approximation can be done by properly discretizing the HJB equation and interpreting the result as a Markov chain (see Kushner and Dupuis, 1992; Haurie et al., 1994). Under rather mild conditions a classical iterative method, called approximation in policy function space and introduced by Bellman (see Bellman, 1957; Howard, 1960), is globally convergent, and as the approximation parameter \( h \) approaches 0, the value function converges to the value function of the continuous time control problem. This method can be readily extended to our Nash feedback equilibrium problem. In our case the required approximation can be done directly without first discretizing HJB equations (22)–(24). This direct approach makes it straightforward to see the connection between the true value functions \( V^i \) and the Markov chain approximations.

Extending from Kushner and Dupuis (1992, Chapter 4.5, 5.1–2 and 6.2), let \( K > 1 \) be an integer, \( N = K^2, h = X/(K - 1) \) denote an approximation parameter, \( S_h \) denote the discrete set \( \{(ih, jh)|i, j = 1, \ldots, K - 1\} \subset G \), \( B_{1h} = \{(ih, 0)|i = 1, \ldots, K - 1\} \) and \( B_{2h} = \{(0, ih)|i = 1, \ldots, K - 1\} \) denote the discrete boundary sets, \( \bar{B}_{ih} = B_{ih} \cup \{(0, 0)\} \) and \( \bar{S}_h = S_h \cup B_{1h} \cup B_{2h} \). Let \( e_i, i = 1, 2 \) denote the unit vectors of \( R^2 \). By assuming \( p(q) - c(x) < 0 \) for all \( q > Q \) and for all \( x, 0 \leq x \leq X \), we may restrict ourselves to supplies satisfying \( q_1 + q_2 \leq Q \) for all \( (x_1, x_2) \in \bar{S}_h, q_1 = 0 \) for \( (x_1, x_2) \in \bar{B}_{ih}, i \neq j \). Let time be discretized using a constant time step \( \Delta t_h = h/Q \). If \( (q_{1n}, q_{2n}) \) is a constant pair of supplies during time interval \([t_n, t_{n+1}) = [n\Delta t_h, (n + 1)\Delta t_h) \) and \( x_{in} \) denotes the stock of producer \( i \) at moment \( t_n \) then these stocks satisfy

\[
    x_{i, n+1} = x_{in} - q_{in} \Delta t_h, \quad i = 1, 2, \quad n = 0, 1, \ldots
\]

Then, as corresponding discrete time approximations to the values \( J^i, i = 1, 2 \) we have

\[
    J^{ih}(x_{10}, x_{20}, q_1, q_2) = \sum_{n=0}^{\infty} e^{-\delta \Delta t} [p(q_{1n} + q_{2n}) - c(x_{in})]q_{in} \Delta t_h
\]
and the optimal value functions $\tilde{V}^i_{th}$, given the other producer $j$’s supply strategy $q_j$, satisfy dynamic programming equations

$$
\tilde{V}^i_{th}(x_1, x_2; q_j) = \max_{q_i \geq 0} \left\{ e^{- \delta \Delta t} \tilde{V}^i_{th}(x_1 - q_i \Delta t^h, x_2 - q_2 \Delta t^h, q_j) + [p(q_1 + q_2) - c(x_i)]q_i \Delta t^h \right\}.
$$

(27)

Markov chain approximations $V^i_{th}$ for value functions $\tilde{V}^i_{th}$ are obtained if we let approximations $\tilde{V}^i_{th}$ be continuous functions that are linear in each triangle on $G$ of the type shown in Fig. 6.

Then all point $(z_1, z_2) = (x_1 - q_1 \Delta t^h, x_2 - q_2 \Delta t^h)$ reachable from point $(x_1, x_2)$ in time $\Delta t^h = h/Q$ with supplies $q_1, q_2 \geq 0, q_1, q_2 \leq Q$ are convex combinations of the corner points $(x_1 - h, x_2), (x_1, x_2 - h)$ and $(x_1, x_2)$ with weights $q_1/Q, q_2/Q$ and $1 - (q_1 + q_2)/Q$, respectively. Accordingly, $V^i_{th}(z_1, z_2)$ is the convex combination of values of $V^i_{th}$ at the same corner points with the same weights. Let us take these weights as transition probabilities for a controlled discrete Markov chain on $S_h$. Let $p(x_1, x_2; z_1, z_2)$ denote the transition probability from state $(x_1, x_2)$ to state $(z_1, z_2)$. Then

$$
p(ih, jh; (i - 1)h, jh) = \frac{q_i}{Q} \text{ for } i \geq 1, j \geq 0, p(ih, jh; ih, (j - 1)h) = \frac{q_j}{Q} \text{ for } i \geq 0, j \geq 1, p(ih, jh; ih, jh) = 1 - (q_1 + q_2)/Q \text{ for } i \geq 0, j \geq 0, \text{ and } p(x_1, x_2; z_1, z_2) = 0 \text{ otherwise.}
$$

Using this notation, we can approximate (27) on $S_h$ by

$$
V^i_{th}(x_1, x_2; q_j) = \max_{q_i \geq 0} \left\{ e^{- \delta \Delta t^h} \sum_{(z_1, z_2) \in S_h} p(x_1, x_2; z_1, z_2) V^i_{th}(z_1, z_2; q_j) + [p(q_1 + q_2) - c(x_i)]q_i \Delta t^h \right\}.
$$

(28)

Eq. (28) can also be interpreted as a Markov chain approximation of the value function $V^i$, where the chain is defined on states $S_h$. Following Kushner and Dupuis (1992), it can be shown that, given a feasible supply strategy $q_j(x_1, x_2)$, as $h \to 0$, the solution $V^i_{th}$ of (28) converges to the value function $V^i$ of producer $i$, given strategy $q_j(x_1, x_2)$ of the other producer.

Let us index the states $x_k = (x_{1k}, x_{2k}) = (ih, jh), k = 1, \ldots, N = K^2$ in $S_h$ so that $k = iK + j + 1, i, j = 0, 1, \ldots, K - 1$. For $i = 1, 2$, define $N$ vectors $V^i, Q^i$ and $\Pi^i$ having $k$th components $V^i_k = V^i_{th}(x_k), Q^i_k = q_i(x_k)$ and $\Pi^i_k = [p(q_{1k} + q_{2k}) - c(x_k)]q_{ik} \Delta t^h, k = 1, \ldots, N$. The state transition probabilities then define state transition probability matrix $P \in R^{K \times N}$ with elements,

$$
P_{kk}' = \begin{cases} 
1 - (q_{1k} + q_{2k})/Q, & k' = k, \\
q_{1k}/Q, & k' = k - 1, k' \neq iK, i = 0, 1, \ldots, K - 1, \\
q_{2k}/Q, & k' = k - K, k' \geq 1, \\
0, & \text{otherwise}, \\
k = 1, \ldots, N. 
\end{cases}
$$

(29)
Finally, let \( R(Q^1, Q^2) = e^{-\delta t}P(Q^1, Q^2) \) denote the discounted state transition matrix. Using this notation and (28) a Markov chain approximation on \( \bar{S}_h \) of our Nash feedback equilibrium can then be written so as to find supply strategies \( \bar{Q}^i \) and value vectors \( \bar{V}^i \), \( i = 1, 2 \) which satisfy the equations

\[
\bar{V}^i = R(\bar{Q}^1, \bar{Q}^2)\bar{V}^i + \Pi^i(\bar{Q}^1, \bar{Q}^2), \quad i = 1, 2, \tag{30}
\]

\[
\bar{Q}^i = \arg\max_{Q^i \geq 0} \{ R(\bar{Q}^{-1})\bar{V}^i + \Pi^i(\bar{Q}^{-1}) \}, \quad i = 1, 2, \tag{31}
\]

where \( \bar{Q}^{-1} = (Q^1, Q^2) \) and \( \bar{Q}^{-2} = (\bar{Q}^1, Q^2) \). Note that (31) is a pair \((i = 1, 2)\) of vector maximization problems where maximization is to be performed component by component for each \( k \). For the \( k \)th component this pair is a static Nash equilibrium problem for \((q_1^k, q_2^k)\).

The iterative method of approximation in policy space is to pick feasible initial supply strategies \( Q^{i0} \) such that value vectors \( \bar{V}^{i0} \) are bounded solutions of (30) for \( i = 1, 2 \). We can let \( Q^{i0} = 0 \), which gives \( \bar{V}^{i0} = 0 \), \( i = 1, 2 \). Then for \( n \geq 1 \), define the sequence of feedback controls \( Q^{in} \) and value vectors \( \bar{V}^{in} \) recursively from the formulas

\[
Q^{in} = \arg\max_{Q^i \geq 0} \{ R(Q^{1n}, Q^{2n})\bar{V}^{n-1} + \Pi^i(Q^{1n}, Q^{2n}) \}, \quad i = 1, 2, \tag{32}
\]

\[
\bar{V}^{in} = R(Q^{1n}, Q^{2n})\bar{V}^{in} + \Pi^i(Q^{1n} + Q^{2n}), \quad i = 1, 2. \tag{33}
\]

Note that the matrix \( R \) is a lower triangular band matrix with bandwidth \( K + 1 \) (and three nonzero diagonal rows). Therefore solving (33) for \( \bar{V}^{in} \) is easily done by Gaussian elimination. The fact that \( \Pi^i_k \leq 0 \) on the boundary set \( \bar{B}_{jh}, i \neq j \), and the structure of \( R \) ensure fulfillment of the boundary conditions (24).

Because (32)–(33) represents a process of finding an equilibrium for a game instead of finding an optimal solution for a control problem, we can no longer
Fig. 7. Equilibrium strategies with different functional forms. (a) Linear demand, linear costs, \( \delta = \frac{1}{2}, \bar{p} = 10, p(q) = \bar{p} - q_1 - q_2, c(x) = \bar{p} - cx. \) (b) Nonlinear demand, linear costs, \( \delta = \frac{1}{2}, \bar{p} = 10, p(q) = \bar{p} \exp(-q_1 - q_2), c(x) = \bar{p} - cx. \) (c) Linear demand, nonlinear costs, \( \delta = \frac{1}{2}, \bar{p} = 10, \bar{p} = 5, p(q) = (\bar{p} - \bar{p}) \exp(-x_i) + \bar{p} . \) (d) Nonlinear demand, nonlinear costs, \( \delta = \frac{1}{2}, \bar{p} = 10, \bar{p} = 5, p(q) = \bar{p} \exp(-q_1 - q_2), c(x) = (\bar{p} - \bar{p}) \exp(-x_i) + \bar{p} . \)

guarantee, that the above interactive method converges. In our test problems the convergence turned out to be very rapid, requiring only about 20 iterations to reach accuracy \( 10^{-8} \) for the largest change in \( q_{ik} \) values in a grid with \( K = 1001 \) and \( h = 0.01. \)

4.2. Interpretation of the numerical solutions

Fig. 7 shows the equilibrium strategies. Using the linear quadratic functional forms we obtain Fig. 7a, which approximates our analytical solution with high accuracy. Note that extraction is a linear function of the state given that both
players extract the resource. Over the whole state space, the strategy is non-linear, since there are regimes where one of the players does not extract the resource. In Fig. 7b the demand function is non-linear. The figure suggests that this reduces the region of states where the supply is simultaneous. Fig. 7c and d are based on a strictly convex extraction unit cost function. The Nash feedback strategy is now more clearly a nonlinear function of the resource stocks. In all four cases one can observe the three supply regimes. In addition, the extraction level always increases with the player’s own stock level and decreases with the rival’s stock level, implying that the player with the higher resource stock must have the bigger market share.

Fig. 8 shows the computation results in resource stocks state space. Recall that paths reaching either of the axes outside the origin are ruled out as equilibrium candidates because they inevitably imply \( p(q_i, q_j) - c(x_i) < 0 \) as \( x_i \to 0 \), meaning that player \( i \) would sell his last units at a price lower than marginal extraction costs. Thus any equilibrium with strictly positive initial resource levels must converge toward the origin along a regime where both players supply the resource. The affine-quadratic formulation can be understood as a linearization of the equilibria based on nonlinear demand and cost functions. Thus, if we consider a region arbitrarily close to the origin, the affine-quadratic case must yield an approximation for the other equilibria with arbitrarily high accuracy. This shows that close to the origin any equilibrium must converge toward a turnpike trajectory where the market shares of the players coincide. Such a feature can be observed in Fig. 8b–d.

Comparison of Fig. 8a and b suggests that a change in the demand function does not substantially change the equilibrium time paths but may strengthen the convergence toward the turnpike. Fig. 8c and d show that strictly convex unit cost function imply major changes for the equilibrium properties when the state is not near the origin. This follows because with high stock levels the unit extraction cost is approximately constant. Thus with high stock levels, our strictly convex cost function resembles the constant unit costs of the ‘oil’igopoly theory. In line with this, Fig. 8c and d indicate that with high stock levels also the equilibrium properties resemble the equilibrium properties of the ‘oil’igopoly theory (cf. Fig. 2). Recall that the lower the slope of any trajectory, the higher player 1’s market share. Fig. 8a–d again suggest that the player with the greater resource stock has the bigger market share. If the initial difference in the stock levels is large enough, the player with the greater stock has at first increasing market share, but when the dependence of extraction costs on remaining reserves increases, the market share starts to decrease and finally converges toward \( \frac{1}{2} \). Similar analysis can be done concerning the evolution of the production/reserves ratio (cf. Proposition 2). These examples suggest that as resources are depleted the equilibrium properties may vary from those of the ‘oil’igopoly theory towards the properties determined by the turnpike property and economic depletion.
Fig. 8. Duopoly equilibria with different functional forms. (a) Linear demand, linear costs, \( \delta = \frac{1}{10}, c = \frac{1}{10}, p = 10, p = \hat{p} - q_1 - q_2, c(x) = \hat{p} - cx. \) (b) Nonlinear demand, linear costs, \( \delta = \frac{1}{10}, c = \frac{1}{10}, p = 10, p(q) = \hat{p} \exp(-q_1 - q_2), c(x) = \hat{p} - cx. \) (c) Linear demand, nonlinear costs, \( \delta = \frac{1}{10}, \hat{p} = 10, \hat{p} = 5, p(q) = \hat{p} - q_1 - q_2, c(x) = (\hat{p} - \hat{p}) \exp(-x_i) + \hat{p}. \) (d) Nonlinear demand, nonlinear costs, \( \delta = \frac{1}{10}, \hat{p} = 10, \hat{p} = 5, p(q) = \hat{p} \exp(-q_1 - q_2), c(x_i) = (\hat{p} - \hat{p}) \exp(-x_i) + \hat{p}. \)

5. Discussion

We have analyzed an oligopoly model for nonrenewable resources, assuming economic depletion and a feedback equilibrium concept. The case of an affine-quadratic version of this problem leads to explicitly Nash–Cournot feedback equilibria in both the case of duopoly or \( n \)-player oligopoly. Specifications based on nonlinear demand and nonquadratic costs were solved by numerical
methods. The earliest models for oligopolistic nonrenewable resource markets predict that the concentration in supply increases in time and that a nonrenewable resource market typically evolves toward a monopoly. In contrast, our model suggests that increasing price induces new producers whose extraction costs were initially too high for entering the market. In addition, with symmetric suppliers, the equilibrium always approaches a turnpike where symmetric producers have equal market shares. However, numerical computation with nonlinear demand and nonquadratic costs shows that if extraction costs are initially independent of the reserves level, the market shares may temporarily diverge.

We emphasize that our study however neglects the possibility that producers may use memory-dependent trigger strategies. Within such a more general framework incentives to form a joint maximization strategy that resembles monopoly supply may strengthen as time evolves and producers become more symmetric with respect to extraction costs. Analysing this question within a nonlinear multiple resource stock framework is an interesting issue for future research.

Polansky (1992) concludes that ‘oligopoly exhaustible resource theory predicts that the percentage of reserves produced should decline with reserves size’ and that this hypothesis is supported by the oil market data. This may suggest that the future development of oil markets can be viewed according to the ‘oil oligopoly theory. We have shown that his empirical finding may as well be consistent with such a version of the oligopoly exhaustible resource model (Section 4) that gives completely different future predictions. In addition, the hypothesis is conditional on extraction cost specification in a way that is not controlled by Polansky (1992). More generally the interesting question whether oil and some other nonrenewable resource markets should be viewed using Nash–Cournot or Stackelberg model may be impossible to solve before developing a consistent version of the Stackelberg equilibrium for resource markets. It is also clear that oligopoly models for nonrenewable resources should be extended to include aspects like the discovery of new deposits and technological change, so that they more accurately describe the evolution of these markets.

Appendix A. Proofs of Propositions 1 and 3

Define $\omega = \Omega/c^2$, $\xi = \lambda/c^4$, $D = \sqrt{\omega^2 - 4\xi}$, $\gamma = \delta/c$, $\lambda_i = \mu_i/c$ ($i = 1, 2$) and $\rho_i = r_i/c$ ($i = 1, 2$). Eqs. (20) and (21) imply $r_1 = \mu_1 - \mu_2$ and $r_2 = (1 - n)\mu_1 - \mu_2$. Thus, $\lambda_2 + (n - 1)\lambda_1 > 0$ is necessary for $r_2 < 0$. We next show that $r_1, r_2$ are real for all $\mu_1 < 0, \mu_2 + (n - 1)\mu_1 > 0, n \geq 2$, i.e. for all $\lambda_1 < 0, \lambda_2 + (n - 1)\lambda_1 > 0$. For $n = 2, D^2 = \omega^2 - 4\xi = k_0 \lambda_1^2 + k_1 \lambda_1 + k_2$, where $k_0 = (\lambda_2 + 2\gamma)^2 > 0$, $k_1 = -\frac{1}{2} \gamma^2 - \gamma \lambda_2 - \gamma^2 \lambda_2 - 3\gamma \lambda_2 - 2\lambda_2^2 < 0$ and $k_2 = (1 - \frac{1}{2} \gamma^2 - 3\gamma \lambda_2 + 2\lambda_2^2)^2 \geq 0$. Therefore $\omega^2 - 4\xi > 0$, implying that $D$ is real.
For \( n \geq 3 \), denote \( \lambda_2 = \tau - (n-1)\lambda_1 \), where \( \tau > 0 \). Using the fact that \( \tau > 0 \) and \( \lambda_1 < 0 \), we obtain \( \omega^2 - 4\xi > \tau^4 + 2\gamma\tau^3 + \gamma(\gamma - 1)\tau^2 - \gamma^2\tau + \gamma^2/4 = Y^2[Y + \sqrt{\gamma(2 + \gamma)}] \geq 0 \), where \( Y = \tau + \frac{1}{2}\gamma - \frac{1}{2}\sqrt{\gamma(2 + \gamma)} \). Because, in addition, \( \omega = -\frac{1}{2}\gamma - \tau^2 + \frac{1}{2}\gamma(n + 1)\lambda_1 - n\lambda_1^2 + [(n + 1)\lambda_1 - \gamma] \tau < 0 \) and \( \zeta = \gamma(\tau - \lambda_1) \), \( \gamma + \tau - \lambda_1 > 0 \) for \( n \geq 2 \), \( \lambda_1 < 0 \), \( \lambda_2 + \lambda_1(n - 1) > 0 \), it follows that \( r_1 \) and \( r_2 \) are real.

Next, we show that (20), (21) have a unique solution when \( \lambda_2 + (n - 1)\lambda_1 > 0 \), \( \lambda_1 < 0 \). Using the above definitions, we can write

\[
\rho_1 = \lambda_1 - \lambda_2 = \frac{1}{2}\gamma - \sqrt{\gamma^2/4 - \omega/2 + D/2},
\]

\[
\rho_2 = (1 - n)\lambda_1 - \lambda_2 = \frac{1}{2}\gamma - \sqrt{\gamma^2/4 - \omega/2 - D/2}.
\]

Eqs. (A.1) and (A.2) are equivalent to

\[
(2\lambda_1 - 2\lambda_2 - \gamma)^2 + [2(1 - n)\lambda_1 - 2\lambda_2 - \gamma]^2 = 2\gamma^2 - 4\omega,
\]

\[
[(2\lambda_1 - 2\lambda_2 - \gamma)^2 - [2(1 - n)\lambda_1 - 2\lambda_2 - \gamma]^2 = 16D^2.
\]

\[
\Rightarrow \lambda_1^2 + (n - 1)\lambda_1\lambda_2 + n(n - 1)\lambda_1^2 + \gamma\lambda_2 + \frac{1}{2}\gamma(n - 1)\lambda_1 - \frac{1}{2}\gamma = 0,
\]

\[
n^2\lambda_1^2[(2 - n)\lambda_1 - 2\lambda_2 - \gamma]^2 = D^2.
\]

Note that \( \lambda_2 + (n - 1)\lambda_1 > 0 \) implies that \( (2 - n)\lambda_1 - 2\lambda_2 - \gamma < 0 \) when \( \lambda_1 < 0 \). Eq. (A.3') defines an ellipse. Let \( a_0 = 1/\sqrt{(3n + 1)(n - 1)} \), \( a_1 = a_0\gamma^2(2 + \gamma) \), \( a_2 = 1/\sqrt{2 + \gamma} \). Then, the arc of this ellipse can be given in parametric form as

\[
\lambda_1 = -a_1\sqrt{(1 + s^2)}, \lambda_2 = a_2s + (n - 1)\frac{1}{2}a_1\sqrt{1 + s^2} - \frac{1}{2}\gamma,
\]

where \( s > 0 \). Let us denote: \( b_0 = -1 \), \( b_1 = -1 + \gamma + 3n + \gamma n \), \( b_2 = 5 + 2\gamma + 10n + 4\gamma n - 3n^2 \), \( b_3 = 5 + \gamma + 11n + 3\gamma n - 9n^2 + \gamma n^2 + n^3 - n^3 \). The restriction \( \lambda_2 + (n - 1)\lambda_1 > 0 \) takes the form

\[
s > s = a_0\frac{1}{2} \{ (2 + \gamma)(n - 1) + \sqrt{\gamma(n - 1)(8n - \gamma + \gamma n)} \}
\]

and equation (A.4') \( c(s, \gamma, n)h(s, \gamma, n) = 0 \), where \( c(s, \gamma, n) = \gamma^2/[3n + 1]^2 \), \( (2 + \gamma)(n - 1)(1 + s^2) \) and \( h(s, \gamma, n) = b_0(s/a_0)^3 + b_1(s/a_0)^3 + b_2(s/a_0) + b_3 \).

Because \( b_0 = -1 \) and \( a_0 > 0 \), \( h \rightarrow \infty \) as \( s \rightarrow \infty \). On the other hand, \( h(s, \gamma, n) = (2 + \gamma)[4n + \gamma(1 - 5n + 6n^2) + \gamma^2(1 - 2n + n^2) + (1 - \gamma + 2n + \gamma n)\sqrt{\gamma(n - 1)(8n - \gamma + \gamma n)}] > 0 \). Thus, there always exists a root \( s_1 > s \) for \( h(s, \gamma, n) = 0 \). Denote the roots of \( c\hat{h}(s, \gamma, n) = 0 \) by \( s_a \) and \( s_b \). One obtains \( s_a s_b = b_2 a_0^2/3b_0 \) and \( (s^2 - s_a s_b)a_0^2 = \frac{4}{3}(2 + n) + \frac{2}{3}(5 - 8n + 9n^2) + \frac{1}{3}\gamma^2(1 - 2n + n^2) + \frac{1}{3}\gamma^2(1 - 2n + n^2) + \frac{1}{3}(2 + \gamma)(n - 1)\sqrt{\gamma(n - 1)(8n - \gamma + \gamma n)} > 0 \). This excludes \( s_a > s_0 \) and proves the uniqueness of the solution for \( h(s, \gamma, n) = 0 \) above \( s_0 \).
Using the solution procedure for third order polynomials, the solution $s_1 > s$ for $h(s, \gamma, n) = 0$ is

$$s_1 = \frac{a_0}{3} \{ p_0 + 2p_1 \cos[\frac{1}{3} \arccos(p_2/p_1^3)] \},$$

where $p_0 = -1 + \gamma + (3 + \gamma)n$, $p_1 = [16 + 4\gamma + \gamma^2 + (24 + 16\gamma + 2\gamma^2)n + (6\gamma + \gamma^2)n^2]^{1/2}$, $p_2 = 44 + 30\gamma + 6\gamma^2 + \gamma^3 + (180 + 102\gamma + 30\gamma^2 + 3\gamma^3)n + (108\gamma + 33\gamma^2 + 3\gamma^3)n^2 + (9\gamma^2 + \gamma^3)n^3$. □

### Appendix B. Proof for Lemma 1

Necessary conditions for the efficient equilibrium include

$$q_i^e = \frac{1}{2} (cx_i^e - \varphi_i^e), \quad \dot{\varphi}_i^e = -cq_i^e + \delta \varphi_i^e, \quad \dot{x}_i^e = -q_i^e, \quad i = 1, 2,$$

where the subscript $e$ refers to the efficient solution. This yields $q_i^e = -wx_i^e e^{w t}$, where $w = \frac{1}{2}[\delta - (\delta^2 + 2\delta c)^{1/2}]$. Let the subscript $d$ refer to duopoly where $q_i^d = (\mu_1 + \mu_2)w^i_0 e^{-i(\mu_1 + \mu_2)h}$, $i = 1, 2$, and where the necessary conditions for problem (1)–(3) must hold. We show that $q_i^d \geq q_i^e$ for any $x > 0$ yields a contradiction. The solutions for $q_i^d$ and $q_i^e$ given above imply $\dot{q}_i^d \leq \dot{q}_i^e < 0 \forall x > 0$ when $q_i^d \geq q_i^e$. By the Hamiltonian maximization conditions, we obtain

$$q_i^e = \frac{1}{2} (cx_i^e - \varphi_i^e) \leq q_i^d = (cx_i^d - \varphi_i^d)/3 \iff \varphi_i^d/3 - \frac{1}{2} \varphi_i^e \leq -cx/6 < 0 \forall x > 0,$$

where $x_i^d = x_i^e \equiv x$. However,

$$d[\frac{1}{2} (cx_i^e - \varphi_i^e)]/dt \geq d[(cx_i^d - \varphi_i^d)/3]/dt \Rightarrow \delta(\varphi_i^d/3 - \frac{1}{2} \varphi_i^e) \geq -\mu_1(q_i^d + \psi_i^d) > 0,$$

a contradiction because by (7) $q_i^d + \psi_i^d > 0$ along the turnpike. The last claim holds because $q_i^d + \psi_i^d \leq 0$ would imply $\psi \rightarrow -\infty$ as $t \rightarrow \infty$, which contradicts (6) and $q_i^d \rightarrow 0$ as $t \rightarrow \infty$. □

### Appendix C. Proof of Proposition 2

**Proof.** (A) By $x_1 > x_2$, we obtain $q_1 - q_2 = \mu_2(x_1 - x_2) + \mu_1(x_2 - x_1) > 0$. In the $x_1 - x_2$ state space, the slope of any path equals $q_2/q_1$. The time derivative of this slope $(\dot{q}_2q_1 - \dot{q}_1q_2)/q_1^2$ is positive if $x_1 > x_2$ because then the solution approaches the turnpike from below. Thus, the time derivative of agent 1’s market share, i.e.

$$d[q_1/(q_1 + q_2)]/dt = [\dot{q}_1(q_1 + q_2) - q_1(\dot{q}_1 + \dot{q}_2)]/(q_1 + q_2)^2,$$
is negative. An analogous result holds when \( x_1 > x_1 \). (B) When \( x_{10} > x_{20} \), the path converges toward the turnpike from below and is concave. This and the fact that the trajectory converges toward the origin implies \( x_1/x_{10} < x_2/x_{20} \ \forall t \in (0, \infty) \). (C) We show that \( x_{10} > x_{20} \) implies that \( q_1/x_1 > q_2/x_2 \) for \( \forall t \in [0, \infty) \). Writing the extraction rates in feedback form gives

\[
q_1/x_1 > q_2/x_2 \iff (\mu_2x_1 + \mu_1x_2)/x_1 > (\mu_1x_1 + \mu_2x_2)/x_2 \iff x_2/x_1 < x_1/x_2.
\]

By the turnpike property, \( x_{10} > x_{20} \) implies \( x_1 > x_2 \ \forall t \in [0, \infty) \), which proves the last claim. □

Appendix D. Proof of Proposition 4

The market share of agent \( i \) is constant along the linear turnpike. Because \( r_1 < r_2 < 0 \), the turnpike is defined by \( V_1 = 0 \) (Eqs. (17) and (18)). \( V_1 = 0 \) when \( z = x_i\mu_1(1 - n)/[\mu_1(1 - n) + \mu_2 + r_2 - \mu_1] \). By (17), (18), and \( \mu_1 > 0, \mu_2 > 0 \), we know that \( \mu_1(n - 1) + \mu_2 + r_2 = 0 \), implying \( z = x_i(n - 1) \) in order for \( V_1 = 0 \) to hold. When \( x_i > z/(n - 1) \), the solution path is below the linear turnpike and, because \( r_1 < r_2 < 0 \), it must converge toward the turnpike, implying that the slope of the trajectory increases; i.e. the market share of agent \( i \) decreases. A perfectly analogous argument applies when \( x_i < z/(n - 1) \). □

Appendix E. Proof of Proposition 5

(A) In \( x_i - z \) space, the slope of the trajectories equals \( \sum_{j \neq i} q_j/q_i \). Along the turnpike,

\[ x_i = z/(n - 1) \Rightarrow q_i = \sum_{j \neq i} q_j/(n - 1); \]

i.e. the slope equals \( n - 1 \). When \( x_i > z/(n - 1) \), the path is below the turnpike and the slope is smaller than \( n - 1 \); i.e.

\[ \sum_{j \neq i} q_j/q_i < n - 1 \iff q_i > \sum_{j \neq i} q_j/(n - 1). \]

(B) \( q_i/x_1 > \sum_{j \neq i} q_j/z \) is equivalent to

\[ (\mu_1z + \mu_2x_i)/x_i > [(n - 1)\mu_1x_i + [\mu_1(n - 2) + \mu_2]z]/z. \]

The latter inequality simplifies to \( (n - 1)x_i^2 + zx_i(n - 2) - z^2 > 0 \). By \( x_i(n - 1) - z > 0 \), we obtain that \( (n - 1)x_i^2 + zx_i(n - 2) - z^2 > 0 \) is equivalent to \( z[x_i + x_i(n - 2) - z] > 0 \), which holds by \( x_i(n - 1) - z > 0 \).
(C) The implication $x_i/x_{10} < z/z_0 \forall t \in (0, \infty)$ follows because, in $x_i - z$ state space, the trajectory is concave and converges toward the origin. □

References


