The importance of the number of different agents in a heterogeneous asset-pricing model

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Abstract

Models with heterogeneous agents and incomplete markets often only have two types of agents to limit the computational complexity. The question arises whether equilibrium models with a realistic number of types have the same implications as models with a small number of types. In the asset-pricing model considered in this paper, several properties depend crucially on the number of types. For example, in the economy with only two types interest rates respond to ‘idiosyncratic’ income shocks which makes it easier to smooth consumption. Moreover these effects can be so strong that it is possible that a relaxation of the borrowing constraint reduces an agent’s utility. Average interest rates on the other hand are not very sensitive to the number of types. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Dynamic equilibrium models with heterogeneous agents and incomplete markets have become popular tools in the macro asset-pricing literature. Heterogeneity usually arises because an agent’s income is affected not only by aggregate but also by idiosyncratic income shocks, but other forms of heterogeneity have also been considered.\(^1\) These models have been shown to be an improvement over standard representative agent models in several dimensions. For example, they provide an explanation for the low level of real interest rates observed in the US data.\(^2\) The reason is that the presence of idiosyncratic risk provides an additional incentive to save which lowers real interest rates. Many models analyzed in the literature have only two different types of agents.\(^3\) In these models, individual specific shocks are not truly idiosyncratic since all agents of the same type receive the same shock. One could interpret an agent’s idiosyncratic income shock in an economy with two types as a sector-specific shock. This has two disadvantages. First, although moral hazard can be used to explain why contracts that are contingent on an individual’s income realization do not exist, it cannot be used to rule out contracts contingent on the performance of the sector. The second disadvantage of interpreting the idiosyncratic shocks as sector specific shocks is that the variability of the average income in a sector is a lot smaller than the variability of individual income. Without having a substantial amount of idiosyncratic uncertainty, however, the predictions of heterogeneous agent models are not that different from models with a continuum of agents.

Papers that have analyzed models with two agents and incomplete markets deserve a lot of credit for improving the predictions of general equilibrium models. Using recently developed algorithms to accurately solve models with a large number of different agents\(^4\) we can now answer the question whether models with a large number of types also imply low real interest rates and whether the qualitative and quantitative properties are different from the models with two types of agents that are studied in the literature.

\(^1\) For example, Judd et al. (1998) and Krusell and Smith (1998) consider heterogeneous preferences.


\(^3\) Examples are Heaton and Lucas (1992, 1996), Lucas (1994), Marcet and Singleton (1999), Telmer (1993), and Zhang (2000). Alternatively, it is assumed that there is no aggregate uncertainty and a continuum of agents. In this case asset prices are constant over time. See, for example, Aiyagari (1994) and Aiyagari and Gertler (1991).

In this paper, I analyze an infinite-horizon equilibrium model of the short-term interest rate similar to that used in the literature except that it has a continuum of types instead of two types. Markets are assumed to be incomplete, but agents can smooth their consumption by trading in risk-free bonds. For the parameter values considered in this paper, average interest rates in the model with a large number of different agents are very similar to average interest rates in the corresponding model with only two types of agents. In the presence of borrowing constraints, the model with a continuum of types, thus, also predicts low average real interest rates. Several other properties of the model studied in this paper, however, depend crucially on the number of types.

The first important difference between the two models is that the amount of cross-sectional wealth dispersion is a lot more volatile in the economy with only two types because in economies with only two types ‘idiosyncratic’ shocks affect the amount of cross-sectional dispersion. This paper documents how and why these differences in the time-series properties of the amount of cross-sectional wealth dispersion can affect the time-series properties of interest rates, consumption, and bond purchases in asset-pricing models with heterogeneous agents. Not surprisingly, the higher volatility of cross-sectional dispersion in the economies with two types implies a much higher volatility of interest rates as well. Heaton and Lucas (1996) mention that ‘typically in models that fit the equity premium, the resulting volatility of the bond return is too high’. This issue has only been addressed in economies with two types. In this paper I show that economies with a large number of different agents can have very low average real interest rates without having an excessively high volatility for bond returns.

Another important difference between the economy with two types and the economy with a continuum of different types is the ability of the agent to insure himself against idiosyncratic shocks. In the economy studied in this paper, it is easier to smooth consumption when there are only two types than when there are a continuum of types because when there are only two types the interest rate responds to ‘idiosyncratic’ shocks. The explanation is the following. Consider the case where an agent receives a low realization of the idiosyncratic shock for several periods. In the economy with two types this means that half the population receives a low realization for several periods. This increases the amount of cross-sectional dispersion and this causes the interest rate to drop. This reduction in the interest rate due to ‘idiosyncratic’ shocks works like a transfer from the rich agents (the lenders) to the poor agents (the borrowers). In an economy with a continuum of types, an agent who faces the same sequence of low-income realizations is not so lucky to see the interest rate drop at the same time. Consequently his consumption decreases more than the consumption of the agent in the economy with only two types. These effects are magnified when one of the two agents is at the borrowing constraint. Moreover, they can be so strong that a relaxation of the borrowing constraint can actually reduce an agent’s welfare in an economy with two types.
In this paper, stationary variables are denoted by small characters and non-stationary variables are denoted by capital characters. Wang (1995) shows that contracts contingent on the realization of the shock are possible when multi-period contracts are enforceable.

The organization of this paper is as follows. In the next section, an infinite-horizon endowment economy with incomplete markets will be discussed. In Section 3, I analyze the properties of the economy with a continuum of types and those of the economy with two types. The last section concludes.

2. An equilibrium model with heterogeneous agents

In this section, I develop an infinite horizon model of the short-term interest rate similar to the models in Deaton (1991) and Pischke (1995). In contrast to Deaton (1991) and Pischke (1995), the interest rate is not constant but varies to ensure that the bond market is in equilibrium. In Section 2.1, I discuss the optimization problem of the individual agent. In Section 2.2, I discuss the equilibrium condition, and in Section 2.3, I discuss the parameter values used.

2.1. The individual agent’s problem

Ex ante agents are exactly the same, but ex post differ due to the presence of idiosyncratic shocks. In particular, the endowment of agent $i$ relative to the per capita endowment, $y_i^t$, can take on a low value, $y^L$, and a high value, $y^H$. The (gross) growth rate of the per capita endowment, $a_t = A_t / A_{t-1}$, can also take on a low (or recession) value, $a^R$, and a high (or boom) value, $a^B$. Both processes are assumed to be first-order Markov processes.

In an economy in which all shocks are observed without costs, it would be optimal to write contracts contingent on the realization of the idiosyncratic shock. If it is impossible for the lender to verify the realization of the borrower’s income, the borrower would always report that he received the lowest possible realization. In this case, the optimal one-period contract is a bond with a fixed payment. It is assumed here that agents can only smooth their consumption by trading in a one-period risk-free bond. Markets are, thus, incomplete.

Agent $i$’s maximization problem is as follows:

$$\max \sum_{t=0}^{\infty} \beta^t \frac{C_i^t \gamma - 1 - 1}{1 - \gamma}$$

s.t. $C_i^t + q_t B_i^t = y_i^t A_t + B_{i-1}^t$, \hspace{1cm} (1)

5 In this paper, stationary variables are denoted by small characters and non-stationary variables are denoted by capital characters.

6 Wang (1995) shows that contracts contingent on the realization of the shock are possible when multi-period contracts are enforceable.
where \( C^i_t \) is the amount of consumption of agent \( i \) in period \( t \), \( B^i_t \) is the demand for one-period bonds that pay one unit of the consumption commodity in the next period, \( q_t \) is the price of this one-period bond, and \( B^i_{t-1} \) is given. The agent takes the bond price as given. The interest rate, \( r_t \), is defined as \((1 - q_t)/q_t\).

The question arises whether a lender would want to restrict the amount he is willing to lend to a borrower. It would make sense to limit the amount of debt by the net present value of the borrower’s endowment stream. In this economy, in which agents live and receive an endowment stream for ever, this constraint is unlikely to seriously restrict the demand for loans. In reality, however, consumers and firms do seem to face borrowing constraints. More restrictive borrowing constraints arise in our model if one makes the assumption that the borrower faces limited costs of defaulting on the loan. Suppose that the default costs are equal to \( \bar{B}_i \). In this case, borrowers will default whenever \(- B_t \geq \bar{B}_i \). This, of course, implies that the lender will never lend more than \( \bar{B}_i \). I assume that \( \bar{B}_i \) (scaled by the per capita endowment) is a constant. Thus, the following borrowing constraint is added to the maximization problem:

\[
B_t^i \geq - \bar{b}A_t. \tag{2}
\]

Note that \( B_t^i \) includes the interest payments on the bond. Let \( \bar{B}_t^i \) denote the amount of savings net of interest payments. Then Eq. (2) can be rewritten as follows:

\[
\bar{B}_t^i \geq - \frac{\bar{b}A_t}{1 + r_t}. \tag{3}
\]

Eq. (3) is a typical formulation of a borrowing constraint. It has the property that the maximum amount an agent can borrow is decreasing with the interest rate. The first-order conditions for the maximization problem are the two-part Kuhn–Tucker conditions:

\[
q_t[C^i_t]^{-\gamma} \geq \beta E_t[C^i_{t+1}]^{-\gamma} \quad \text{and} \quad (B_t^i + \bar{b}A_t)(q_t[C^i_t]^{-\gamma} - \beta E_t[C^i_{t+1}]^{-\gamma}) = 0. \tag{4}
\]

The equations of this model can easily be transformed to a system of equations that contains only stationary variables. To see this, define \( z_t = Z_t/A_t \) for any non-stationary variable \( Z_t \). Eqs. (1), (2) and (4) can then be written as

\[
c^i_t + q_t b^i_t = y^i_t + b^i_{t-1}/a_t, \tag{5}
\]

\[
b_t^i \geq - \bar{b} \tag{6}
\]

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7. See Jaffee and Stiglitz (1990, Section 5.3) and the references therein.
8. See, for example, Bernanke et al. (1996).
See Judd et al. (1998) for a discussion on state variables in this type of model.

\[
q_t[c_t^{\gamma}]^{-\gamma} \geq \beta E_t[c_{t+1}^{\gamma} a_{t+1}]^{-\gamma} \quad \text{and} \\
(b_t^i + \bar{b})(q_t[c_t^{\gamma}]^{-\gamma} - \beta E_t[c_{t+1}^{\gamma} a_{t+1}]^{-\gamma}) = 0.
\]  

(7)

2.2. Equilibrium condition

Two different versions of the model are considered. In both versions there are a continuum of agents with unit mass. In the first version, the idiosyncratic draws, \( y_i^t \), are distributed independently across agents and across time. In the second version, there are only two types of agents and each agent always receive the same realization of the idiosyncratic shock as the other agents of the same type. Note that in the economy with two types \( y_1^t \) and \( y_2^t \) are perfectly negatively correlated since \( y_1^t + y_2^t = 2 \). This restriction will be discussed in detail in the next section.

Let \( F^L_t \) and \( F^H_t \) be the cumulative distribution function of the cross-sectional beginning-of-period bond holdings of the agents who receive the low-income shock and the high-income shock, respectively. In the economy with only two types, all agents who receive the low realization for the idiosyncratic shock have the same amount of beginning-of-period bond holdings since they all have the same history of idiosyncratic shocks. The same is true for the agents who receive the high value. Consequently, \( dF^L \) and \( dF^H \) have mass at only one level of \( b_{t-1} \) in the economy with two agents. In contrast, in the economy with a continuum of types, there will be a wide variety of bond holdings at each point in time.

The state variables of agent \( i \) are \( b_{t-1}^i, y_i^t \), and the aggregate state variables \( \bar{a}_t \). Agent \( i \)'s demand for bond holdings is assumed to be a function of the state variables.\(^9\) The equilibrium condition for the bond market is given by

\[
\int_{-\delta/a_t}^{\infty} b(b_{t-1}, y_t^L, \bar{a}_t) \, dF^L_t(b_{t-1}) + \int_{-\delta/a_t}^{\infty} b(b_{t-1}, y_t^H, \bar{a}_t) \, dF^H_t(b_{t-1}) = 0. 
\]  

(8)

In the economy with two types, information about the idiosyncratic shock and bond holdings of one type disclose the corresponding values of the other type. The set of aggregate state variables (\( \bar{a}_t \)) is, thus, equal to the growth rate of the aggregate endowment, \( a_t \). In the economy with a continuum of different agents, the aggregate state variables are \( a_t \), the cross-sectional distribution of the bond holdings of the low-income agents, and the cross-sectional distribution of the bond holdings of the high-income agents. A solution to the model, in this case, consists of a consumption function, \( c(b_{t-1}, y_t, a_t, F^L_t, F^H_t) \), an investment function, \( b(b_{t-1}, y_t, a_t, F^L_t, F^H_t) \), a bond price function, \( q(a_t, F^L_t, F^H_t) \), and the functionals \( F^L(a_t, a_{t-1}, F^L_{t-1}, F^H_{t-1}) \) and \( F^H(a_t, a_{t-1}, F^L_{t-1}, F^H_{t-1}) \) that describe the

\(^9\)See Judd et al. (1998) for a discussion on state variables in this type of model.
law of motion of $F_t^1$ and $F_t^H$, respectively. The only reason why the current value of $a_t$ is an argument of the transition functionals is that the beginning-of-period bond holdings are scaled relative to the per capita endowment. For example, suppose that in period $t - 1$ an agent borrows the maximum amount $\bar{b}_{t-1} A_{t-1}$. Thus, $b_{t-1} = -\bar{b}$. This means that in period $t$, beginning-of-period bond holdings, relative to the aggregate endowment, are equal to $-\bar{b}/a_t$.

Several papers in the literature discuss the existence of equilibrium. Existence of equilibrium with borrowing constraints in an economy like ours but with a finite number of agents is discussed in Magill and Quinzii (1994). Den Haan (1997) compares the numerical solution to the model with a continuum of agents with the simulated results of an economy with 100,000 agents and finds that the results are very similar. This suggests that there are no differences between an economy with a continuum of agents and an economy with a large but finite number of agents. Related papers are Duffie et al. (1994), Levine (1989), and Levine and Zame (1993).

2.3. Parameter values

The time period in the model corresponds to a year and the discount rate is set equal to 0.965. Asset prices and consumption behavior crucially depend on the assumed values for the degree of relative risk aversion, $\gamma$, and the borrowing constraint parameter, $\bar{b}$. I, therefore, consider values for $\gamma$ equal to 1, 3, and 5 and values for $\bar{b}$ ranging from 0.2 to 2.0.

The parameter values of the stochastic driving processes for $A_t$ and $y_t^i$ are identical to those used in Heaton and Lucas (1996). Heaton and Lucas (1996) obtained estimates for these processes using the income series from the Panel Studies of Income Dynamics (PSID) under the assumption that both processes are two-state first-order Markov processes. The two values for the idiosyncratic shock, $y_t$, are 0.7544 and 1.2456 and the two values for the aggregate (gross) growth rate, $a_t$, are 0.9904 and 1.0470. The probability of receiving in this period the same idiosyncratic shock as was received in the last period is equal to 0.7412 and the probability of receiving the same aggregate shock is equal to 0.5473.

Note that the time-series specifications for $A_t$ and $y_t^i$ are the same in both economies. The amount of idiosyncratic uncertainty as well as the amount of aggregate uncertainty, therefore, does not depend on the number of types in the model. In both economies they match the corresponding empirical estimates. It is important to understand that matching the empirical value for the amount of

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10The reported values are two times those reported in Heaton and Lucas (1996) since Heaton and Lucas (1996) define $y_t$ as the share of the aggregate endowment one agent receives, while this paper defines $y_t$ as the endowment relative to the per capita endowment to ensure that the definition also makes sense in the case with a continuum of types.
idiosyncratic uncertainty as well as the empirical value for the amount of aggregate uncertainty imposes restrictions on the joint distribution of $Y_i^1$ and $Y_i^2$ in the economy with two agents. To understand this consider the case where there is no aggregate uncertainty and $A_i = \bar{A}$ in every period.\textsuperscript{11} To ensure that $A_i = \bar{A}$, it must be the case in the economy with two agents that $Y_i^1$ and $Y_i^2$ are perfectly negatively correlated. In contrast, in the economy with a continuum of agents $Y_i^1$ and $Y_i^2$ can be independent for $i \neq j$. A law of large numbers guarantees that $A_i = \bar{A}$. If one would assume that $Y_i^1$ and $Y_i^2$ are independent in the economy with only two agents as well then the amount of aggregate uncertainty would depend on the number of agents. In this case, the properties of the model would depend on the number of agents even when markets are complete because the amount of aggregate uncertainty depends on the number of agents.

3. The properties of diverse and less-diverse economies

In this section, I analyze the properties of the two equilibrium models described in Section 2. Agents in the two economies face exactly the same amount of idiosyncratic and aggregate endowment risk. The joint time-series properties differ considerably, however. In the economy with two types, an agent's idiosyncratic shock is perfectly positively correlated with the shocks of half of the population. In the economy with a continuum of different agents, an agent's idiosyncratic shock is distributed as an independent random variable. In Section 3.1, I analyze the time-series properties of the interest rate. In Section 3.2, I analyze the time-series properties of consumption and bond purchases in the two economies. In Section 3.3, I analyze the ability to insure against idiosyncratic shocks in the two economies.

3.1. Interest rates in diverse and less-diverse economies

In this section, I analyze the behavior of the interest rate in the economy with two types as well as in the economy with a continuum of types. Important for the time-series properties of the interest rate are the time-series properties of the amount of cross-sectional dispersion and the relationship between the amount of cross-sectional dispersion and the interest rate. In Section 3.1.1, I discuss the relationship between the amount of cross-sectional dispersion and the interest rate. In Section 3.1.2, I analyze the behavior of interest rates in the two models.

\textsuperscript{11}Since the amount of aggregate risk is small relative to the amount of idiosyncratic risk this example is not that different from the actual model used.
Fig. 1. Cross-sectional dispersion and the interest rate. This figure plots the interest rate as a function of the low-income agent’s beginning-of-period debt level in the economy with two types. As the debt of the low-income agent increases, the amount of cross-sectional dispersion increases. The borrowing constraint parameter is equal to 1.4 and the parameter of relative risk-aversion is equal to 3. The graph is drawn for the low realization of the aggregate growth rate.

3.1.1. The amount of cross-sectional dispersion and the interest rate

In this section, I discuss the relationship between the amount of cross-sectional dispersion and the interest rate. An example of this relationship is given in Fig. 1 that plots the interest rate as a function of the low-income agent’s beginning-of-period debt in the economy with two types. Note that as the debt of the low-income agent increases, the amount of cross-sectional dispersion increases. The decreasing concave relationship between the interest rate and the amount of cross-sectional dispersion depicted in Fig. 1 is robust to changes in the parameter values. This relationship can be easily understood when one recalls the property that the marginal propensity to save is increasing in wealth.\(^\text{12}\) Consider a wealth transfer from a ‘poor’ agent to a ‘rich’ agent. In response to this transfer the poor agent would want to save less (or borrow more) and the rich agent would want to save more. Since the rich agent’s marginal propensity to save is higher than the poor agent’s marginal propensity to save, the interest rate has to decrease to keep the bond market in equilibrium. Moreover, the effect becomes smaller when the amount of cross-sectional dispersion decreases, i.e., the relationship is concave. To understand this,

\(^{12}\)Carroll and Kimball (1996) showed that this property holds in intertemporal optimization models under very general conditions.
consider the extreme case when there is no cross-sectional dispersion. In this case, the marginal propensities to save are equal for all agents and a marginal wealth transfer would increase the amount of dispersion but would have no effect on the interest rate.

The effects are magnified in the presence of borrowing constraints since borrowing constraints reduce the marginal propensity to save of the less wealthy even further. When a poor agent has reached his debt limit, a wealth transfer from this agent to a wealthier agent requires a large drop in the interest rate since the new interest rate has to be such that the wealthier agent is willing to consume the total wealth transfer. This explains the kink in Fig. 1. At the kink, the poor agent has reached his borrowing constraint and the slope of the function sharply decreases.

3.1.2. Time-series behavior of the interest rate

In this section, I analyze the time-series behavior of the interest rate in an economy with two types of agents and in an economy with a continuum of different types. In particular, I consider the average, the standard deviation, the autocorrelation coefficient, and the correlation with the growth rate of the aggregate endowment. I start with the averages of the bond price and the interest rate. Fig. 2 plots the average bond price in both economies as a function of the borrowing constraint parameter. The borrowing constraint parameter indicates the maximum amount an agent is allowed to borrow relative to the per capita endowment. The parameter of relative risk aversion is equal to one (three) in the case of low (high) risk-aversion. The other parameter values are reported in Section 2.3. Population moments are approximated using the moments calculated with a simulated draw of 100,000 observations.
of the borrowing constraint parameter. I choose to plot the average bond price because the differences are bigger for the bond price than for the interest rate.\textsuperscript{13} Nevertheless, the results are remarkably similar for most parameter values. This is good news for several papers in the asset-pricing literature that mainly focused on average returns.\textsuperscript{14} As documented in the graph, some differences can be found when the borrowing constraint parameter is low and the parameter of relative risk aversion is high. When the borrowing constraint parameter is equal to 0.2 and the parameter of relative risk aversion is equal to three, for example, then the average bond price in the economy with two types is 1.4\% higher than the average bond price in the economy with a continuum of types.\textsuperscript{15} For many purposes these differences are of little importance. When one wants to know the effect of a change in the borrowing constraint parameter from 0.2 to 2.0, for example, then the economy with two types predicts a decrease in the average bond price of 13.3\% and the economy with a continuum of types predicts a decrease of 12.0\%.

The results from the last subsection provide an intuitive explanation for the finding that the average interest rate (bond price) is lower (higher) in the economy with two types. Since the interest rate is a concave function of the amount of cross-sectional dispersion and the amount of cross-sectional dispersion is considerably more volatile in economies with two types of agents, Jensen’s inequality suggests that the average interest rate should be lower in the economy with two types.\textsuperscript{16}

Table 1 documents the differences across the two types of economies for the following time series statistics of the interest rate: the standard deviation, the first-order autocorrelation coefficient, and the correlation with the growth rate of the aggregate endowment. The following observations can be made from the table. First, big differences are observed when one compares the standard

\textsuperscript{13} Jensen’s inequality implies that the differences are smaller for average interest rates than for average bond prices because the interest rate is a concave function of the bond price.

\textsuperscript{14} Examples are Heaton and Lucas (1996), Lucas (1994), and Telmer (1993).

\textsuperscript{15} Not shown in the graph is the result that when the borrowing constraint parameter is equal to 0.2 and the parameter of relative risk aversion is equal to five, then the difference in average bond prices is equal to 2.9 percentage points.

\textsuperscript{16} Note that this is not a formal argument. One reason is that characterizing the amount of cross-sectional dispersion is much more complex in the economy with a continuum of agents. For example, in the economy with two types the amount of beginning-of-period bond holdings of the low-income agents completely describes the amount of cross-sectional dispersion. In the economy with a continuum of types this is not the case. Also, even if the same measure of cross-sectional dispersion is used, the relationship between the interest rate and the amount of cross-sectional dispersion does not have to be the same in the two types of economies. Finally, the average amount of cross-sectional dispersion may differ across the two types of economies.
Table 1
Time-series behavior of the interest rate*

<table>
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<tr>
<th>γ</th>
<th>$\bar{b}$</th>
<th>SD</th>
<th>Autocorrelation</th>
<th>Correlation with $a_i$</th>
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<td></td>
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<td>types 2 types</td>
<td>$\infty$ types 2 types</td>
</tr>
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<td>0.03067</td>
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</tbody>
</table>

*Note: This table reports time-series statistics of the interest rate on a one-period bond. $\gamma$ denotes the parameter of relative risk aversion and $\bar{b}$ indicates the amount the agent is allowed to borrow relative to the per capita endowment. The other parameter values are reported in Section 2.3. Population moments are approximated using the sample moments of a simulated draw of 100,000 observations.

deviation of interest rates across the two types of economies. This can be explained by the large differences in the volatility of the amount of cross-sectional dispersion across the two types of economies. Heaton and Lucas (1996) report that in models with two types of agents it is not possible to fit the equity premium without bond returns being excessively volatile. Heaton and Lucas (1996) report a standard deviation for US bond returns equal to 0.026. As documented in Table 1, for values of $\gamma$ and $\bar{b}$ that produce low average interest rates (i.e. high values of $\gamma$ and low values of $\bar{b}$), the volatility in the economy with two types is indeed much higher than what is observed in the data. In the corresponding economies with a continuum of agents, however, the volatility of interest rates is never larger than what is observed in the data.

Second, the autocorrelation of the interest rate is much higher and much closer to the observed autocorrelation in the economy with two types. The reason is that in this type of economy the interest rate is influenced by realizations of the idiosyncratic shock which are much more persistent than shocks to the aggregate growth rate. Of course, this does not seem like a plausible explanation for the observed persistence.

Third, the correlation with the growth rate of the aggregate endowment is close to one in the economy with a continuum of types because this random variable is the only shock that affects the interest rate. In the economy with only two types this coefficient is much lower because the interest rate is also affected by idiosyncratic shocks.
Note that in the presence of complete markets, the interest rates would be exactly the same in both types of economies. Consistent with this is the observation that the differences across the two types of economies become smaller when the borrowing constraint parameter increases, that is, when the financial frictions become smaller.

3.2. Consumption and bond holdings

In this section, I analyze the time-series behavior of consumption and bond purchases in the economy with two types and the behavior of these variables in the economy with a continuum of types. The consumption variable that I focus on is the percentage change in individual consumption, \( \Delta \ln C_t = \Delta \ln (c_t, A_t) \). Since bond purchases can take on negative values and values close to zero, it does not make sense to use the percentage change. To study the behavior of bond purchases, I, therefore, focus on the amount of bonds purchased relative to the per capita endowment, \( b_t = B_t/A_t \). This variable has the disadvantage that its unconditional covariance with any aggregate variable is by construction equal to zero because agents are ex ante identical. The statistics considered for the consumption variable are the standard deviation, the correlation with the individual endowment growth rate, the correlation with the aggregate consumption growth rate, and the correlation with the interest rate. The statistics considered for the bond purchases are the standard deviation, the first-order autocorrelation coefficient, the correlation with the idiosyncratic endowment shock, and the fraction of times the agents is at the constraint. The results are reported in Tables 2 and 3 for consumption and bond purchases, respectively.

The following observations can be made. First, in both types of economies, the correlation between individual consumption growth and income growth decreases and the correlation between individual and aggregate consumption growth increases when the borrowing constraint parameter increases. Results not reported here show that the amount of serial correlation in consumption growth reduces when the borrowing constraint parameter increases. These findings, thus, document that as financial frictions weaken agents are better able to smooth consumption. Second, the correlation of consumption growth with the interest rate in the economy with two types is much smaller than the correlation in the economy with a continuum of types. Given the much higher volatility of interest rates in the economy with two types, it is no surprise that the correlation coefficients differ across economies. Third, the qualitative change in the time-series statistics in response to changes in the parameter values is very similar. For example, the correlation between individual consumption growth and the interest rate increases in both types of economies with an increase in the borrowing constraint parameter, although the amount of correlation differs substantially in both types of economies.
### Table 2
Time-series behavior of individual consumption growth

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\delta$</th>
<th>SD</th>
<th>Correlation with $\Delta \ln (Y_t)$</th>
<th>Correlation with $a_t$</th>
<th>Correlation with $r_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$\infty$ types</td>
<td>2 types</td>
<td>$\infty$ types</td>
<td>2 types</td>
</tr>
<tr>
<td>0.2</td>
<td>1.0</td>
<td>0.140</td>
<td>0.135</td>
<td>0.888</td>
<td>0.924</td>
</tr>
<tr>
<td>1</td>
<td>2.0</td>
<td>0.070</td>
<td>0.065</td>
<td>0.724</td>
<td>0.783</td>
</tr>
<tr>
<td>2.0</td>
<td>0.2</td>
<td>0.051</td>
<td>0.045</td>
<td>0.646</td>
<td>0.687</td>
</tr>
<tr>
<td>3</td>
<td>1.0</td>
<td>0.131</td>
<td>0.122</td>
<td>0.891</td>
<td>0.927</td>
</tr>
<tr>
<td>2.0</td>
<td>0.071</td>
<td>0.047</td>
<td>0.713</td>
<td>0.736</td>
<td>0.490</td>
</tr>
</tbody>
</table>

*Note: This table reports the time-series statistics for the percentage change in individual consumption growth, $\Delta \ln C_t$. The variable $\Delta \ln Y_t$ denotes the percentage change in the individual endowment. $a_t$ is the growth rate of aggregate endowment which equals the growth rate of aggregate consumption. $r_t$ is the interest rate. $\gamma$ denotes the parameter of relative risk aversion and $\delta$ indicates the amount the agent is allowed to borrow relative to the per capita endowment. The other parameter values are reported in Section 2.3. Population moments are approximated using the sample moments of a simulated draw of 100,000 observations.*
Table 3
Time-series behavior of individual bond purchases

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\bar{b}$</th>
<th>SD</th>
<th>Autocorrelation</th>
<th>Correlation with $y_t$</th>
<th>Fraction at constraint</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$\infty$ types</td>
<td>2 types</td>
<td>$\infty$ types</td>
<td>2 types</td>
</tr>
<tr>
<td>0.2</td>
<td>1.0</td>
<td>2.0</td>
<td>0.2</td>
<td>0.198</td>
<td>0.166</td>
</tr>
<tr>
<td>1</td>
<td>0.775</td>
<td>0.636</td>
<td>0.967</td>
<td>0.954</td>
<td>0.967</td>
</tr>
<tr>
<td>2</td>
<td>1.349</td>
<td>1.142</td>
<td>0.986</td>
<td>0.981</td>
<td>0.986</td>
</tr>
<tr>
<td>0.2</td>
<td>0.190</td>
<td>0.166</td>
<td>0.831</td>
<td>0.781</td>
<td>0.831</td>
</tr>
<tr>
<td>3</td>
<td>0.747</td>
<td>0.660</td>
<td>0.966</td>
<td>0.956</td>
<td>0.966</td>
</tr>
<tr>
<td>2</td>
<td>1.319</td>
<td>1.192</td>
<td>0.986</td>
<td>0.983</td>
<td>0.986</td>
</tr>
</tbody>
</table>

*Note*: This table reports the time-series statistic for the bond purchases relative to the current per capita endowment, $b_t$. The variable $y_t$ denotes the agent's endowment relative to the per capita endowment. $\gamma$ denotes the parameter of relative risk aversion and $\bar{b}$ indicates the amount the agent is allowed to borrow relative to the per capita endowment. The other parameter values are reported in Section 2.3. Population moments are approximated using the sample moments of a simulated draw of 100,000 observations.
3.3. Insurance against idiosyncratic risk

A crucial feature of the economy with two types is that the interest rate drops when an agent receives the low realization for several periods because of the increase in cross-sectional dispersion. This works like a transfer from the rich agents (the lender) to the poor agents (the borrower) and suggests that agents in economies with only two types are better off than agents in economies with a continuum of types. There is another effect, however, that makes it harder to smooth consumption in economies with two types. It is harder to smooth consumption in an economy with two types because an agent will not lend to another agent of the same type and can never lend more than the agent of the other type is allowed to borrow. This will prevent him from building up a large buffer stock during good times. Agents in an economy with a continuum of types can lend to a wide variety of different agents and at times accumulate assets well in excess over the maximum amount that is possible in an economy with two types.

To document the quantitative importance of the interest rate effect, I plot in Fig. 3 the impulse response function of the interest rate when the same agent receives the low-income realization for several periods. The graph plots the interest rate for three levels of the borrowing constraint parameter and a
parameter of relative risk aversion equal to three. The agents in the economy with two types start out with zero bond holdings, and without loss of generality, I consider the case where the economy is in a recession. The graph also plots the interest rate in the economy with a continuum of types that, of course, does not respond to the realizations of an individual’s income shock. Consider the initial period in the two models. Since the initial levels of bond holdings are equal to zero in the economy with two types, there is little cross-sectional dispersion when the agent receives his first low-income realization. In the economy with a continuum of types there always is a certain amount of dispersion. Initially, therefore, the interest rate is lower in the economy with a continuum of types, since higher cross-sectional dispersion corresponds to lower interest rates. When the agent receives additional negative shocks, the amount of cross-sectional dispersion in the economy with a continuum of types is not affected but increases in the economy with two types. As documented in the graph, this causes the interest rate to drop. Quantitatively, these effects are enormous. When the agent reaches his constraint in the economy with two types, then the interest rate is equal to $-15\%$, $-3.7\%$, and $-1.8\%$, for values of the borrowing constraint parameter equal to 0.2, 1.0, and 2.0, respectively. These negative interest rates imply that the agent consumes more than his income level when he reaches the constraint. Not bad, to live in a world in which changes in the interest rate provide this kind of insurance!

The graph does not reveal the ability of an agent in the economy with a continuum of types to accumulate more assets during good times. The next two figures reveal both this effect and the interest rate effect discussed above. In particular, Figs. 4 and 5 plot for the economy with a continuum of types and the economy with two types a realization for consumption (relative to per capita income) and bond purchases (relative to per capita income), respectively. The parameter of risk aversion is equal to one and the borrowing constraint parameter is equal to one. During the first 30 periods, the agent is hit several times by negative shocks and is repeatedly at the borrowing constraint. As documented in the graph, the behavior of bond purchases is very similar in both economies during this period. In contrast, as documented in Fig. 4 the consumption level in the economy with two types is higher than the consumption level in the economy with a continuum of types during this period with frequent negative shocks because of reductions in the interest rate. The behavior of bonds is quite different across the two types of economies when the agent has received a series of high realizations. This happens around periods 50, 110, and 130. In the economy with two types this means that the agents of the other type have received a series of low realizations and have reached their borrowing

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\(^{17}\) It is assumed that the economy has been in a recession for several periods so that there are no more dynamic adjustments in the interest rate in the economy with a continuum of agents.
Fig. 4. A realization for individual consumption. This graph plots the consumption levels relative to the per capita endowment for the indicated economy. The parameter of relative risk aversion and the borrowing constraint parameter are equal to one. The values of the other parameters are reported in Section 2.3.

Fig. 5. A realization for individual savings. This graph plots the savings decision relative to the per capita endowment for the indicated economy. The parameter of relative risk aversion and the borrowing constraint parameter are equal to one. The values of the other parameters are reported in Section 2.3.

constraints. Consequently, equilibrium on the bond market limits the amount of savings that the agent with the positive shocks can accumulate. In the economy with a continuum of types, there is no such limitation and the agent accumulates assets well in excess of the borrowing constraint parameter. The graph for
Table 4
Welfare comparisons

A. Permanent percentage increase of consumption to become as well off as in complete markets economy

<table>
<thead>
<tr>
<th>$\bar{b}$</th>
<th>$\gamma = 1$</th>
<th>$\gamma = 3$</th>
<th>$\gamma = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\infty$ types</td>
<td>2 types</td>
<td>$\infty$ types</td>
</tr>
<tr>
<td>0.2</td>
<td>1.70</td>
<td>1.71</td>
<td>4.43</td>
</tr>
<tr>
<td>1.0</td>
<td>0.70</td>
<td>0.71</td>
<td>2.13</td>
</tr>
<tr>
<td>2.0</td>
<td>0.46</td>
<td>0.36</td>
<td>1.68</td>
</tr>
</tbody>
</table>

B. Permanent percentage increase of consumption to become as well off as in the economy with two types

<table>
<thead>
<tr>
<th>$\bar{b}$</th>
<th>$\gamma = 1$</th>
<th>$\gamma = 3$</th>
<th>$\gamma = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\infty$ types</td>
<td>2 types</td>
<td>$\infty$ types</td>
</tr>
<tr>
<td>0.2</td>
<td>$-0.01$</td>
<td>0.27</td>
<td>0.96</td>
</tr>
<tr>
<td>1.0</td>
<td>$-0.01$</td>
<td>0.15</td>
<td>0.49</td>
</tr>
<tr>
<td>2.0</td>
<td>0.07</td>
<td>0.02</td>
<td>$-0.45$</td>
</tr>
</tbody>
</table>

*Note: Panel A reports the permanent increase in consumption that is required to make an agent in the indicated economy as well off as an agent in the complete markets economy. Panel B reports the permanent change in consumption that is required to make an agent in the economy with a continuum of types as well off as an agent in the economy with two types. To calculate the (unconditional) expected discounted utility the average of the actual discounted utility levels across 50,000 replications of 250 observations is used. In each replication, the initial conditions are drawn from the ergodic distribution. $\gamma$ is the parameter of relative risk aversion and $\bar{b}$ indicates the amount an agent is allowed to borrow relative to the per capita endowment. The values of the other parameters are reported in Section 2.3.

consumption shows that the agent in the economy with a continuum of types is better able to smooth his consumption in the periods after which he has built up a large buffer stock of savings.

The question arises how these differences in the ability to insure against idiosyncratic shocks affect the agents’ welfare. In Table 4, I report the welfare differences between the different economies. To do this I calculate the (unconditional) expected discounted utility for the economies considered. The expected utility is calculated as the average of the actual utility across 50,000 replications of 250 observations. In each replication, the initial conditions are drawn from the ergodic distribution. Panel A of Table 4 compares the incomplete markets

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18 The discounted utility of consumption in periods 251 and higher is so small that ignoring it does not affect the results.
economies relative to the complete markets economies. In particular, it reports the permanent percentage increase in consumption that would make the agents in the incomplete markets economies as well off as the agents in the complete markets economy. The table documents that agents in the incomplete markets economies are substantially worse off at the lower levels of the borrowing constraint parameters, especially for the higher levels of risk aversion.

Panel B reports the permanent percentage change in consumption that would make an agent in the economy with a continuum of types as well off as an agent in the economy with two types. For most parameter values, this number is positive, which means that an agent is better off in an economy with only two different types of agents. In these cases, the interest rate effects are more important than the ability to accumulate high levels of assets. The results are quantitatively important for some parameter values. When the parameter of relative risk aversion is equal to five (three) and the borrowing constraint parameter is equal to 0.2, for example, then an agent in the economy with two types is willing to permanently reduce his consumption by 0.96 (0.27)% to avoid being in an economy with a continuum of types. An important exception is the case when the parameter of risk aversion is equal to five and the borrowing constraint parameter is equal to two. In this case, the agent is better off in the economy with a continuum of types.

One more interesting observation can be made about the economy with two types. As documented in Panel A, when the parameter of risk aversion is equal to five, then an increase in the borrowing constraint parameter from 0.2 to 1.0 makes the agent better off, but a further increase leads to a decrease in the agent’s expected utility. The reason is that for higher levels of risk-aversion, the interest rate effects are so helpful in smoothing consumption that being in an economy with a higher borrowing constraint parameter does not make you necessarily better off. To understand this result, I plot in Fig. 6 the impulse response function of consumption (relative to the per capita endowment) when the agent receives the low-income realization for several periods.

The graph makes clear how it can be possible that an agent’s expected utility is higher when the borrowing constraint parameter is equal to 1.0 than when the borrowing constraint parameter is equal to 0.2 or 2.0. Note that the agent’s consumption when \( b = 1.0 \) is never far below the maximum consumption level for the other two values of \( b \). But when \( b = 0.2 \) an agent’s consumption is considerably lower than the alternatives if he has been hit a few times by a low realization. Similarly, when \( b = 2.0 \) an agent’s consumption is considerably lower when he has been hit by a large string of low realizations.

Note that when the agent reaches his constraint in the economy with two types, he consumes more than his endowment level because the interest rate is

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19 This paper, thus, provides the first politically correct argument against diversity.
negative. One might think that the agent that is allowed to borrow less would benefit less from the negative interest rates. In fact, the opposite is true and the agents are consuming more when they face tighter borrowing constraints. The reason is that the interest rate drops to a much more negative value in economies with tight borrowing constraints.

4. Concluding comments

In this paper, I analyze the properties of an asset-pricing model with a continuum of types and the properties of the corresponding model with two types. Besides the number of types, the two models are exactly identical. In particular, the univariate time-series specifications of the individual and aggregate endowment are the same in the two models.

The following properties for the time-series behavior of interest rates are found.

- Average interest rates are remarkably similar for the parameters considered here. For some parameter values interest rates are on average somewhat lower in economies with two types. This is consistent with the finding that the interest rate is a concave function of the amount of cross-sectional dispersion and cross-sectional dispersion is more volatile in economies with two types.
Interest rates are much more volatile in economies with only two types because an important part of the fluctuations in interest rates is due to idiosyncratic shocks. Unlike economies with two types, economies with a continuum of types can have low real interest rates without having excessively volatile bond returns.

The different time-series behavior of interest rates causes the behavior of consumption and bond purchases to differ as well. The following differences are found.

- In an economy with two types, each agent can never invest more than the agents of the other types are allowed to borrow. In an economy with a continuum of types, equilibrium on the bond market does not create such a constraint. Consequently, in the economy with a continuum of types, agents accumulate during good times assets well in excess of the amount they are allowed to borrow. This makes it easier to smooth consumption.
- There is also a reason why it is easier to smooth consumption in the economy with two types. If, in an economy with two types, the same agent is hit by a series of negative ‘idiosyncratic’ shocks, the amount of cross-sectional dispersion increases which leads to a drop in the interest rate. The reduction in the interest rate works like a transfer from the rich agents (the lenders) to the poor agents (the borrowers). For most parameter values, this effect is more important and the agent’s expected utility is higher in economies with only two types.
- For higher values of the parameter of relative risk aversion these interest rate effects become so strong that the effect of financial frictions on agent’s welfare has a different sign in the two types of economies. In economies with a continuum of types, an agent’s welfare is always increasing when the maximum amount he is allowed to borrow increases. In the economy with two types, however, there are parameter values for which an agent’s welfare is decreasing when the borrowing constraint is relaxed.

Appendix A. The solution algorithm

In Den Haan (1997) it is shown in detail how the model in Section 2 with a continuum of agents is solved. In this section, a brief outline of the algorithm is given. Den Haan (1996) and Krusell and Smith (1998) deal with the infinite dimensional set of state variables by approximating the cross-sectional distribution with a finite set of moments. Den Haan (1997) follows the same approach but associates a density with these moments. This makes it possible to avoid the Monte Carlo integration techniques used in Den Haan (1996) and Krusell and Smith (1998). A notable paper that discusses solution techniques for models with
a large (but finite) number of state variables without reducing the dimension of state variables is Gaspar and Judd (1997).

In the model, the bond price at period $t$ is a function of $a_t$ and the cross-sectional distribution of bond holdings. A key feature of the proposed algorithm is to approximate the cross-sectional distribution of wealth and income holdings at period $t$ using an $(M \times 1)$ vector, $\phi_t$, containing moments of the cross-sectional distribution. To approximate the bond price function $I$, therefore, use the function $H(a_t, \phi_t; \delta^o)$, where $\delta^o$ is a vector of parameters and $H(\cdot)$ is chosen from a class of functions that can approximate any function arbitrarily well. Similarly, I use the function $\Phi(a_{t+1}, a_t, \phi_t; \delta^\phi)$ to approximate the transition law of $\phi_t$. Since $\phi_t$ is a vector, $\Phi(\cdot)$ is a vector-valued function.

Solving the individual problem requires approximating one more function. I approximate the conditional expectation, $E_t[\mathcal{C}_{t+1}^{i}]^{-\gamma}$, and denote the approximating function by $W(y_{it}, b_{it-1}, a_t, \phi_t; \delta^y)$. For a particular functional form for $H(\cdot)$, $\Phi(\cdot)$, and $W(\cdot)$, the algorithm solves for the parameter values $\delta^\phi$, $\delta^\theta$, and $\delta^y$ with the following iteration scheme:

1. Given parameter values for $\delta^\phi$ and $\delta^\theta$, solve the individual's problem, that is, obtain parameter values for $\delta^y$. The number of state variables for this problem is equal to six. Other than the somewhat high number of state variables, this is a straightforward numerical problem.

2. Given the decision rules for the individuals, solve the aggregate problem, that is, obtain parameter values for $\delta^\phi$ and $\delta^\theta$. Since the cross-sectional density is assumed to belong to a certain class of densities, knowing the moments is the same as knowing the density. At each point in the state space one can use numerical integration techniques and the individual policy functions derived in step 1 to calculate the equilibrium interest rate and the moments of next period’s cross-sectional distribution. A simple projection is used to calculate the values for $\delta^\phi$ and $\delta^\theta$. If these parameter values are ‘close’ to the ones used to solve the individuals problem in step 1, then the algorithm has converged. If not, one has to repeat steps 1 and 2.

In this paper, I use the first two moments of the bond holdings of the agents who receive the low-income shock and the first two moments of the bond holdings of the agents who receive the high-income shock to approximate the cross-sectional distributions. In this case $M$ equals 3 since the means of the two cross-sectional distributions add up to zero. Other choices of the numerical solution procedure are the same as those for EXP2Q500 described in Table 2 of Den Haan (1997).

The numerical procedure to solve the model with only two types is relatively straightforward and requires only obtaining a numerical approximation $\Psi^*(y_{it}, b_{it-1}, a_t; \delta^y)$ to $E_t[\mathcal{C}_{t+1}^{i}]^{-\gamma}$. Note that $\phi_t$ is not an argument of $\Psi^*(\cdot)$ because the variables $y_{it}, b_{it-1},$ and $a_t$ completely describe the state of the economy at period $t$. The parameter values for $\delta^y$ are obtained in a manner similar to the procedure used to solve $\delta^y$ in the economy with a continuum of
Table 5
Euler equation errors

<table>
<thead>
<tr>
<th>Risk aversion</th>
<th>Borrowing constraint</th>
<th>Maximum % error</th>
<th>Average % error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.2</td>
<td>0.16%</td>
<td>0.0096%</td>
</tr>
<tr>
<td>1</td>
<td>1.0</td>
<td>0.24%</td>
<td>0.0056%</td>
</tr>
<tr>
<td>1</td>
<td>2.0</td>
<td>0.60%</td>
<td>0.0085%</td>
</tr>
<tr>
<td>3</td>
<td>0.2</td>
<td>1.22%</td>
<td>0.0936%</td>
</tr>
<tr>
<td>3</td>
<td>1.0</td>
<td>1.96%</td>
<td>0.0692%</td>
</tr>
<tr>
<td>3</td>
<td>2.0</td>
<td>1.81%</td>
<td>0.0538%</td>
</tr>
<tr>
<td>5</td>
<td>0.2</td>
<td>2.17%</td>
<td>0.1469%</td>
</tr>
<tr>
<td>5</td>
<td>1.0</td>
<td>2.51%</td>
<td>0.0873%</td>
</tr>
<tr>
<td>5</td>
<td>2.0</td>
<td>4.19%</td>
<td>0.0629%</td>
</tr>
</tbody>
</table>

*Note:* This table reports for different parameter values the maximum and the average Euler equation errors calculated over a grid of 1000 beginning-of-period bond holdings for each of the four exogenous states.

types. The only difference is that now the bond price is solved simultaneously using a nonlinear equation solver by imposing that the demand for bonds of the two types of agents adds up to zero. As in Den Haan (1997) orthogonal Chebyshev polynomials are used for the approximating function. Here a 49th-order polynomial is used.

Accuracy of the solution procedure used to solve the model with a continuum of types is discussed in Den Haan (1997). Since solving the model with two types requires only obtaining an approximation to the conditional expectation one can use the Euler equation errors calculated over a fine grid to assess the accuracy of the solution procedure. These errors are reported in Table 5 and are calculated as follows.

1. Using the approximation to the conditional expectations for the two agents, I calculate the equilibrium bond price and the consumption level for agent 1.
2. I recalculate the conditional expectations for the two agents. To calculate the current-period savings decision, tomorrow’s bond price, and tomorrow’s consumption levels I use the obtained numerical approximation.
3. Using the recalculated conditions expectations I resolve the equilibrium bond price and the consumption level for agent 1.
4. I calculate the percentage (absolute) difference between the consumption levels obtained in steps 1 and 3.

Fig. 7 reports the two consumption levels for those parameter values and those realizations of the exogenous state variables for which the largest average error is observed. As indicated in the graph, the approximation is very good except in that area of the state space where one of the agents is close to the point where the constraint becomes binding. It is important to note that the smooth line is the
Fig. 7. The accuracy of the numerical solution for the consumption function. This graph plots the numerical solution of the consumption function and the consumption level that corresponds to the conditional expectation that is recalculated at the indicated level of beginning-of-period bond holdings. The parameter of relative risk aversion is equal to 5 and the borrowing constraint parameter is equal to 2.

one corresponding to the numerical solution. The one with tiny blips corresponds to the consumption level based on the conditional expectation that is recalculated at the indicated beginning-of-period bond holdings. Since these blips are unlikely to be part of the true rational expectations solution, the differences reported here are likely to overstate the errors of the numerical solution procedure.

It is interesting to note that for this value of the parameter of risk aversion the non-differentiability in the consumption function is very slight. The savings function has a strong non-differentiability because of the borrowing constraint and the consumption function would have one as well if interest rates would be constant. The reduction in the interest rate when the agent becomes constrained, however, reduces the drop in consumption and reduces the non-differentiability.

References


