Optimal pricing strategy for durable-goods monopoly

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Abstract

In this paper, we reconsider the profitability of a durable-good monopoly when the seller’s discount rate may be different from the buyers’. In an infinite-horizon continuous-time full-commitment model, the monopoly can achieve more than the static monopoly profit if and only if the seller is more patient than the buyers. Under this condition, the price function is strictly decreasing over time. These results remain valid in models with buyers arriving sequentially, where the seller may have only one unit to sell or can produce more at constant marginal cost. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

The issue of durable-goods monopoly has been studied extensively in the literature. The focus has been on whether and how the monopoly earns its static
monopoly profit in a dynamic model.\textsuperscript{1} There is a presumption that the static monopoly profit is the maximal profit a (static or dynamic) monopoly can obtain. This paper shows that this speculation is true only in some cases. The condition for its validity closely associates with the relative discount rates of the monopoly and the buyers.

In most of the literature, it is assumed that the discount rates for the buyers and the seller are exactly the same. Evidence indicates otherwise. One can argue that a firm has better access to low-interest-rate loans than the consumer, and therefore the firm’s discount rate is often lower than the consumer’s.\textsuperscript{2} While it is not the focus of this paper to examine how to measure the discount rates and who have the higher numbers, the effects of different discount rates still need to be investigated. The issue then becomes how much more the monopoly can earn in a dynamic model when its discount rate is different than its buyers?

In this paper, we answer this question in an infinite horizon model with a continuum of buyers where the monopoly has full commitment power. It is easy to see that the monopolist can earn at least the static monopoly profit by offering the monopoly price at all times. Is that the maximum profit the monopolist can earn? The answer is no. We find that when the monopolist is more patient than the buyers, which is usually the case, the monopolist can extract more profit from the buyers by offering a strictly declining pricing path. Even though delay is costly to the seller, it is even more costly to the buyers. A monopolist can exploit this property to discriminate among buyers with different willingnesses to pay. This result justifies the conventional wisdom: a dynamic monopoly ought to do ‘better’ than a static monopoly. If we consider the extreme case where the buyers are infinitely impatient, the seller can discriminate perfectly between each buyer and earn all the surplus.

The property that a dynamic monopoly can earn more than a static monopoly is first exhibited in Sobel and Takahashi (1983) in a discrete-time model for the class of demand functions: \( q = 1 - F(p) \), where \( F(p) = p \). When the seller and the buyers’ discount rates are equal, Gul (1987) and Ausubel and Deneckere (1987) show that the static monopoly profit is the maximum a dynamic monopoly can get. This paper generalizes these results to a general demand function and different discount rates in a continuous-time model. A simple necessary and

\textsuperscript{1} See representative work by Coase (1972), Gul et al. (1986), and Ausubel and Deneckere (1992a). One of the exceptions is Bagnoli et al. (1989), where perfect discrimination by the monopoly is facilitated in a dynamic model.

\textsuperscript{2} This is evidenced in the advertisement of some furniture stores: ‘Do not pay for one full year.’ It is argued in Sobel (1991) that the seller can make a profit by lending money to consumers in this case. Given the risks involved, other firms usually do not engage in this practice. In the case of furniture stores, it seems likely that furniture purchasers usually stay in the local area longer and have higher incomes and a lower risk of defaulting than other groups of consumers.
sufficient condition for higher profits is provided. This condition is proved to be quite robust. It remains valid in models with sequential stochastic arrival of new buyers, such as the one considered by Sobel (1991). It is also valid in a related bargaining model.\footnote{When there are (random) arrivals of new buyers, the bargaining problem and the monopoly problem are slightly different. In the bargaining problem, the arrival of a new buyer is known to the seller before the transaction price is finalized. In the monopoly problem the seller does not know when a new buyer arrives.}

The rest of the paper is organized as follows. In Section 2, the basic model is established and the major result regarding the seller’s profit is proved. In Sections 3 and 4, random arrivals of new buyers are considered, in a monopoly setting and in a bargaining setting respectively. In Section 5, we conclude.

2. A model of a durable-goods monopoly

Suppose that a monopoly seller has an infinitely durable good to sell to a pool of buyers. The monopolist’s marginal cost of production is $s$, which is common knowledge. At the beginning, there is a continuum of buyers, each infinitesimal but measuring one unit in total. A buyer demands at most one unit of the good. The valuations (or willingness to pay) of the population are characterized by c.d.f. $F(\cdot)$, forming a total demand function $1 - F(p)$. Assume that the support of $F(\cdot)$ is $[\underline{v}, \overline{v}]$ and that $f(v) = F'(v)$ is positive, continuous, and differentiable on $(\underline{v}, \overline{v})$.$\footnote{Distributions that are not well behaved are not discussed in this paper. But as we will see later, the qualitative results are not different.}$

This monopoly seller has full commitment power and at time 0 specifies a pricing path $p(t), t \in [0, +\infty)$. Given this path, the buyers decide when to purchase this infinitely durable good. If a buyer with willingness to pay $v$ buys at $t$ at the price $p(t)$, then the utility of that buyer is given by $e^{-\delta_b t} [v - p(t)]$, where $\delta_b$ is the buyer’s discount rate, and the profit for the seller from that transaction is given by $e^{-\delta_s t} [p(t) - s]$, where $\delta_s$ is the seller’s discount rate. We intend to investigate how the relative magnitudes of $\delta_b$ and $\delta_s$ affect the seller’s optimal pricing pattern and the corresponding profit.

We first examine the seller’s static monopoly profit, which can be achieved in this dynamic model by charging the monopoly price at all times. For a price $p$, any buyer with a willingness to pay more than $p$ buys and the seller earns a profit of $(p - s)[1 - F(p)]$. The first-order condition is given by

$$1 - F(p) - (p - s)f(p) = 0.$$ 

\footnote{When there are (random) arrivals of new buyers, the bargaining problem and the monopoly problem are slightly different. In the bargaining problem, the arrival of a new buyer is known to the seller before the transaction price is finalized. In the monopoly problem the seller does not know when a new buyer arrives.}

\footnote{Distributions that are not well behaved are not discussed in this paper. But as we will see later, the qualitative results are not different.}
In fact, the results in this section require only a weaker condition:

\[ -2f(p) - (p - s)f'(p) < 0, \quad \forall s \leq p \leq \bar{v}. \]  

Let \( p^m \) denote the optimal (monopoly) price and \( \pi^m = (p^m - s)[1 - F(p^m)] \) denote the static monopoly profit. By charging \( p^m \) all the time, the seller obtains this static monopoly profit.

We now consider a general \( p(t) \). We shall focus on the optimal path, the path that maximizes the seller’s profit. Specifically, we are interested in the possibility of earning more than \( \pi^m \) along some \( p(t) \).

Suppose that \( p(t) \) is weakly monotone decreasing and twice differentiable.\(^6\) Given that \( p'(t) \leq 0 \), a buyer with a willingness to pay \( v \) maximizes his utility \( e^{-\delta t}[v - p(t)] \) by choosing a \( t \) satisfying

\[ v = p(t) - \frac{1}{\delta_b}p'(t), \]  
given that the second-order condition for the maximization, \( \delta_b p' - p'' < 0 \), is satisfied by \( p(t) \). Note that (3) denotes the willingness to pay of a buyer who buys optimally at \( t \). Let \( \nu(t) \) denote that willingness to pay. From the second-order condition, we have \( \nu'(t) < 0 \); that is, a buyer with a higher willingness to pay will choose to buy the good earlier. Note that (3) is valid only for those buyers with a willingness to pay less than \( \nu(0) \). If a buyer is willing to pay more than \( \nu(0) \), he will buy the good right at \( t = 0 \).

The profit for the seller is given by

\[ \Pi(p(t)) = [p(0) - s][1 - F(\nu(0))] + \int_0^\infty e^{-\delta_b t}[p(t) - s]d[1 - F(\nu(t))], \]  

where \( d[1 - F(\nu(t))] \) gives the density function for \( t \), noting that \( \nu'(t) < 0 \). Integrating by parts, we have

\[ \Pi(p(t)) = \int_0^\infty e^{-\delta_b t}[1 - F(\nu(t))][\delta_b(p(t) - s) - p'(t)] dt. \]  

\(^5\)In fact, the results in this section require only a weaker condition: \( -2f(p) - (p - s)f'(p) < 0 \) for all solutions of (1). It implies that the profit function is concave at all of its stationary points and thus any local maximum is also a global maximum. This condition is equivalent to \( 3 + [(1 - F(p))/f(p)] > 0 \), which is satisfied by many widely used distributions, such as normal, uniform, and exponential distributions. The condition becomes slightly different in later sections.

\(^6\)Even though non-twice-differentiable functions should not be excluded from the analysis, every such function can be approximated arbitrarily close by twice-differentiable functions.
Let \( G(t, p(t), p'(t)) \) denote the integrand of (4). Recalling (3), maximizing (4) gives us the following Euler equation:

\[
G_p(t, p, p') = \frac{dG_p(t, p, p')}{dt},
\]

where

\[
G_p(t, p, p') = e^{-\delta t} \left( \delta_s [1 - F(v)] - f(v) [\delta_s (p - s) - p'] \right),
\]

and

\[
G_p'(t, p, p') = e^{-\delta t} \left\{ -[1 - F(v)] + \frac{1}{\delta_b} f(v) [\delta_s (p - s) - p'] \right\}.
\]

Straightforward calculations show that

\[
f(v) \left[ -2p'' + 2\delta_s p' + \delta_s (\delta_b - \delta_s) (p - s) \right] + f'(v) [\delta_s (p - s) - p']
\]

\[
\times \left( p' - \frac{1}{\delta_b} p'' \right) = 0.
\]

We have the following theorem.

**Theorem 1.** If \( \delta_s \geq \delta_b \), then the optimal pricing path for the seller is given by \( p(t) \equiv p^m \). If \( \delta_s < \delta_b \), then the optimal pricing path is characterized by a strictly decreasing \( p(t) \) which converges to \( s \), yielding a profit higher than \( \pi^m \).

**Proof.** First, we consider the case of \( \delta_s = \delta_b \). In this case, (6) becomes

\[
(-p'' + \delta_b p') [2\delta_b f(v) + f'(v) (\delta_b (p - s) - p')] = 0.
\]

Therefore, either \(-p'' + \delta_b p' = 0\), which implies that \(-p' + \delta_b p = \) constant, or \(2\delta_b f(v) + f'(v) (\delta_b (p - s) - p') = 0\), which together with \( \nu(t) = p - (1/\delta_b) p' \) implies that \( \nu(t) = \) constant; that is, again, \(-p' + \delta_b p = \) constant.

Solving the above first-order differential equation, we have \( p(t) = c_1 e^{\delta t} + c_2 \). Because \( p(t) \) is bounded by \( p(0) \) and \( s \) as \( t \) goes to infinity, we have \( p(t) \equiv c_2 \). Maximizing \( \Pi(p(t)) \) results in \( c_2 = p^m \); that is, \( p(t) \equiv p^m \) is the optimal path. \( c_2 \) can also be derived from the transversality condition \( G_p'(0, p(0), p'(0)) = 0 \).

Next, we consider the case of \( \delta_s > \delta_b \). Compared to the case of \( \delta_s = \delta_b \), an increase in \( \delta_s \) results in less profit for the seller, strictly less if some transactions

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\(^7\) See Kamien and Schwartz (1991) for a reference of the method of calculus of variations.
take place after \( t = 0 \) along the optimal pricing path. It is clear that \( \pi^m \), the maximal profit when \( \delta_s = \delta_b \), is the upper bound of the seller’s profit in the case of \( \delta_s > \delta_b \). Since \( p(t) \equiv p^m \) gives the seller a profit of \( \pi^m \) and all sales occur at \( t = 0 \), it is the (unique) optimal pricing path.

Finally, we consider the case of \( \delta_s < \delta_b \). In this case, the optimal pricing path is again determined by the solution to (6), together with the transversality condition \( G_p'(0, p(0), p'(0)) = 0 \). Replacing \( 2\delta_s p' \) by \( 2\delta_b p' + 2(\delta_s - \delta_b) p' \) in (6) and rewriting, we have

\[
p' - \frac{1}{\delta_b} p'' = \frac{f(v) (\delta_s - \delta_b) (\delta_s (p - s) - p')}{2\delta_b f(v) + f'(v) (\delta_s (p - s) - p')},
\]

(7)

Since \( \delta_s < \delta_b \), \( \delta_s (p - s) - p' \leq \delta_b (p - s) - p' = \delta_b (v - s) \). If \( f'(v) \geq 0 \), the denominator of (7) is positive. If \( f'(v) < 0 \),

\[
2\delta_b f(v) + f'(v) (\delta_s (p - s) - p') \geq 2\delta_b f(v) + f'(v) (\delta_b (p - s) - p')
\]

\[
= \delta_b [2f(v) + (v - s)f'(v)] > 0,
\]

where the last inequality is given by the assumption in (2). Therefore, \( v'(t) = p' - (1/\delta_b) p'' < 0 \). It is easy to see that \( p(t) \) is strictly decreasing. First of all, \( p'(t) \) cannot be positive. If \( p'(t) = 0 \), then \( p''(t) > 0 \), which in turn implies \( p'(t + dt) > 0 \), violating the condition of \( p'(t) \leq 0, \forall t \). Therefore, \( p'(t) < 0, \forall t \).

Let us now consider the seller’s maximal profit in this case. We concluded from the above arguments that the optimal path \( p(t) \) is characterized by (6), with the property that \( p'(t) < 0 \) and \( v'(t) < 0 \). Since \( p(t) \equiv p^m \) does not satisfy the Euler equation, the optimal path we obtained above must generate an expected profit higher than the static monopoly profit.

We can also prove that the optimal price must converge to \( s \) as \( t \) goes to infinity. Since \( p(t) \) is bounded by \( p(0) \) and \( p'(t) < 0 \), we have \( \lim_{t \to \infty} p'(t) = 0 \) and \( \lim_{t \to \infty} p''(t) = 0 \). \( p(t) \) must converge to some finite value; if it is not \( s \), (7) will not be satisfied. Therefore, \( p(t) \) must converge to \( s \).

This theorem characterizes the pricing paths that generate the highest profit for the seller under different circumstances. The main result is that if (and only if) the seller is more patient than the buyers, he can earn more than the static monopoly profit by offering a decreasing price path and discriminating buyers with different willingnesses to pay, since buyers with higher willingness to pay are more eager to buy early. One may wonder if there is any discontinuity in the seller’s profit at \( \delta_s = \delta_b \); that is, will a slightly lower seller’s discount rate generate a significantly higher profit? The following corollary states that the answer is negative.
Corollary 1. As $\delta_s$ converges to $\delta_b$, the optimal price path converges to $p(t) \equiv p^m$. 

Proof. Recalling (3), as $\delta_s$ converges to $\delta_b$, $\forall v \in [s, \bar{v}]$,

$$\frac{f(v)(\delta_s(p - s) - p')}{2\delta_b f(v) + f'(v)(\delta_s(p - s) - p')} \approx \frac{\delta_b vf(v)}{2\delta_b f(v) + vf'(v)},$$

which is bounded (say, by $M$) as the denominator is positive over the compact set $[s, \bar{v}]$ (cf. (2)). If $\delta_s \geq \delta_b$, $p(t) \equiv p^m$ is the optimal path. If $\delta_s < \delta_b$, from (7) and above,

$$\left| p' - \frac{1}{\delta_b} p'' \right| \leq |\delta_s - \delta_b| M.$$ 

Let $\delta_s \to \delta_b$. We conclude that $p'(t) - (1/\delta_b)p''$ converges to zero uniformly, which in turn implies that $p - (1/\delta_b)p' \to$ constant, $\forall t$. We can prove that in the limit $p'(t)$ must be zero. If $p'(t) = -\epsilon < 0$ for some $t$, $p(t)$ is decreasing. To keep $p - (1/\delta_b)p'$ constant, $p(t)$ becomes more and more negative as $t$ increases. Therefore, $p(t)$ itself soon becomes negative, which is impossible. Thus $p(t)$ becomes a constant as $\delta_s$ converges to $\delta_b$. From the transversality condition $G_p(0, p(0), p'(0)) = 0$ (cf. (5)), $v(0)$ converges to $p^m$ as $\delta_s \to \delta_b$, which means that $p(0)$ converges to $p^m$ (since $p'$ converges to zero). Therefore, the whole optimal pricing path converges to $p^m$. $\square$

It is not difficult to prove that the seller’s maximal profit increases in $\delta_b$ and decreases in $\delta_s$. This is because a seller with a lower discount rate simply has a larger present discounted profit for the same transaction path, while buyers with a higher discount rate will buy sooner. When the buyers become infinitely impatient, all transactions take place almost instantaneously.

Corollary 2. If $\delta_b$ goes to $+\infty$, the profit obtained by the optimal pricing path converges to $\int_s^\bar{v} (v - s)f(v)dv$, that is, the monopolist can perfectly discriminate among the buyers in the limit.

Proof. Consider the following pricing path (probably not exactly the optimal path): $p(t) = s + (\bar{v} - s)e^{-\sqrt{\delta_b}t}$. From (3), a buyer who buys optimally at $t$ (and at price $p(t)$) has a willingness to pay

$$v = p(t) - \frac{1}{\delta_b} p'(t) = s + \left(1 + \frac{1}{\sqrt{\delta_b}}\right)(\bar{v} - s)e^{-\sqrt{\delta_b}t}$$

$$= s + \left(1 + \frac{1}{\sqrt{\delta_b}}\right)(p(t) - s).$$
Let \( T(v) \) be the time at which a buyer with willingness to pay \( v \) buys and \( P(v) \) be the corresponding transaction price. We have

\[
T(v) = \frac{1}{\sqrt{\delta_b}} \ln \left( \frac{1 + (1/\sqrt{\delta_b})(\bar{v} - s)}{v - s} \right),
\]

and

\[
P(v) = s + \frac{v - s}{1 + (1/\sqrt{\delta_b})}.
\]

The seller’s total present discounted profit is then given by

\[
\int_s^v e^{-\delta_b T(v)}(P(v) - s)f(v) \, dv.
\]

As \( \delta_b \to \infty \), \( T(v) \to 0 \) and \( P(v) \to v \). As a result, in the limit of this sequence of pricing paths, the seller obtains the maximal profit he can get: the perfectly discriminatory profit. Therefore, this profit must also be the optimal profit as \( \delta_b \to \infty \). \( \square \)

The following example illustrates the way to calculate the seller’s optimal pricing path and the corresponding optimal profit when \( \delta_s < \delta_b \).

**Example.** Suppose that the seller’s valuation \( s = 0 \) and the buyer’s valuation is distributed uniformly on \([0, 1]\), so that \( f(v) = 1, v \in [0, 1] \). Let \( \delta_s = 1 \) and \( \delta_b = 5 \). From (6), we have

\[
-2p'' + 2p' + 4p = 0.
\]

Therefore, \( p(t) = c_1 e^{-t} + c_2 e^{2t} \). Since \( p(t) \) must be bounded by \( p(0) \) and 0, we have \( c_2 = 0 \), and

\[
\Pi(p(t)) = \int_0^\infty e^{-t}[1 - v(t)][p(t) - p'(t)] \, dt,
\]

where

\[
v(t) = p(t) - \frac{1}{2}p'(t) = \frac{5}{8}c_1 e^{-t}.
\]

Therefore,

\[
\Pi(p(t)) = \int_0^\infty e^{-t}\left[1 - \frac{5}{8}c_1 e^{-t}\right]\left[c_1 e^{-t} + c_1 e^{-t}\right] \, dt = c_1 - \frac{49}{64}c_1^2.
\]

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8 Another way to get the value for \( c_1 \) is through the transversality condition \( G_p(0, p(0), p'(0)) = 0 \). Similarly, we obtain \( c_1 = 5/8 \).
Maximizing $\Pi(p(t))$ results in $c_1 = \frac{5}{8}$, and $\Pi(p(t)) = \frac{5}{2}$. Thus, the static monopoly profit $\frac{1}{2}$ is less than the profit obtained by this optimal pricing path $p(t) = \frac{5}{8}e^{-t}$.

3. Stochastic arrivals of new buyers

In Section 2, we considered a pool of buyers which are fixed at $t = 0$. Suppose now that we modify the model to allow for new buyers to arrive from time to time during the selling process. We intend to examine the robustness of the results in the previous section; that is, whether the possibility of arriving buyers affects the validity of the qualitative results in Theorem 1.

The model we shall consider is similar to the one in Sobel (1991), except that we examine continuous-time price functions, that the buyer’s arrival is random, and that the discount rates for the seller and the buyers may be different. In order to accommodate the arrival of new buyers, we need to modify the basic model in Section 2. In that model, we considered a continuum of buyers with different willingnesses to pay. The analysis there would be the same if we were to consider just one buyer, but with private willingness to pay drawn from a distribution characterized by c.d.f. $F(\cdot)$. In this section, we adopt the latter definition and assume that each arriving buyer has a private willingness to pay that is drawn independently according to c.d.f. $F(\cdot)$. (If we adopted the former definition, the analysis below would be similar given that new cohorts of buyers, instead of new individual buyers, would arrive randomly.)

Suppose that new buyers arrive according to the Poisson stochastic process with rate of arrival $\lambda$. The exact time of arrival, however, is not known to the seller. The monopoly can produce more output at the constant marginal cost $s$ to meet any demand. Let $p(t)$ be the price charged by the monopoly. As each arrival is history independent, past sales provide no information on either the residual demand in the market or future demand. Therefore, only time and past prices are relevant to the monopolist’s profit.

Suppose that a buyer arrives at $t_0$. Given price path $p(t)$, the profit associated with that buyer from $t_0$ is given by

$$\hat{\pi}(t_0) = e^{-\delta t_0}[p(t_0) - s][1 - F(v(t_0))] + \int_{t_0}^{\infty} e^{-\delta t}[p(t) - s][1 - F(v(t))] dt$$

$$= \int_{t_0}^{\infty} e^{-\delta t}[1 - F(v(t))][\delta_s(p(t) - s) - p'(t)] dt,$$

where, again, $v(t) = p(t) - (1/\delta_s)p'(t)$.

Between $t_0$ and $t_0 + dt_0$, the probability that a new buyer arrives is $\lambda dt_0$. (The probability that more than one buyer arrives is negligible.) To make the
model compatible with the models in previous sections, suppose that there is already one buyer waiting at \( t = 0 \). (Replacing \((1 + \lambda t)\) by \( \lambda t \) in the rest of this section, we will get the analysis for the model without the initial buyer). The total profit for the monopolist is then given by

\[
\Pi(p(t)) = \hat{\pi}(0) + \int_0^\infty \hat{\pi}(t_0) \lambda \, dt_0.
\]  

(8)

Changing the order of integration in the above implicit double integral, we have

\[
\Pi(p(t)) = \int_0^\infty (1 + \lambda t)e^{-\lambda t}[1 - F(v(t))][\delta_s(p(t) - s) - p'(t)] \, dt.
\]

Let \( \hat{G}(t, p(t), p'(t)) \) denote the integrand of the above equation. The optimal price path \( p(t) \) must satisfy the following Euler equation:

\[
\frac{d\hat{G}_p(t, p(t), p'(t))}{dt} = \hat{G}_p(t, p(t), p'(t)) = \frac{d\hat{G}_p(t, p(t), p'(t))}{dt},
\]

where

\[
\hat{G}_p(t, p, p') = (1 + \lambda t)e^{-\lambda t}\{(\delta_s[1 - F(v)] - f(v)[\delta_s(p - s) - p'])
\]

and

\[
\hat{G}_p(t, p, p') = (1 + \lambda t)e^{-\lambda t}\{-[1 - F(v)] + \frac{1}{\delta_b} f(v)[\delta_s(p - s) - p']\}.
\]  

(9)

Straightforward calculations show that

\[
f(v)[-2p'' + 2\delta_s p' + \delta_s(\delta_b - \delta_s)(p - s)] + f'(v)[\delta_s(p - s) - p']
\]

\[
\times \left( p' - \frac{1}{\delta_b} p'' \right) + \frac{\lambda}{1 + \lambda t} \left[ f(v)[\delta_s(p - s) - p'] - \delta_b[1 - F(v)] \right] = 0.
\]

(10)

Let \( \hat{\pi}^m \) denote the optimal conventional monopoly profit for the seller generated by charging a single optimal price \( \hat{p}^m \) forever. It is easy to show that \( \hat{p}^m = p^m \), the static monopoly price, and \( \hat{\pi}^m = (1 + (\lambda/\delta_s))\pi^m \), where \( \pi^m \) is the static monopoly profit. This is because \( p^m \) maximizes each \( \hat{\pi}(t_0) \) in (8). We have the following theorem.

**Theorem 2.** If \( \delta_s \geq \delta_b \), then optimal pricing path for the seller is given by \( p(t) \equiv p^m \). If \( \delta_s < \delta_b \), then the optimal pricing path is characterized by a weakly decreasing \( p(t) \) which converges to \( s \), yielding a profit higher than \( \hat{\pi}^m \).
Proof. We first consider the case of $\delta_s = \delta_b$. In this case, the Euler equation (10) can be rewritten as

$$(1 + \lambda t)v'(t)\pi''(v(t)) + \lambda \pi'(v(t)) = 0,$$  

(11) implies that

$$(1 + \lambda t)v'(v) = \text{constant}.$$  

(12)

Recalling (9), the transversality condition for this maximization $\hat{G}_p(0, p(0), p'(0)) = 0$ implies $\pi'(v(0)) = 0$. Therefore, the constant in (12) is zero and $\pi'(v) = 0$. Thus, $v(t) \equiv p^m$. Since $p(t)$ must be bounded, $v(t) = p(t) - (1/\delta_b)p'(t) = p^m$ implies $p(t) \equiv p^m$, which is therefore the optimal path.

We now consider the case of $\delta_s > \delta_b$. An argument similar to that in the proof of Theorem 1 applies here. Since $p(t) \equiv p^m$ maximizes each $\tilde{\pi}(t)$, it must also maximize $H(p(t))$. Therefore, $p(t) \equiv p^m$ is also the optimal path in this case.

Finally, we consider the case of $\delta_s < \delta_b$. Rewriting (10), we obtain a formula similar to (7):

$$p' - \frac{1}{\delta_b}p'' = \frac{f(v)(\delta_s - \delta_b)(\delta_s(p - s) - p') + (\lambda/(1 + \lambda t))\{\delta_b[1 - F(v)] - f(v)[\delta_s(p - s) - p']\}}{2\delta_b f(v) + f'(v)[\delta_s(p - s) - p']}.$$  

Similarly, we can argue that the denominator is positive. From the transversality condition $\hat{G}_p(0, p(0), p'(0)) = 0$, we have

$$1 - F(v) - \frac{1}{\delta_b}f(v)[\delta_s(p - s) - p'] = 0 \quad \text{at} \quad t = 0.$$  

Therefore, at $t = 0$, the numerator is negative and thus $p(t)$ is decreasing. However, as $p(t)$ decreases, $v(t)$ decreases, implying that the second term in the numerator becomes positive. It is possible that at some $t$, the numerator becomes positive and the restriction of $p' - (1/\delta_b)p'' < 0$ is violated. In this case, $p(t)$ becomes constant. Note that $p(t)$ will not increase, because an increase in $p(t)$ implies an increase in $v(t)$ as well. Therefore, an initially decreasing and then increasing price path implies first a negative, then a zero and then a negative numerator, which yields a contradiction to what the sign of $p' - (1/\delta_b)p''$ indicates. As $t$ increases, $\lambda/(1 + \lambda t)$ converges to zero, and the effect of the second term becomes diminishing, and the price declines again. Because $p'(t)$ and $p''(t)$ go to zero as $t$ increases, $p(t)$ converges to $s$ (from (10)). Therefore, the optimal price initially declines, with possibly a period of constant price, declines again, and converges to $s$ as $t$ increases. Since $p(t) \equiv p^m$ cannot be the optimal path, $\tilde{p}^m$ is not the highest profit the seller can earn when $\delta_s < \delta_b$. \qed
The results in this theorem are quite intuitive. When $\delta_s < \delta_b$, since there is one buyer at the beginning of the selling, the optimal pricing calls for a decreasing price path (cf. Theorem 1). As $t$ increases, more and more buyers arrive, lowering the price further reduces the profits generated from new buyers. An increase in price is obviously not optimal, since the stock of the remaining buyers is higher. Therefore, the price may be held constant for a period of time. As $t$ becomes large, the effect of existing residual buyers becomes dominant, and the optimal pricing calls for a further reduction in prices. Eventually, the price converges to $s$ (as in Theorem 1) and all buyers with a willingness to pay more than $s$ will buy a unit of the good.

4. Bargaining with sequentially arriving buyers

As pointed out by many authors, the problem of a durable-goods monopoly selling the good to a continuum of consumers is mathematically equivalent to the problem of sequential bargaining between a seller (of a single object) who makes all the offers and a single buyer with uncertain valuation. With the arrivals of new buyers, however, the equivalence disappears. When new buyers arrive, the monopoly can produce more output at a constant marginal cost to satisfy their demand, as in the last section. Meanwhile, the seller of a single good cannot provide more units when a new buyer shows up.

In this section, we consider yet another variation of the basic model. Suppose that the seller can only bargain with one buyer at a time. Can the seller now earn more than the conventional monopoly profits even when $\delta_s > \delta_b$? Suppose that buyers arrive according to a Poisson process with rate of arrival $\lambda$ and that the seller can only negotiate with one buyer at a time. We should note the informational difference between a monopoly that produces output at a constant marginal cost $s$ and a seller who values a single object at $s$ and negotiates with potential buyers. In the former case, the monopolist does not know instantly when a new buyer arrives, while in the latter case he does.

As in Section 2, we restrict our attention to a weakly decreasing path of prices offered to one particular buyer by the seller. A buyer with high valuation will buy early. If during the price reduction process a new buyer arrives, it is obvious that the seller will drop the current buyer and negotiate with the new buyer.

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9This kind of model is used to study bargaining problems involving uncertainty, as Rubinstein’s (1982) alternative-offer model becomes too complicated to analyze. See Cramton (1984), for example. As demonstrated by Ausubel and Deneckere (1992b), however, if only one party possesses private information, that party will remain silent during the bargaining process and let the other party make all the offers in an alternative-offer model.
since the existing buyer has signaled his low valuation by waiting for a lower price. With the assumption of Poisson arrivals, the price path for each new buyer is exactly the same. (See Fig. 1.)

Given that the seller’s pricing path is $p(t)$ until a new buyer arrives, a buyer with valuation $v$ maximizes the following expected utility:

$$\max e^{-\delta_b t}(v - p(t))e^{-\lambda t},$$

where $e^{-\lambda t}$ is the probability that no buyers arrive by $t$. Therefore, the optimal time to buy is determined by

$$v = p(t) - \frac{1}{\delta_b + \lambda} p'(t).$$

Since now the real discount factor for the buyer has increased to $\delta_b + \lambda$, we wonder if the seller can make more than the conventional profit when $\delta_b < \delta_s < \delta_b + \lambda$. To calculate the profit for the seller, assume that the maximum present discounted profit is $P^*$ when the seller meets with a new buyer. Since arrivals are independent, the profit for the seller adopting pricing path $p(t)$ is given by

$$P(p(t)) = [p(0) - s][1 - F(v(0))] + \int_0^T \int_0^T e^{-\delta_b t'}[p(t) - s]d[1 - F(v(t))]$$

$$+ e^{-\delta_b T}\int_T^\infty P^*d[1 - F(v(t))],$$
noting that the maximum of \( \Pi(p(t)) \) is equal to \( \Pi^* \). By changing the order of integration, the above equation can be rewritten as

\[
\Pi(p(t)) = \left[ p(0) - s \right][1 - F(v(0))] + \int_0^\infty e^{-(\delta_s + \lambda) t} [p(t) - s] d[1 - F(v(t))]
\]

\[
+ \int_0^\infty \frac{\lambda \Pi^*}{\delta_s + \lambda} [1 - e^{-(\delta_s + \lambda) t}] d[1 - F(v(t))].
\]

Integrating the above by parts, we have

\[
\Pi(p(t)) = \int_0^\infty e^{-(\delta_s + \lambda) t} [1 - F(v(t))] [(\delta_s + \lambda)(p(t) - s) - p'(t)] dt
\]

\[
+ \int_0^\infty e^{-(\delta_s + \lambda) t} \Pi^* F(v(t)) \lambda e^{-\lambda t} dt
\]

\[
= \int_0^\infty e^{-(\delta_s + \lambda) t} [1 - F(v(t))] [(\delta_s + \lambda)(p(t) - s) - p'(t)]
\]

\[
+ \lambda \Pi^* F(v(t)) dt.
\]

Let \( \tilde{G}(t, p(t), p'(t)) \) denote the integrand of the above integral. From the Euler equation \( \tilde{G}_p(t, p, p') = (d \tilde{G}_p(t, p, p'))/dt \), we have

\[
f(v)\left[ -2p'' + 2(\delta_s + \lambda)p' + (\delta_b - \delta_s)[(\delta_s + \lambda)(p - s) - \lambda \Pi^*] \right]
\]

\[
+ f'(v)[(\delta_s + \lambda)(p - s) - \lambda \Pi^*]\left(p' - \frac{1}{\delta_s + \lambda} p''\right) = 0.
\]

Since \( \Pi^* \) is what the seller expects when a new buyer arrives, the seller would never set a price below his expected revenue from rejecting the current buyer; that is,

\[
p(t) \geq s + \Pi^* \int_0^\infty e^{-\delta_s t} e^{-\lambda t} dt = s + \frac{\lambda}{\delta_s + \lambda} \Pi^*.
\]

Hence, \( (\delta_s + \lambda)(p - s) - \lambda \Pi^* \) is always nonnegative.

We shall investigate under what circumstance the seller can obtain more than the conventional profit. To calculate the conventional monopoly profit, let \( p(t) \equiv p \), \( v(t) = p \), and \( \Pi^* = \Pi(p(t)) = \Pi(p) \) in (13) and solve for \( \Pi(p) \). We have

\[
\Pi(p) = \frac{p(1 - F(p))}{1 - (\lambda/\delta_s + \lambda)F(p)}.
\]

Let \( \bar{p}^m \) maximize \( \Pi(p) \) and \( \bar{p}^m = \Pi(\bar{p}^m) \) be the conventional monopoly profit. We have the following theorem, the proof of which is similar to that of Theorem 1.
Theorem 3. If $\delta_s \geq \delta_b$, then the optimal pricing path for the seller is given by $p(t) = \hat{p}^m$. If $\delta_s < \delta_b$, then the optimal pricing path is characterized by a strictly decreasing $p(t)$ which converges to $s + (\lambda \tilde{\pi}^m) / (\delta_s + \lambda)$, yielding a profit higher than $\pi^m$.

In the presence of new arrivals, a buyer faces the risk of being dropped from the current negotiation. Therefore, the discount rate for a buyer increases to $\delta_b + \lambda$ from $\delta_b$. This increase, however, is not enough to make the seller more patient than the buyer. As the seller also faces the risk of losing the current buyer, his discount rate also increases to $\delta_s + \lambda$. Therefore, the arrival of new buyers does not change the relative discount rates between the seller and the buyer and hence the results of Theorem 1 continue to hold. Note that Bagnoli, Salant and Swierzbinski’s (1989) techniques do not apply to this model (nor to any other bargaining models), as there is only one buyer and one object.

5. Conclusions

In this paper, we obtain an important result in a very general setting. When a durable-goods monopoly has full commitment power, it can earn more than the static monopoly profit by decreasing its price over time if it is more patient than the buyers. This justifies the common observation that the prices for most durable goods are indeed declining over time. Of course, there could be other explanations for this phenomenon, such as decreasing marginal cost of production because of learning by doing or technological advances. This paper assesses that even if the cost of production is constant, the prices are expected to decline, as long as the monopoly is more patient than the buyers. This result seems to be very robust, as it remains valid in a variety of models studied in the paper.

References


