Non-steady-state equilibrium solution of a class of dynamic models

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Abstract

We study a class of monetary and growth models by using both analytic and numerical tools. In particular, we consider a model that was studied by Rotemberg, (Journal of Political Economy, 92 (1984) 40–58) and provide a solution to the open problem concerning the existence of the non-steady-state equilibrium of the model. We investigate the stable manifold solution to the underlying dynamical system and then use it to generate the equilibrium path. We use a fixed point iteration to numerically evaluate the stable manifold solution and eventually discover the exact solution. This investigation gives another illustration of the potential power of the general approach developed by the author that combines mathematical analysis and numerical simulations. © 2001 Elsevier Science B.V. All rights reserved.

1. Introduction

In this paper, we shall present a mathematical method and some numerical algorithm for computing equilibria of a class of dynamic models. In particular, we are interested in monetary and growth models. The equilibria of many of such models are often described by nonlinear difference equations of second or higher order. One interesting phenomenon is that only one boundary condition (plus another condition at infinity) can be imposed for these equations.

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Examples of such models include Baumol (1952), Grossman (1982), Grossman and Weiss (1983), Grossman (1985), Rotemberg (1984), Stockman (1980), Polemarchakis (1990) and Jose-Victor and Rios-Rull (1996). The usual treatments in the literature for this type of problems are: consider the steady-state equilibrium only, or linearize the nonlinear difference equation around the steady-state and find a local solution, or use the shooting method to find the numerical equilibrium.

For a dynamic model, however, a steady-state equilibrium does not provide complete information about the considered economy, especially when we are interested in the effects of such non-steady-state events as open-market operations. As for the local linearization method, we know that sometimes the higher-order terms matter so the local result cannot be extended. The shooting algorithm, on the other hand, has at least two major problems. First, there is little theoretical basis for the proposed terminal values and the model can easily get polluted by parameters that are artificially introduced. Second, the computational effort associated with the artificial two-point boundary-value approach grows super-linearly with the distance of the terminating value from the initial value if one is trying to find the dynamics for an interval of initial conditions.

In this paper, we shall propose a numerical method that is based on our new mathematical understanding of the model. There are two important features of the proposed method. First, it provides the global closed-form solution of non-steady-state equilibrium sequences for many existing models. Second, the numerical approach has strong theoretical background and very efficient. We shall describe our method using a particular model proposed by Rotemberg (1984).

The rest of the paper is organized as follows. Section 2 briefly presents the model and algorithm which we will apply for our method. Section 3 presents the perfect foresight equilibrium of the economy, where the equilibrium sequence was governed by a third difference equation with only one initial value. Section 4 proposes the mathematical method and the numerical algorithm and establishes the existence of the dynamic equilibrium sequence. Section 5 gives some concluding remarks.

2. A special employed model

As mentioned earlier, we shall demonstrate our new approach by studying a particular model proposed by Rotemberg (1984). For clarity, we shall now give a brief description of this model. For details, the interested readers are referred to the original paper.

There are three actors in this model: firms, consumers and government. Competitive firms in period $t$ produce only one good that can be both consumed and invested. The total output $Q_t$ depends on the amount of labor $L$ hired at
the amount of good that was produced but not consumed at previous period. Such a dependence is described by constant-returns-to-scale production function $f$ as follows:

$$Q_t = L(f(K_{t-1}/L)).$$

We assume that the labor is supplied inelastically $L = \bar{L}$, $f' > 0$ and $f'' < 0$. To maximize the profit, the workers are assumed to be paid their marginal product. Therefore, the total return, denominated in period $t$ goods, from forgoing the consumption of an additional good at $t - 1$ is given by $1 + r_{t-1}$, where

$$1 + r_{t-1} = \bar{L} f' \left( \frac{K_{t-1}}{L} \right).$$

(1)

Consumers pick the path of consumption optimally. Because of the transactions cost, consumers visit their financial intermediary only occasionally. So, we assume that there are two types of consumers, $n$ of them in each type and total $2n$ of them. Consumer type ‘a’ engages in financial transactions in the even periods, $t, t + 2, t + 4, \ldots$, type ‘b’ engages in financial transactions in the odd periods, $t + 1, t + 3, t + 5, \ldots$, and consumer $i$ withdraws an amount $M^i_\tau$ of money to support the consumption at period $\tau$ and $\tau + 1$. Consumer $i$ faces the optimal problem:

$$\max \sum_{\tau=t}^{\infty} \beta^{\tau-t} \ln C^i_\tau$$

s.t. $M^i_\tau = P_\tau C^i_\tau + P_{\tau+1} C^i_{\tau+1}$ for $\tau = t, t + 1, \ldots$

(2)

where $0 < \beta < 1$ is the discount factor, $C^i_\tau$ is the consumption of consumer $i$ at time $\tau$ and $P_\tau$ is the nominal price of the consumption and investment good at $\tau$. The consumers have access to two assets, money and claims on capital. But money is the only asset with which goods can be bought.

The lifetime budget constraint of the consumer at $t$ is given in terms of the consumers monetary withdrawals:

$$\sum_{\tau=0}^{\infty} \frac{M^i_{t+2\tau}/P_{t+2\tau}}{\prod_{\tau=0}^{t-1}(1 + r_{t+i})} = \sum_{\tau=0}^{\infty} \frac{Y^i_{t+\tau}}{\prod_{\tau=0}^{t-1}(1 + r_{t+i})} + K^i_{t-1}(1 + r_{t-1}) - \sum_{\tau=0}^{\infty} \frac{B}{\prod_{\tau=0}^{t-1}(1 + r_{t+i})}$$

(3)

where $K^i_\tau$ are the claims on capital of consumer $i$ at $t$, $B$ is the real cost of visiting the financial intermediary, and $Y^i_\tau$ is the non-investment income of the consumer $i$ at time $t$. 
To find the optimal path of consumption, one usually considers the auxiliary optimization problem:

$$\max \ln C_t^i + \beta \ln C_{t+1}^i$$

subject to $M_t^i = P_t C_t^i + P_{t+1} C_{t+1}^i$. This yields

$$C_{t+1}^i = \frac{\beta P_t}{P_{t+1}} C_t^i = \frac{\beta M_t^i}{1 + \beta P_{t+1}}.$$

Using the just established relation above, we then have

$$\ln C_t^i + \beta \ln C_{t+1}^i = (1 + \beta) \ln \left(\frac{M_t^i}{P_t}\right) + \beta \ln \beta - (1 + \beta) \ln(1 + \beta) - \beta \ln \frac{P_{t+1}}{P_t}.$$

(4)

Substituting (4) into (2), we obtain

$$\max \sum_{k=0}^{\infty} \beta^{2k} \left( (1 + \beta) \ln \left(\frac{M_{t+k}^i}{P_{t+k}}\right) + \beta \ln \beta - (1 + \beta) \ln(1 + \beta) - \beta \ln \frac{P_{t+k+1}}{P_{t+k}} \right)$$

subject to the lifetime budget constraint (3). This maximization yields

$$\frac{C_{t+2k+2}^i}{C_{t+2k}^i} = \beta^2(1 + r_{t+2k})(1 + r_{t+2k+1}).$$

The above relation states the rate of return on capital.

The government in this model does not have expenditures. It only levies taxes, issues money and holds capital. The evolution of the capital held by the government is

$$K_{t+1}^g = f(K_t/L)K_t^g + \frac{M_{t+1} - M_t}{P_{t+1}} + T_{t+1}$$

where $T_{t+1}$ is the real taxes levied at time $t + 1$, $K_t^g$ is the government’s real holding of capital at $t$. An increase in money supply means $M_{t+1} > M_t$.

The equilibrium for the economy is a path of $\{P_t, r_t\}$ such that consumers maximize utility and firms maximize profits using these price sequence $P_t$ and satisfies the goods market and money market clearing conditions:

$$C_t + K_t = L f \left(\frac{K_{t-1}}{L}\right)$$

(5)

and

$$nP_t C_t^{i-1} = M_{t-1},$$

(6)

$$nP_t C_t^i = M_t.$$  

(7)
By (6) and (7) we then obtain

\[ C_\tau = n \left( 1 + \frac{\beta M_{\tau-1}}{M_\tau} \right) C^*_\tau, \]  

(8)

where \( C_\tau \) is the total consumption at \( \tau \) and \( C^*_\tau \) is the consumption at \( \tau \) of consumers that visit the financial intermediary at \( \tau \) and \( K^*_\tau + K^*_\tau = K_\tau \).

Using (8) and (5), and the transversality condition \( \lim_{T \to \infty} (\beta^T K_T^T)/(M_T^T/P_T) = 0 \). We obtain the nonlinear third-order difference equation that governs the equilibrium level of capital sequence:

\[
\begin{align*}
\ell f\left( \frac{K_{\tau+2}}{L} \right) - K_{\tau+3} &= \beta^2 \frac{1 + \beta (M_{\tau+2})/(M_{\tau+3})}{1 + \beta (M_\tau)/(M_{\tau+1})} f\left( \frac{K_{\tau+1}}{L} \right) f'\left( \frac{K_{\tau+2}}{L} \right) \left( \ell f\left( \frac{K_{\tau}}{L} \right) - K_{\tau+1} \right),
\end{align*}
\]

where \( \tau = t - 1, t, t + 1, \ldots \). It is clear the sequence of capitals provides sequence of rates of return by (1), the total consumptions by (5), the individual consumption by (8) and the equilibrium price sequence by (7).

First, we consider the existence of the equilibrium sequence that converges to a steady state with positive consumption. In this case, \( M_t/M_{t+1} \) converges to a constant, so the steady state values of capital \( K \) then satisfies the following relation:

\[
\left[ \ell f\left( \frac{K}{L} \right) - K \right] (1 - \beta^2 \left( f'\left( \frac{K}{L} \right) \right)^2) = 0.
\]

It follows that the unique steady state with positive consumption satisfies:

\[
f'\left( \frac{K}{L} \right) = \frac{1}{\beta^2}.
\]

(9)

Now, we shall consider some specific production function, namely the Cobb–Douglas production \( f(K_t) = K_t^x \) where \( 0 < x < 1 \). Without loss of generality, we assume that the labor supply constant \( L = 1 \). For this type of production function the equilibrium level of capital sequence satisfies:

\[
K_{t+2}^x - K_{t+3} = x^2 \beta^2 \frac{1 + \beta (M_{t+2})/(M_{t+3})}{1 + \beta M_t/(M_{t+1})} K_{t+1}^{x-1} K_{t+2}^{x-1} (K_t^x - K_{t+1}).
\]

(10)

In virtue of (9), for the Cobb–Douglas production function, the steady-state equilibrium is given by

\[
K = (x\beta)^{1/(1-x)}.
\]

(11)
Let us now begin to consider the non-steady-state equilibrium. We assume that no money stock increase will occur after time period \( t_0 \geq 1 \), namely \( M_{t+1} = M_t \) for \( t \geq t_0 \). In this case, the finite difference equation is reduced to

\[
K_{t+2} - K_{t+3} = \frac{\beta^2 (M_{t+2})/(M_{t+3})}{1 + \beta M_t/(M_{t+1})} K_{t+1}^{-1} (K_t - K_{t+1}),
\]

\( 0 \leq t \leq t_0 \) (12)

and

\[
K_{t+2} - K_{t+3} = (\beta^2 K_{t+1}^{-1} (K_t^2 - K_{t+1}), \quad t \geq t_0.
\]

The very basic question for this reduced model is, for a feasible initial \( K_0 \), if there exists a path \( \{K_t\} \) that satisfies (12) and (13) and converges to \( \bar{K} \), furthermore, if such a path is unique. These important questions were first addressed in the original work of Rotemberg (1984), but up to now, the answers to these questions still remain open.

Here we shall provide an affirmative answer to the question on the existence. The answer to the uniqueness question is much more involved and we shall not get into the details here.

3. On the existence of non-steady-state equilibrium path

The main idea is that we shall look for equilibrium path in a special form, namely

\[
K_{t+1} = g(K_t), \quad t \geq t_0
\]

for some function \( g \). Then for such a special path, the finite-difference equation (10) is satisfied, if \( g \) satisfies the following functional equation:

\[
g(g(x))^2 - g(g(g(x))) = \beta^2 g(x) g(x)^{-1} g(g(x))^2 g(x)^{-1} [x^2 - g(x)]
\]

together with the following two conditions:

C1: \( g(\bar{K}) = \bar{K} \);
C2: \( \lim_{t \to \infty} g(K_t) = \bar{K} \) for any \( K_1 \) if \( K_t \) is given by recursion: \( K_{t+1} = g(K_t) \) for \( t \geq t_0 \).

We note that the trajectory given by the aforementioned function \( g \) corresponds to a stable manifold of the dynamical system. For convenience, we shall call this function to be the equilibrium function.

We shall now discuss a simple but fairly general technique to obtain the appropriate equilibrium function \( g \). Let us re-write the equation in the following
equivalent form:

\[ F(g(x)) = g(x), \]

where

\[ F(g(x)) = x^2 - (x\beta)^{-2}(g(g(x))^a - g(g(g(x))))g(g(x))^{1-x}g(x)^{1-x}. \] (16)

The existence of \( g \) of (15) is then equivalent to the existence of the fixed point for \( F \). Among many mathematical techniques for establishing the existence of fixed points, the contraction mapping theorem oftentimes proves to be effective and furthermore this technique also gives rise to a simple numerical algorithm to approximately compute the fixed point. We have successfully applied this technique before, see, for example, Li (1998).

The idea is that we can show that the nonlinear operator \( F \) is contractive with respect to certain measurement, then the fixed-pointed theorem claims that a unique fixed point \( g \) exist for \( F \) and furthermore this fixed point can be computed as a limit of the sequence that are recursively defined

\[ g^{k+1} = F(g^k), \quad k \geq 0. \] (17)

The nonlinear functional \( F \) given by (16) does indeed appear to be contractive in a neighborhood of \( x = \bar{K} \) (notice that \( F(\bar{K}) = \bar{K} \)). We have carried out some extensive numerical experiments and found that the sequence given (17) did indeed converge to a simple-looking function.

The iteration is based on a linear finite element discretization. Let us now briefly describe this procedure. The first step is, for a given integer \( m \), to divide the interval \([0,1]\) into \( m-1 \) with the following nodal points:

\[ x_i = (i-1)h, \quad i = 1, 2, \ldots, m \quad \text{with} \quad h = \frac{1}{m-1}. \]

We then consider continuous piecewise (with respect to the above subintervals) linear function to approximate the function \( g \). Let us denote the approximation by \( g_h \). Then the fixed point iteration can be realized by the following relation:

\[ g_h^{k+1}(x_i) = F(g_h^k(x_i)), \quad k = 1, 2, \ldots. \]

A natural initial guess of this iteration is the constant function \( g_h^0 = \bar{K} \). Our numerical experiments have shown that this procedure converges rapidly.

From the practical point of view, it is perhaps sufficient to use this numerically computed function, but we carried out our investigation one step further. Judging from the shape of the computed function (see Fig. 1), we used the date-fitting technique to try to guess the exact formulation of this function. We found an exact fitting (first for \( x = \frac{3}{2} \)) by using the following function:

\[ g(x) = \gamma x^a, \] (18)
where $\gamma = \alpha \beta$. As a matter of fact, it is rather easy to verify directly that the function given by (18) does indeed satisfy (15) and both conditions C1 and C2. With a more careful theoretical manipulation, furthermore, it is actually possible to derive this close-form solution analytically. We shall not get into the details of these specific theoretical techniques here and instead we shall stay focused on the numerical methods that can be applied to this type of problems and can also be easily extended to more general problems.

4. Evaluation of non-steady-state equilibrium path

In this section we present some numerical results by using different monetary injections. We first study the change of the equilibrium capital sequence when the money supply is increased by 2% at first period, and then study the change when the money supply is increased in two consequent periods but with total amount remaining the same, namely 1% each at periods 1 and 2.
To be specific, we present our results for some special values of \( \alpha \) and \( \beta \), namely \( \alpha = \frac{3}{4} \) and \( \beta = 0.99 \). In this case,

\[ \bar{K} = 0.3039. \]

4.1. One injection

Now we consider the first case. We assume that in period 0, the economy is at the steady state, namely \( K_0 = \bar{K} \), and assume that

\[ M_1 = (1 + 2\%)M_0, \quad M_{t+1} = M_t, \quad t \geq 1. \]

In view of (12) and (13), we have \( t_0 = 2 \) and Eq. (12) is reduced to an equation for \( K_1 \):

\[ g(K_1)^{3/4} - g(g(K_1)) = \frac{9}{16} \beta^2 \frac{1 + \beta}{1 + \beta M_0/M_1} K_1^{-1/4} g(K_1)^{-1/4}(K_0^{3/4} - K_1). \]

(19)

With the explicit solution \( g \) given by (18), we can solve for \( K_1 \) from (19) by the following very simple identity:

\[ K_1 = \frac{\sigma}{\gamma - \gamma^2 + \sigma} K_0^{3/4} \quad \text{with} \quad \sigma = \gamma^2 \frac{1 + \beta}{1 + \beta M_0/M_1}. \]

(20)

With \( K_1 \) given above, we then use (14) to evaluate \( K_2 \) for \( t \geq 2 \). The resulting equilibrium path is illustrated in Fig. 2.

4.2. Two injections

Now we consider the second case. Again we assume that \( K_0 = \bar{K} \), but for periods 1 and 2, we assume that

\[ M_1 = (1 + 1\%)M_0, \quad M_2 = (1 + 1\%)M_1, \quad M_{t+1} = M_t, \quad t \geq 1. \]

In view of (12) and (13), we have \( t_0 = 1 \) and Eq. (12) is reduced to two equations for \( K_1 \) and \( K_2 \) as follows:

\[ K_2^{3/4} - g(K_2) = \sigma_0 K_1^{-1/4} K_2^{-1/4}(K_0^{3/4} - K_1) \quad \text{(21)} \]

and

\[ g(K_2)^{3/4} - g(g(K_2)) = \sigma_1 K_2^{-1/4} g(K_2)^{-1/4}(K_1^{3/4} - K_2) \quad \text{(22)} \]

where

\[ \sigma_0 = \frac{9}{16} \beta^2 \frac{1 + \beta}{1 + \beta M_0/M_1}, \quad \sigma_1 = \frac{9}{16} \beta^2 \frac{1 + \beta}{1 + \beta M_1/M_2}. \]
A direct calculation shows that

\[ K_1 = \delta_0 K_0^{3/4}, \quad K_2 = \delta_1 K_1^{3/4}, \]  

where

\[ \delta_0 = \frac{\sigma_0}{\delta_1(1 - \gamma) + \sigma_0}, \quad \delta_1 = \frac{\sigma_1}{\gamma - \gamma^2 + \sigma_1}. \]

With \( K_1 \) and \( K_2 \) given above, we then use (14) to evaluate \( K_t \) for \( t \geq 3 \). The resulting equilibrium path is illustrated in Fig. 3.

We observe that when the money is injected to the market two times, then the equilibrium capital path has two period oscillations before it approaches the steady state.

The effect of the open-market operation we obtain here is in agreement with the result in Rotemberg (1984), where a shooting method was used to solve the underlying finite difference equations.

We remark that it is clear that the procedure described above can be used similarly to obtain equilibrium path for more general money injection policy,
namely to solve Eqs. (12) and (13) for any $t_0 \geq 1$ and any distributions of $M_t$ for $1 \leq t \leq t_0$.

5. Concluding remarks

One major mathematical problem involved in the treatment of forward-looking models is the lack of determination and the absence of dynamic recursivity: forward-looking models typically yield finite-difference systems whose dynamic order is greater than the dimension of the given boundary values.

In this paper we proposed a method that proves to be very effective for this kind of problem. We first introduce an equilibrium function (which corresponds to the stable manifold of the underlying dynamical system) to delineate the dynamics and to change the difference equation to an operator equation. Then we apply a proper fixed point iteration to numerically obtain the equilibrium function. While this method is applied to Rotemberg’s monetary model (1984), we are able to use the numerical solution as a guidance to obtain the closed-form equilibrium successfully.
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