Economic rates of return: an extension

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Abstract

This paper extends the cash-based rate of return models of Ijiri (1978) and Salamon (1982) by allowing two classes of assets and three classes of sources of funds. The use of debt as a source of funds permits a return on equity calculation. Subsequently, the return on equity calculation is modified to obtain a return on total assets profitability measure. These extensions lead to a solution of the cash-flow observability problem first noted by Stark (1987). A second analytical result shows an equivalence between the accounting rate of return and the estimated rate of return developed in this paper if a particular cash-flow pattern is used. © 2001 Elsevier Science Ltd. All rights reserved.

1. Introduction

One of the uses of accounting is to measure the rate of return of a firm and its investment centers. 1 Remarkng on his celebrated monograph with A.C. Littleton, An Introduction to Corporate Accounting Standards (Paton and Littleton, 1940), Paton (1980, p. 630) comments that the work did not “call

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1 Situations which can require measures of profitability include antitrust litigation (Fisher and McGowan, 1983, p. 82), rate determination for regulated industries, evaluation of initial public offerings, and measuring the performance of management at the firm, division or project level. For some of these situations, market-based profitability measures may be available and sufficient but for others, regulators, creditors, and investors rely heavily on accounting-based measures.

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attention” to “the earning rate”. Furthermore, concern about the return on
capital of a firm, industry or nation is strong (e.g., Porter, 1992, pp. 67–68) and
drives the demand for better profitability measures. The financial accounting
measure of profitability is the accounting rate of return. However, as Horngren
et al. (1997, p. 799) observe, “Obviously, there is an inconsistency between
citing DCF methods as being best for capital budgeting decisions and then
using different concepts to evaluate subsequent performance.” Fisher and
McGowan (1983, p. 84) fault the accounting rate of return because it employs
accounting depreciation rather than economic depreciation. The common
thread of these criticisms is that the accounting rate of return calculations are
not based on the present value of cash flows. A response to this critique whose
origins date back to Hotelling (1925) has been research into cash-based mea-
sures of performance starting with Ijiri (1978). Our paper continues that line of
research.

Our model extends the cash-based rate of return models of Ijiri (1978),
Salamon (1982) and Shinnar et al. (1989) in two important ways. First, we
distinguish between depreciable and non-depreciable assets. Second, by al-
lowing three sources of funds, we build into our model the flexibility to cal-
culate a rate of return on equity measure as well as the rate of return on total
assets measure developed in the extant models.

Our generalization of the Ijiri–Salamon model leads to two analytical re-
results. Our formulation provides a solution to the cash-flow observability
problem attributed to cash-based models (Stark, 1987, pp. 101–102). This is
important because the cash flow observability problem is associated with a
firm’s working capital, typically a significant class of assets. We also show
(Theorem 2) that for a particular cash-flow pattern our economic return on
equity measure equals the accounting return on equity; furthermore this cash-
flow pattern is shown to have some appealing characteristics. Thus Theorem 2
provides a potential explanation for the continued popularity of the accounting
return on equity in practice despite its seeming inconsistency with a discounted
cash-flow approach to investment. 2

The paper is organized as follows. In Section 2, we review the profitability
literature, and in Section 3, we present the model. In Section 4, we adapt the
model for application to financial statements, and in Section 5, we show the
implications for economic depreciation. In Section 6, this approach is extended
to a return on total assets calculation, and we summarize our results in Section
7. Appendix C shows that our approach provides a solution to Stark’s (1987,

2 A result analogous to Theorem 2 has not been established for other cash-based models, and
there is no reason to believe that such a result holds for those models.
2. Related previous research on profitability

The financial accounting measure of profitability, the accounting rate of return, has both critics and defenders. Critics (Atkinson et al., 1997, p. 475; Brealey et al., 1999, p. 161; Fisher and McGowan, 1983, p. 86) point to two theoretical deficiencies. The first is that, for a single completed investment project, the annual accounting rates of return vary and do not equal the economic or internal rate of return (Atkinson et al., 1997, p. 475; Brealey et al., 1999, p. 161). The second is that, for an infinite horizon reinvestment model, the annual accounting rate of return can converge to a number quite different from the common economic rate of return of the individual investments (Fisher and McGowan, 1983, p. 86). The critics find these theoretical deficiencies telling: if the accounting rate of return can fail in these relatively simple settings what confidence can one have for an actual firm which is a complex aggregation of projects (Fisher and McGowan, 1983, p. 83)? The defenders of the accounting rate of return refer to the desirable theoretical properties of a weighted average of accounting rates of return (e.g., Edwards et al., 1987, pp. 65–68; Brief and Lawson, 1992, pp. 416–419) and empirical results relating the accounting rate of return to various stock market measures (e.g., Board and Walker, 1990, pp. 186–187; Landsman and Shapiro, 1995, pp. 111–115).

An important response to this controversy, which goes back many decades (Hotelling, 1925), was to introduce a cash-based rate of return which is calculated from a firm’s cash recovery rate (Ijiri, 1978). The intuition behind Ijiri’s approach is that if decision-makers use cash-flow forecasts to evaluate possible investments and acquisitions, then cash flows rather than earnings should be used for performance evaluation. Ijiri’s cash-based rate of return, which has been modified and refined by Salamon (1982) and Shinnar et al. (1989), is the internal rate of return under the assumptions of particular reinvestment models. When applied to an actual firm’s set of financial statements, the cash-based rate of return formulas are, of course, only approximations or estimates of the firm’s internal rate of return.

The important question of the ability of the Ijiri–Salamon calculations to yield a good approximation of a firm’s internal rate of return has been addressed by focusing on the various important assumptions and input parameters of the Ijiri–Salamon approach. Lee and Stark (1987, pp. 126–127) question Ijiri’s cash flow definition and propose modifications. Ismail (1987, pp. 82–85) challenges the assumption that the cash recovery rates are stable.

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3 Evidence in Kaplan and Ruback (1995, p. 1071) demonstrates both the widespread use and effectiveness of the cash-flow approach in highly leveraged transactions.

4 It is this equality under these restricted conditions which led writers (e.g., Salamon, 1982, p. 294) to call the cash-based rate of return the internal rate of return.
over time. Hubbard and Jensen (1991, pp. 233–234) show that the Ijiri–Salomon estimates of the internal rate of return can be sensitive to an incorrect specification of the expected life of a composite asset. Griner and Stark (1991, pp. 210–212) examine the error in the Ijiri–Salomon estimates resulting from incorrectly specified cash inflow profiles. This question was further considered in Stark et al. (1992, p. 412) where it is shown that Ijiri–Salomon estimates may be flawed as a means to assess systematic error in the accounting rate of return, and in Stark (1994, p. 224) where the resulting errors from incorrectly specified cash flow profiles are connected to the important economic/finance concept of duration. Also, Stark (1987, pp. 101–102) has shown that Salomon’s cash-based rate of return calculation can lead to an incorrect calculation of the internal rate of return for firms with current assets. Peasnell (1996, pp. 293–297) is a survey which considers many of these issues.

3. Economic rate of return on equity model

The objective of our model is to develop a cash-based economic rate of return measure which extends previous research in two ways: (1) the model explicitly considers non-depreciable assets and (2) it permits a rate of return on equity calculation as well as a rate of return on total assets calculation. It is also important that our approach be adaptable to financial statements, a valuable property of previous cash-based models. Initially, we follow the model of Salomon (1982, pp. 294–295) which generalizes that of Ijiri (1978, pp. 338–342). Salomon’s model of a firm makes rather strong assumptions, such as a single investment opportunity, when compared with those of Miller and Modigliani (1961, pp. 412–414), Ohlson (1995, pp. 666–668), and Feltham and Ohlson (1999, pp. 169–171). However, only the Ijiri–Salomon approach addresses the issue of calculating a firm’s economic rate of return from financial statements. The model’s development and its application to financial statements combine to produce considerable notation. Appendix A comprises a list of terms which the reader may want to refer to occasionally.

Each year a firm invests in a single investment project. These investment projects have the same useful life, cash-flow pattern and internal rate of return. Cash flows from each year’s investment are proportional to the size of the investment.

To describe the model let \( g \) be one plus the annual rate of change in gross investment, \( n \) the useful life of each year’s project, \( r \) one plus economic rate of return of each year’s project, \( C_0^j \) the investment in year \( j \), where \( j = 0, 1, \ldots \) and \( C_i^j \) be nominal cash flow to the firm in year \( j + i \) resulting from investment made in year \( j \), where \( i = 1, \ldots, n \), and \( j = 0, 1, \ldots \).

It is assumed that \( C_0^0 > 0, C_i^j > 0 \) for \( 0 < i \leq n, r > 0, \) and \( g > 0 \). The assumption of non-negative cash flows is a sufficient condition for there to be a
unique internal rate of return. The assumption that each year investment grows at a constant rate of \( g - 1 \) is a strong assumption, but not an uncommon one (see, e.g., Brennan and Schwartz, 1982, p. 511). The constant growth rate assumption implies that investment in year \( j \), \( C^j_0 \), equals \( g^j C^0_0 \). Since cash flows from each year’s investment are proportional to the size of the investment \( C^j_0 = g^j C^0_0 \). A difference of this approach and that of Salamon (1982, p. 294) is that all cash flows and calculations are in nominal terms.  

The Salamon (1982, pp. 294–295) model has two important strengths. The firm’s rate of return is well-defined since it equals the common internal rate of return of each year’s investment. Second, the model can be readily adapted to financial statements. Two limitations of the model are that all assets are assumed to be depreciable, and there is only the possibility of a return on total assets calculation.

To address these two limitations, we generalize Salamon (1982, pp. 294–295) by allowing three sources and two uses of investment.  

Let \( C_{0E}/C^0_0 \) be the fixed fraction of funds for investment that comes from equity, \( C_{0D}/C^0_0 \) the fixed fraction of funds for investment that comes from debt, and \( C_{0N}/C^0_0 \) be the fixed fraction of funds for investment that comes from non-interest bearing liabilities such as accounts payable. Also let \( C_{DA}/C^0_0 \) be the fixed fraction of investment in depreciable assets, and \( C_{NA}/C^0_0 \) the fixed fraction of investment in non-depreciable assets.

Thus, depreciable and non-depreciable assets are joint inputs for the single stream of cash flows. Since the sources and uses of investment are equal, each year we have the equality

\[
C_{OE} + C_{OD} + C_{ON} = C_{DA} + C_{NA}.
\]

(1)

While we have generalized the Iijiri–Salamon approach in important ways, it is important to point out two strong assumptions that are maintained. We assume that all categories of sources and uses of investment remain at the same level for the life of the project. For example, debt repayment occurs

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5 This will help us make comparisons with the accounting rate of return which is also in nominal terms. Also see Copeland et al. (1990, p. 104) for a discussion of the advantages of using nominal terms rather than real terms in cash-flow analysis.

6 De Villiers (1989, pp. 497–498) has allowed three types of investment in a cash-based rate of return model. Our types of investment and the questions we address are different from De Villiers (1989). Equity is interpreted in our paper as common shareholder’s equity. Preferred stock could be incorporated into the model without difficulty as a fourth source of investment so that, say, \( C_{0P}/C^0_0 \) is the fixed fraction of funds for investment that comes from preferred stock. The model would then assume that preferred stock is repaid at the end of the project, the same assumption that is made for debt and non-interest-bearing liabilities. The definition of cash flow to equity in Section 4 would remain unchanged since it already subtracts preferred dividends. The definition of cash flow to the firm in Section 6 would remain unchanged, and the main results of the paper would continue to hold.
only at the end of the project. In the case of $C_{NA}$, it means that both the working capital component (primarily inventory, accounts receivable and cash) and the non-depreciable non-current assets component such as land remain at the same level for the life of the project. The cash component of $C_{NA}$ is interpreted as the cash balance needed to absorb the random element of cash inflows and outflows associated with that year’s investment. However, Appendix C of our paper does consider the case where non-depreciable assets such as inventory decrease over the life of the project. A second strong assumption, also made in Salamon (1982, pp. 294–295), is that there is a single representative project rather than a composite project with individual projects of varying useful lives. See Stark (1993) for an analysis of the composite project issue. Each of these issues suggests possibilities for future research.

When year $j$’s investment is made, depreciable assets and non-depreciable assets are acquired in the given proportions, and these assets generate cash flows over the next $n$ periods. At the end of the $n$ periods the debt and non-interest bearing liabilities must be repaid, and depreciable assets have no value. However, starting in Section 4, we consider the case where depreciable assets may have a salvage value. Non-depreciable assets can be redeployed as investment in period $j + n$. The amount available to be redeployed is the amount invested in period $j$. Each year $j$, $j \geq 0$, $C_{NA}g^j$ is invested in non-depreciable assets. For $j > n$, $C_{NA}g^{j-n}$ will come from newly available non-depreciable assets and the rest will be acquired by new investment.

The classification of sources of investment allows one to differentiate between return on equity, return on invested capital (debt plus equity), and return on total assets. To focus our paper, we first develop the return on equity measure. Subsequently, we develop the return on total assets measure. For the return on equity measure the amount invested is the equity portion only and the relevant cash flows are cash flows to equity investors. Let $C_{i0}^0$ be the nominal cash flow to the equity holders of the firm in date $i$ resulting from investment made at date 0, where $i = 1, \ldots, n$. The $C_{i0}^0$ are the same as $C_{i0}$ except that the $C_{i0}^0$ adjusts for interest payments to the holders of debt. Since each project has the same internal rate of return, the return on equity of the firm is $r - 1$ where $r$ satisfies:

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7 This approach to cash is consistent with Brealey et al. (1999, pp. 189, 523).
8 It might be desirable to allow for an inflation factor for some portion of non-depreciable assets such as land. This possibility is not explored in this paper.
9 A rate of return on equity has not previously been calculated using a cash-based approach. Palepu et al. (1996, pp. 4–3) consider the accounting return on equity to be the starting point for a systematic analysis of a firm’s performance. Bushman et al. (1995a, Table 4, Panel B), report that return on equity is the most commonly used rate of return measure for determining annual bonuses at the corporate level. Panel B was not included in the published version (Bushman et al., 1995b).
\[-C_{0E} + \sum_{i=1}^{n} C_{i}^0 r^{-i} + (C_{NA} - C_{0D} - C_{0N}) r^{-n} = 0. \] (2)

The term \((C_{NA} - C_{0D} - C_{0N}) r^{-n}\) in (2) accounts for the redeployment of non-depreciable assets and the repayment of debt and non-interest-bearing liabilities.

The return on equity formula (2) is difficult to apply to an actual firm using financial statements because it requires information on the redeployment of non-depreciable assets as a firm’s projects are completed. This is a variation of the cash-flow observability problem of Stark (1987, pp. 101–102), and leads us to consider a reformulation of the model which is much easier to solve and has the same internal return as (2). \(^{10}\)

### 3.1. A reformulation

Non-depreciable assets are purchased initially and then redeployed \(n\) periods later. The present value of these two transactions nets to the same amount as the present value of an imputed capital charge for non-depreciable assets for each of the \(n\) periods (see Lemma 1 in Appendix B). The reformulation makes this substitution for non-depreciable assets and for debt and non-interest bearing liabilities as well. \(^{11}\) The imputed capital charge equals the return on equity times the non-depreciable asset value, while the imputed charge for a source of capital other than equity is minus the return on equity times the source of capital value.

Let the asterisk notation represent the reformulated problem. Following the above discussion, we define the annual cash flows as

\[ C_{i}^{*0} = C_{i}^0 - (r - 1)(C_{NA} - C_{0D} - C_{0N}) \] (3)

for \(i = 1, 2, \ldots, n\). The term \((r - 1)(C_{NA} - C_{0D} - C_{0N})\) represents the imputed capital charge. Eq. (3) can be simplified using (1) to

\[ C_{i}^{*0} = C_{i}^0 - (r - 1)(C_{0E} - C_{DA}) \] (4)

for \(i = 1, 2, \ldots, n\).

\(^{10}\) Appendix C demonstrates that the reformulation developed below provides a solution to Stark’s cash-flow observability problem.

\(^{11}\) An imputed charge for capital was introduced in the cash-based rate of return literature by Shinnar et al. (1989, pp. 419–420) and applied to working capital. They did not explore the conditions under which their approach would calculate the true internal rate of return. Shinnar et al. (1989, p. 420) define their cash recovery rate as net income plus depreciation plus interest paid less a capital charge for working capital divided by the sum of net investment over the previous \(n\) years where \(n\) is the life of the firm’s projects. This differs from the Ijiri-Salamon definition and the definition of the cash recovery rate in this paper (Eqs. (11) and (11’)) below.
The logic of the reformulation also implies that the formula for the original equity investment, $C_{0E}$, must be revised. $C_{NA}$ is subtracted from $C_{0E}$ since the cost of non-depreciable assets is handled by the imputed capital charge. A similar change is made for non-equity sources of capital since these are also handled by the imputed capital charge. Therefore, the original equity investment in the reformulated problem ($C_{00}^r$) is $C_{0E} - C_{NA} + C_{0D} + C_{0N}$ which by (1) equals $C_{DA}$. Therefore

$$C_{00}^r = C_{DA}. \tag{5}$$

Thus, the reformulation consists of using $C_{00}^r$ and $C_{i}^r$, $1 \leq i \leq n$, to represent initial investment and cash flow, respectively. The reformulation has the apparent difficulty that $r$ is in the definition of the cash flows (3) and is unknown. However, the next section shows that by using results in Salamon (1982, pp. 295–297) we can solve a single equation for the unknown $r$. Note also that the term $(C_{NA} - C_{0D} - C_{0N})r^{-n}$ is eliminated since this term is handled by the imputed capital charge. This elimination represents the major advantage of the reformulation with respect to the goal of developing an expression for a firm’s return on equity which could be evaluated empirically.

The internal rate of return of the reformulation is the $r$ such that

$$-C_{00}^r + \sum_{i=1}^{n} C_{i}^r r^{-i} = 0. \tag{6}$$

Theorem 1 below shows the reformulation is valid since this same $r$ also satisfies $-C_{0E} + \sum_{i=1}^{n} C_{i}^0 r^{-i} + (C_{NA} - C_{0D} - C_{0N})r^{-n} = 0.

**Theorem 1.** The reformulated model has the same internal rate of return as (2).

Proof of Theorem 1 is given in Appendix B.

### 4. Applying the return on equity model

This section shows how the economic rate of return on equity can be determined from financial statements. One of the insights of the Ijiri–Salamon approach is that a firm’s economic rate of return can be estimated using the cash recovery rate, cash inflows divided by gross investment. This is valuable because the cash recovery rate can be calculated from financial statements. Although we employ a reformulated version, we can take advantage of the results in Salamon (1982, pp. 295–297).\(^{12}\)

For dates $j > n$, Salamon (1982, p. 295) shows that the cash recovery rate, is a constant, $\rho$, which is given by

$$\rho = (C_{n}^0 + C_{n-1}^0 g + \cdots + C_{1}^0 g^{n-1})/C_{0}^0 (1 + g + \cdots + g^{n-1}). \tag{6}$$

\(^{12}\) It is easy to adjust for the difference between using nominal and real terms.
Salamon (1982, p. 296) demonstrates that (6) can be rearranged as

\[ \rho = \frac{(g - 1)g^n \sum_{i=1}^{n} C_i^{a0} g^{-i}}{(g^n - 1) \sum_{i=1}^{n} C_i^{a0} r^{-i}}. \]  

(7)

Eq. (7) requires knowledge of the cash-flow pattern. For purposes of empirical application, Salamon introduced a cash-flow parameter \( b \) and set

\[ C_{j+1}^{a0} = b_i C_i^{a0} \]  

for \( j = 0, 1, \ldots, n - 1 \). Eq. (8) says that each year’s cash flows are a constant fraction \( b \) of the previous year’s cash flows. When (8) applies, Eq. (6) simplifies to

\[ \rho = \frac{(g - 1)(g^n - b^n)r^n (r - b)}{(g^n - 1)(g - b)(r^n - b^n)}. \]  

(9)

Eq. (9) simplifies further when cash flows are constant \( (b = 1) \) to

\[ \rho = (r - 1)/(1 - r^{-n}). \]  

(10)

These equations can be used to determine \( r \). For example if \( b = 1 \), then (10) can be solved for \( r \), once \( \rho \) and \( n \) have been determined from the financial statements. In terms of our model (see (4) and (5)), \( \rho = (\text{cash flow} - \text{imputed capital charge})/\text{gross depreciable assets} \) which we restate as

\[ \rho = (\text{CF}_t - (r - 1)(E_t - \text{GDA}_t))/\text{GDA}_t, \]  

(11)

where \( \text{CF}_t \) is the cash flow during period \( t \), \( \text{GDA}_t \) is the gross depreciable assets at the beginning of period \( t \), and \( E_t \) is equity investment at the beginning of period \( t \). \( \text{GDA}_t \) is readily available from the financial statements. Next we turn to the definitions of cash flow and equity investment.

### 4.1. Definitions of cash flow and equity investment

From basic capital budgeting considerations (Brealey et al., 1999, Chapter 6) and (Horngren et al., 1997, Chapter 22), cash flow is net income less the change in working capital plus expenses not requiring cash such as depreciation plus any return from the terminal disposal of the initial investment. Our choice for a cash-flow definition is net income less preferred dividends (since we are developing a return on common share-holder’s equity calculation) plus depreciation plus retirements and sales of depreciable assets at book value. Book

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13 Gordon and Hamer (1988, p. 517) have extended (9) below to include some concave cash-flow patterns. In Appendix B we extend (9) to include linear cash-flow patterns. Griner and Stark (1988, pp. 296–298) show how estimates of economic rates of return can be made using eight possible functional forms of cash-flow patterns including concave cash-flow patterns.
value is used for retirements and sales since any gain or loss on retirements and sales is already included in net income. This choice is consistent with our model which does not admit changes of working capital after the initial investment. Appendix C considers the case where non-depreciable assets including working capital assets decrease over the life of the project. Therefore,

\[ \text{CF}_t \equiv \text{Net Income} - \text{Preferred Dividends} + \text{Depreciation Expense} + \text{Retirements and Sales of Depreciable Assets at Book Value}. \]

\[ \equiv \text{NI}_t - \text{PD}_t + \text{DE}_t + \text{RBV}_t. \]

We employ the definition just given for several reasons besides tractability. First, the Fisher and McGowan (1983, p. 85) critique of the accounting rate of return was exclusively concerned with depreciation. Second, it incorporates the main components of the version of cash flow used by many investment professionals and academics (e.g., Value Line, 1996, p. 6; Lakonishok et al., 1994, p. 1545; Kaplan and Zingales, 1997, p. 178) – net income plus depreciation. And third, by focusing on depreciation, this definition enables us to consider both economic depreciation and depreciation in general. 14 It would be desirable to consider other definitions of cash flow in future research.

We want equity investment to be consistent with cash flow since cash flow, however defined, is either paid out in dividends or reinvested in the firm. In particular, the change in equity investment should equal cash flow less \( D_t \), net dividends to shareholders, and \( R_t \), the retirements and sales of depreciable assets at original cost. Therefore,

\[ E_{t+1} - E_t = \text{CF}_t - D_t - R_t. \] (12)

Eq. (12) is easily seen to be similar to the clean surplus equation.

This leads to the following equity investment definition: 15

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15 To see that the consistency Eq. (12) is satisfied, first observe that the change in equity investment \( (E_t - E_{t-1}) \) equals the change in shareholder’s equity plus the change in accumulated depreciation. Therefore, we must show that \( \text{CF}_t - D_t - R_t \) also equals the change in shareholder’s equity plus the change in accumulated depreciation. We do this by reshuffling the components of \( 
\text{CF}_t \) and subtracting \( D_t \) and \( R_t \). Net income less preferred dividends less common dividends \( (D_t) \) equals the change in shareholder’s equity. Depreciation expense plus the retirements and sales of depreciable assets at book value less their original cost \( (R_t) \) equals the change in accumulated depreciation.
\( E_t \equiv \text{Beginning Common Shareholder’s equity} \\
+ \text{Beginning Accumulated depreciation} \\
\equiv CSE_t + AD_t. \)

### 4.1.1. Values for growth \((g)\) and useful life \((n)\)

Calculation of our economic return on equity measure employ Eqs. (9) and (11) which require values for growth and the useful life of the representative project. As a preliminary to determining growth and useful life we incorporate into the model two items found empirically but not considered in the Salamon (1982) model or our generalization of it until this point: accumulated depreciation and salvage value. This incorporation provides us with relations between the useful life of investments \(n\), one plus the growth rate \(g\), salvage value \(S\), and observable variables which we use to solve for the \(n, g\) and \(S\) values necessary to the return calculation. For notational convenience the dependence of \(n, g\) and \(S\) on the period \(t\) is suppressed.

We illustrate the incorporation of accumulated depreciation and salvage value by employing an example where the useful life, \(n\), is 3 so that steady-state is reached at period (date) 3.

**Example 1.** Let \(S, 0 < S < 1\) be the estimated salvage value as a fraction of the original cost. Thus, the cash flow in the final period is \(C_0^n + SC_0^0\). We assume that the firm uses straight-line depreciation for accounting purposes. Then

<table>
<thead>
<tr>
<th></th>
<th>Period 1</th>
<th>Period 2</th>
<th>Period 3 (Steady-State)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beginning GDA</td>
<td>(C_0^0)</td>
<td>(C_0^0(1 + g))</td>
<td>(C_0^0(1 + g + g^2))</td>
</tr>
<tr>
<td>Beg. Accumulated Dep.</td>
<td>0</td>
<td>((1/3)C_0^0(1 - S))</td>
<td>((1/3)C_0^0(1 - S)(2 + g))</td>
</tr>
<tr>
<td>Period Cash Flow</td>
<td>(C_1^0)</td>
<td>(C_2^0 + C_1^0g)</td>
<td>(C_3^0 + SC_0^0 + C_2^0g + C_1^0g^2)</td>
</tr>
</tbody>
</table>

Note that in this example, the steady-state ratio of cash flow to beginning gross depreciable assets, the cash recovery rate, is \((C_3^0 + SC_0^0 + C_2^0g + C_1^0g^2)/(C_0^0(1 + g + g^2))\) which conforms with (6).

Generalizing from the above example, it is shown in Appendix B that

\[
AD_t/GDA_t = (1 - S)[1/(n(g - 1)) - 1/(g^n - 1)].
\]

\(SC_0^0\), the steady-state salvage value, corresponds to RBV in the financial statements since an asset’s book value at retirement is its estimated salvage value assuming the asset is retired or sold as planned at date \(n\) (and this is assumed by the model). The denominator in (6), \(C_0^0(1 + g + \ldots + g^{n-1})\), corresponds to GDA. Therefore, \(S\) satisfies

\[
S = RBV_t/(GDA_t[(g - 1)/(g^n - 1)]).
\]
The rules of straight-line depreciation imply that

\[ n = \text{GDA}_t (1 - S)/\text{DE}_t. \]  

(15)

We can now state how the economic return on equity is obtained from financial statements. We do this for the case where the cash-flow pattern satisfies (8); the special case where \( b = 1 \) is simpler.

**Step 1.** Substitute for \( \rho \) using (9) into (11) to get

\[
\frac{(g - 1)(g^n - b^n)\rho^n(r - b)(1 - Sr^n)}{(g^n - 1)(g - b)(\rho^n - b^n)} + S(g - 1)/(g^n - 1)
\]

\[ = (\text{CF}_t - (r - 1))(\text{E}_t - \text{GDA}_t)/\text{GDA}_t. \]

The additional terms involving \( S \) come from the new assumption that a salvage value of \( S^0_t \) is obtained in the last year of a project’s useful life (see Appendix B).

**Step 2.** Eqs. (13)–(15) are solved for \( g, n \) and \( S \). Following precedence in the literature, the value of \( b \) is simply assumed. These values are substituted into the equation in Step 1 which leaves us with one equation and one unknown \( r \).

5. Economic depreciation

This section establishes the relation between the accounting return on equity, the economic return on equity and economic depreciation. Economic depreciation has the desirable property that it is the depreciation such that the accounting rate of return is constant and equals the internal (economic) rate of return (Edwards et al., 1987, pp. 21–22). Economic depreciation for an asset is defined as the period to period change in the present value of future net cash flows using the internal rate of return as the discount factor. Thus economic depreciation depends upon the projected cash-flow pattern.

The economic (e.g., Shinnar et al., 1989, p. 189) and accounting literature (e.g., Salamon, 1982, p. 297; Griner and Stark, 1988, p. 298) frequently hypothesize cash flow to be constant or declining over the life of the investment. When cash flow is constant, it is known that economic depreciation equals annuity depreciation (see Stickney and Weil, 1994, pp. 494–495). However, as shown in Theorem 2 below, there is also a decreasing pattern which equates the accounting return on equity with the economic rate of return on equity.

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Note that in each equation \( n \) is treated as a continuous variable. This does not cause a difficulty as long as the hypothesized cash-flow pattern is such that expression (9) or (10) applies.
Therefore, for the cash-flow pattern in Theorem 2, straight-line depreciation is economic depreciation.¹⁷

It is known (Kraus and Huefner, 1972, (6)) that for a single depreciable asset with no salvage value straight-line depreciation is economic depreciation for the following cash-flow pattern:

\[ C_i = \frac{(1 + \text{ARE})}{n} + \text{ARE} - \frac{(\text{ARE}/n)i}{n} \]  (16)

for \( i = 1, \ldots, n \). With this pattern \text{ARE}, the accounting return on equity, does not vary from period to period and equals the internal rate of return. For analytical purposes, the accounting return on equity is defined as net income less preferred dividends divided by beginning book value (see Penman, 1991, p. 235).

For Theorem 2 to hold the required cash-flow pattern is

\[ C_{i}^0 = (1 - S)(1 + \text{ARE}_i)/n + \text{ARE}_i(CSE_t + AD_t)/GDA_t \]

\[ - (\text{ARE}_i(1 - S)/n)i \]  (17)

for \( i = 1, \ldots, n \). The period \( t \) in (17) is the period when the economic rate of return is calculated. We use \( C_{i}^0 \), the cash flow pattern to equity investors, rather than \( C_i \), the reformulated cash-flow pattern, to facilitate comparison with the single asset formula and to avoid using the unknown \( r \) in the statement of the hypothesis. Theorem 2 has not been established for other cash-based models in the literature, and there is no reason to believe it holds for these models.¹⁸

**Theorem 2.** Let cash flow be defined as net income less preferred dividends plus depreciation plus retirements and sales of depreciable assets at book value. Assume that cash flows follow the pattern of Eq. (17). Then, the economic rate of return on equity (calculated for period \( t \)) equals \( \text{ARE}_t \), the accounting return on equity.

¹⁷ Empirically, about 88% of firms use straight-line depreciation (AICPA, 1999, p. 359). Analytically, Eq. (15) means that straight-line depreciation is assumed for all firms.

¹⁸ For example, Table 1 part D in Griner and Stark (1988, p. 302) shows that for all eight functional forms of cash flow patterns they consider, the correlations of the Ijiri estimates of the rate of return and two versions of the accounting rate of return never exceed 0.70 (1988, p. 303). That same Table 1 also shows that for all eight functional forms of cash flow patterns they consider, the correlations of the Lee and Stark estimates of the rate of return and two versions of the accounting rate of return never exceed 0.50 (1988, p. 302).

Statements consistent with Theorem 2 can be found in Chapter 3 of Beaver (1989, pp. 55, 58). Beaver’s (1989, p. 59) statements are based on numerical examples. The conditions of Theorem 2 are more general than Beaver’s examples in four ways: (a) calculations are made from the financial statements of firms, (b) non-depreciable assets are allowed, (c) salvage values are permitted, and (d) debt is allowed.

Theorem 2 has been confirmed computationally by us in calculations employing data from a large sample of firms.
Proof of Theorem 2 is given in Appendix B. The cash-flow pattern of Theorem 2 has two attractive features. First, it is linear and decreasing over time making it consistent with common ideas of typical cash-flow patterns. Second, it decreases more slowly for larger values of $n$. For instance, when $\text{ARE} > 0$, $C_j^{n_0} / C_1^{n_0}, j < n$, is larger for larger values of $n$. In words this says that for a project expected to last 20 years, year 5 cash flow is a larger percentage of year 1 cash flow than if the project was expected to last 6 years.

Theorem 2 clarifies the issue of the accounting rate of return versus an economic rate of return calculated from the model of this paper. For an economic return on equity calculation to be superior to the standard accounting return on equity it must be based on a cash-flow pattern that is more representative/credible than the pattern in Theorem 2.

A cash-flow pattern which has a number of proponents is the inverted U pattern where cash flow is low in the early period, increases and reaches its peak in the middle periods and then declines toward zero. For example, Fisher and McGowan (1983, pp. 84, 86) make their calculations based on the inverted U cash flow pattern. In a survey of firms Gordon and Hamer (1988, p. 515) found that a majority of firms chose an inverted U cash-flow pattern as the one best representing the cash-flow profile of their composite projects. Since the inverted U cash-flow pattern is different from the declining pattern of Theorem 2, this is a situation where there is a potential for an economic rate of return calculation to be superior to the accounting return on equity.

6. Rate of return on total assets

It is desirable to be able to develop an economic rate of return on total assets measure since for some applications (e.g., monopoly profit issues) one wants to calculate returns to the firm independent of the firm’s capital structure. Fortunately, we can derive an economic rate of return on total assets calculation in only one section since the same approach employed for the return on equity measure works for the return on total assets.

Our approach is to go through Eqs. (1)–(15) indicating the changes required for the return on total assets case with some discussion where necessary. The

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19 As an example of the cash flow pattern of Theorem 2, let $S = 0$, $n = 20$, the ratio of equity investment to gross depreciable assets = 2, and the accounting return on equity = 0.15. Then $C_1 = 0.35$, and cash flows decrease linearly to $C_{50} = 0.2075$.

20 For example, suppose that the true cash-flow pattern for a firm is constant or slightly increasing over time. The cash-based estimate of the economic rate of return on equity which assumes a constant cash-flow pattern should be superior to the accounting return on equity. This follows because Theorem 2 shows that the accounting rate of return on equity is equal to the cash-based return on equity which assumes that the cash flow pattern is given by Eq. (17) which is a linear decreasing pattern.
only equation which does not appear to have an analogue in the rate of return on total assets case is Eq. (12). Fortunately, an analogue of Eq. (12) is not needed to determine a return on total assets profitability measure.

Eq. (1) holds without change. Eq. (2) becomes

\[-(C_{DA} + C_{NA}) + \sum_{i=1}^{n} C_{i}^{0}r^{-i} + C_{NA}r^{-n} = 0. \quad (2')\]

In Eq. (2') total assets replace equity, cash flows are the original cash flows to the firm, and the terms involving the redeployment of debt and non-interest-bearing liabilities are no longer needed. Only non-depreciable assets are redeployed. The reformulation is still desirable and Eq. (3) becomes

\[C_{i}^{0} = C_{i}^{0} - (r - 1)C_{NA}. \quad (3')\]

Eq. (3') is simple enough that it does not need further simplification to (4'). Investment in the reformulated problem is total assets less non-depreciable assets (which are handled as redeployed assets) which equals depreciable assets. Therefore, albeit using a different argument, Eq. (5) remains unchanged. It follows that the return on assets analogue of Theorem 1 holds.

Eqs. (6)–(10) hold unchanged. Indeed they were developed by Salamon for a return-on-total-assets type of calculation. Eq. (11) is important because it connects Salamon’s (1982, pp. 295–296) equations with the reformulated model. The return on total assets case Eq. (11), the equation for the cash recovery rate, becomes

\[\rho = (\text{CFF}_{i} - (r - 1)(\text{TA}_{i} + \text{AD}_{i} - \text{GDA}_{i}))/\text{GDA}_{i}, \quad (11')\]

where CFF\(_{i}\) is cash flow to the firm and TA\(_{i}\) is beginning total assets. The numerator of the right-hand side of Eq. (11') comes from substituting financial statement terms into (3'). C\(_{NA}\) represents all assets except depreciable assets. The denominator, as indicated above, is unchanged from (11). Cash flow to the firm, CFF\(_{i}\), is the sum of net income, interest expense (INT\(_{i}\)), depreciation expense, and the retirement and sale of depreciable assets at book value.

It is worthwhile to compare Eq. (11') with the corresponding Eq. (11) for the return on equity case. The right-hand sides of (11') and (11) differ in two ways. First, the return on total assets calculation uses cash flow to the firm, CFF\(_{i}\), rather than cash flow to the common share equity holders. The difference is payments of interest and preferred dividends. Second, the term multiplying \((r - 1)\) in the return on total assets calculation uses TA\(_{i}\) + AD\(_{i}\) in place of E\(_{i}\) = CSE\(_{i}\) + AD\(_{i}\). These differences between Eqs. (11') and (11) are unsurprising because they are similar to the differences one has between the accounting return on equity calculation and the accounting return on total assets calculation.
Returning to our analysis of Eqs. (1)–(15) we find that Eqs. (13)–(15), which are used to calculate growth, useful life, and estimated salvage value as a fraction of original cost remain unchanged. A revised version of Theorem 2 can be derived (but is not derived in our paper) for the return on total asset case when the accounting return on total assets is defined as net income plus interest expense divided by beginning total assets.

We are in a position to state how the economic return on total assets is calculated from financial statements. We do this for the case where the cash-flow pattern satisfies (8); the special case where $b = 1$ is simpler.

Step 1. Substitute for $\rho$ using (9) into (11) to get

$$\frac{(g - 1)(g^n - b^n)r^n(r - b)(1 - Sr^{-n})}{(g^n - 1)(g - b)(r^n - b^n)} + S(g - 1)/(g^n - 1)$$

$= (CFF_t - (r - 1)(TA_t + AD_t - GDA_t))/GDA_t$.

The additional terms involving $S$ come from the assumption that a salvage value of $SC_{t0}$ is obtained in the last year of a project’s useful life (see Appendix B for details).

Step 2. Eqs. (13)–(15) are solved for $g, n,$ and $S$. Following precedence in the literature, the value of $b$ is simply assumed. These values are substituted into the equation in Step 1 which leaves us with one equation and one unknown $r$.

7. Summary and conclusion

An interesting response to criticism that the accounting rate of return is not based on the present value of cash flows was to introduce the concept of a cash-based rate of return (Ijiri, 1978, pp. 331–333). Our paper extends the cash-based rate of return models in several ways. We allow for both depreciable and non-depreciable assets which leads to a solution of the cash-flow observability problem first noted by Stark (1987, pp. 101–102). We include debt in our model, permitting a return on equity calculation. Subsequently the return on equity development is modified to obtain a return on total assets profitability measure. Finally, consistent with accounting practice, we incorporate salvage value into the model.

An important analytical result is Theorem 2 which clarifies the issue of whether the accounting rate of return reflects the economic rate of return. Theorem 2 shows that for a particular linear decreasing cash flow pattern, straight-line depreciation reflects economic depreciation and the economic return on equity equals the accounting return on equity. Theorem 2 helps reconcile the popular use of the accounting rate of return with criticism of its appropriateness by showing that the accounting rate of return is an economic
rate of return for a credible cash-flow pattern. One the other hand, Theorem 2 also suggests that an economic rate of return calculation can be superior to the accounting return on equity when a firm’s cash flow pattern differs in significant ways from the linear decreasing pattern of Theorem 2.

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Appendix A

Notation for the model includes:

- $C_i^0$: the nominal cash flow to the firm in year $i$ resulting from an investment made at year 0
- $C_0^0$: the investment in year 0
- $C_i^{00}$: the nominal cash flow to equity holders in year $i$ resulting from an investment made at year 0
- $C_i^{*0}$: the nominal cash in year $i$ resulting from an investment made at year 0 for the reformulated problem
- $C_0^{*0}$: the investment in year 0 for the reformulated problem
- $g$: one plus the annual rate of change in gross investment
- $r$: one plus economic rate of return of each year’s project
- $n$: the useful life of each year’s project
- $C_{0E}$: funds for investment that comes from equity
- $C_{0D}$: funds for investment that comes from debt
- $C_{0N}$: funds for investment that comes from non-interest-bearing liabilities such as accounts payable
- $C_{DA}$: investment in depreciable assets
- $C_{NA}$: investment in non-depreciable assets
- $b$: cash-flow pattern parameter
- $\rho$: cash recovery rate (cash flow divided by gross investment)
- $S$: estimated salvage value as a fraction of original cost
Notation for the financial statements (year t) includes:

\[ E_t \quad \text{beginning equity } (\text{CSE}_t + \text{AD}_t) \]
\[ \text{ARE}_t \quad \text{accounting return on equity} \]
\[ \text{CSE}_t \quad \text{beginning common shareholder's equity} \]
\[ \text{AD}_t \quad \text{beginning accumulated depreciation} \]
\[ \text{GDA}_t \quad \text{beginning gross depreciable assets} \]
\[ \text{DE}_t \quad \text{depreciation expense} \]
\[ \text{CF}_t \quad \text{cash flow to equity holders } (\text{NI}_t - \text{PD}_t + \text{DE}_t + \text{RBV}_t) \]
\[ \text{CFF}_t \quad \text{cash flow to the firm } (\text{NI}_t + \text{INT}_t + \text{DE}_t + \text{RBV}_t) \]
\[ \text{NI}_t \quad \text{net income} \]
\[ \text{PD}_t \quad \text{preferred dividends} \]
\[ \text{RBV}_t \quad \text{retirements and sale of depreciable assets at book value} \]
\[ \text{INT}_t \quad \text{interest expense} \]
\[ D_t \quad \text{dividends to common shareholders} \]
\[ R_t \quad \text{retirements and sale of depreciable assets at original cost} \]
\[ \text{TA}_t \quad \text{beginning total assets} \]

**Appendix B**

The following Lemma is used in Proof of Theorem 1; its proof is by inspection.

**Lemma 1.** \[ \sum_{i=1}^{n}(r-1)r^{-i} = 1 - r^{-n}. \]

**Proof of Theorem 1.** Let \( r \) satisfy \( -\text{CDA} + \sum_{i=1}^{n} C_i^0 r^{-i} = 0 \). We need to show that \( r \) also satisfies (2), i.e., \( -\text{COE} + \sum_{i=1}^{n} C_i^0 r^{-i} + (\text{CNA} - \text{COD} - \text{CON}) r^{-n} = 0 \). Equivalently, by multiplying (2) by \(-1\) and adding \( -\text{CDA} + \sum_{i=1}^{n} C_i^0 r^{-i} = 0 \), we need to show that \( \text{COE} - \text{CDA} + \sum_{i=1}^{n} (C_i^{x0} - C_i^0) r^{-i} - (\text{CNA} - \text{COD} - \text{CON}) r^{-n} = 0 \). Substituting for \( C_i^{x0} \), the left-hand side of the previous equation becomes \( \text{COE} - \text{CDA} - \sum_{i=1}^{n}(r-1)(\text{CNA} - \text{COD} - \text{CON}) r^{-i} - (\text{CNA} - \text{COD} - \text{CON}) r^{-n} \). By (1) this expression equals \( \text{COE} - \text{CDA} - \sum_{i=1}^{n}(r-1)(\text{COE} - \text{CDA}) r^{-i} - (\text{COE} - \text{CDA}) r^{-n} \). Now apply Lemma 1. \( \square \).

The rest of Appendix B is concerned with the proof of Theorem 2. Two expressions for finite series which hold for \( x \neq 1 \) are useful:

\[ 1 + x + \ldots + x^{n-1} = (x^n - 1)/(x - 1) \quad \text{(B.1)} \]

and

\[ n + (n - 1)x + \ldots + x^{n-1} = x(x^n - 1)/(x - 1)^2 - n/(x - 1). \quad \text{(B.2)} \]
Eq. (B.2) follows from
\[ n + (n-1)x + \cdots + x^{n-1} = (x^n + x^{n-1} + \cdots + x - n)/(x - 1). \]

We first show that Eq. (13) holds. Generalizing from the example in section 4 we set
\[ AD_i/GDA_i = (1 - S)((n - 1) + (n - 2)g + \cdots + g^{n-2})/(n(1 + g + \cdots + g^{n-1})). \]
Now apply Eq. (B.1) to the denominator, and rewrite the numerator as
\[ (n + (n-1)g + \cdots + g^{n-2})/g \] and apply (B.2) to obtain
\[ (g^n - 1)/(g - 1)^2 - n/(g - 1)g - n/g. \] Eq. (13) follows.

The hypothesis of Theorem 2 is stated in terms of the cash pattern \( C^0_i \). It follows from (4) and (5), and our definition of equity investment that
\[ C^0_i = C^0_i - (r - 1)((CSE_i + AD_i)/GDA_i - GDA_i/GDA_i). \]
Therefore if \( C^0_i \) satisfies (17), then \( C^0_i \) is linear in \( i \) and equals
\[ [(1 - S)(1 + ARE_i)/n + (r - 1) + (ARE_i - (r - 1))((CSE_i + AD_i)/GDA_i)] - [ARE_i(1 - S)/n] \] for \( i = 1, \ldots, n \).

As a preliminary to proving Theorem 2 we need to establish (B.3) and Lemma 2 below. Eq. (B.3) extends the Salamon (1982, pp. 295–296) results to the situation where cash flows are linear functions of time. Suppose that \( C^0_i = a - bi \), for \( i = 1, \ldots, n \), and that the salvage value at date \( n \) is \( SC^0_0 \). Then
\[ \rho = (a - b[g/(g - 1) - n/(g^n - 1)])/c + S(g - 1)/(g^n - 1). \] (B.3)

where \( c \) equals
\[ r^{-n}[a(r^n - 1)/(r - 1) - b(r(r^n - 1)/(r - 1))^2 - n/(r - 1)]/(1 - r^{-n}S). \]

Note that Eq. (B.3) is defined for non-integer values of \( n \).

To obtain Eq. (B.3) substitute \( C^0_i = a - bi \) into (7), using (B.1) for the coefficient of \( a \) and (B.2) for the coefficient of \( b \), so that
\[ \rho = [(g - 1)/(g^n - 1)] \]
\[ [a(g^n - 1)/(g - 1) - b(g(g^n - 1)/(g - 1)^2 - n/(g - 1)) + SC^0_0]/(\sum_i^n C^0_i r^{-i} + SC^0_0 r^{-n}). \] To obtain the expression for \( c \), begin with \( c = \sum_i^n C^0_i r^{-i} + SC^0_0 r^{-n} = C^0_0 \). Therefore, \( c = C^0_0 = \sum_i^n C^0_i r^{-i}/(1 - r^{-n}S) \). The last step is to use \( C^0_i = a - bi \), (B.1) for the coefficient of \( a \) and (B.2) for the coefficient of \( b \). The term \( S(g - 1)/(g^n - 1) \) in (B.3) follows from the fact that \( \sum_i^n C^0_i r^{-i} + SC^0_0 r^{-n} \) equals \( C^0_0 \).

**Lemma 2.** Suppose that the cash flow pattern of \( C^0_i \) satisfies (17). If the economic rate of return on equity equals \( ARE_i \), the accounting return on equity, then \( c \) in (B.3) equals 1.

**Proof of Lemma 2.** As noted above, if \( C^0_i \) satisfies (17), then \( C^0_i \) is linear in \( i \). More specifically, \( C^0_i \) can be written as \( a - bi \) where \( a = (1 - S)(1 + ARE_i)/n + (r - 1) + (ARE_i - (r - 1))((CSE_i + AD_i)/GDA_i) \) and \( b = ARE_i(1 - S)/n \). These values of \( a \) and \( b \) are substituted in the expression for \( c \) in the sentence

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21 GDA,

appears in the denominator to properly scale the coefficient of \( (r - 1) \). The cash flows in Theorem 2 correspond to a dollar investment while (4) corresponds to an investment of GDAi.
below Eq. (B.3). We want to show that \( r^{-n}[(1 - S)(1 + \text{ARE}_t)/n + (r - 1) + \\
(\text{ARE}_t - (r - 1))(\text{CSE}_t + \text{AD}_t)/\text{GDA}_t)](r^n - 1)/(r - 1) - r^{-n}[(\text{ARE}_t(1 - S)/n) \\
(r^n - 1)/(r - 1)^2 - n/(r - 1))] = 1 - Sr^{-n}. \) By hypothesis \( r = \text{ARE}_t + 1, \) and we make this substitution in the above equation. The left-hand side becomes \( r^{-n}(1 - S)r(r^n - 1)/(n(r - 1)) + 1 - r^{-n} - r^{-n}(1 - S)r(r^n - 1)/(n(r - 1)) + \\
r^{-n}(1 - S), \) which equals \( 1 - Sr^{-n}. \)

**Proof of Theorem 2.** We begin by using Eq. (11) and the definitions of cash flow and equity investment to obtain

\[
0 = \text{NI}_t - \text{PD}_t + \text{DE}_t + \text{RBV}_t - (r - 1)(\text{CSE}_t + \text{AD}_t - \text{GDA}_t) - \rho \text{GDA}_t.
\]

We substitute for \( \text{AD}_t, \text{RBV}_t \) and \( \text{DE}_t \) using (13)–(15) to obtain:

\[
0 = [\text{NI}_t - \text{PD}_t - (r - 1)\text{CSE}_t] + [1/n - (r - 1)/n(g - 1)] + \\
(r - 1)/(g^n - 1) + (r - 1) - \rho \text{GDA}_t + [-1/n + (g - 1)/(g^n - 1)]
\]

\[
+ (r - 1)/n(g - 1) - (r - 1)/(g^n - 1)]S(\text{GDA}_t).
\]

We want to show \( r = 1 + \text{ARE}_t \) satisfies the previous equation when the cash-flow pattern is given by (17). Substituting for \( a \) and \( b \) in (B.3) and using Lemma 2, we have that

\[
\rho = (1 - S)(1 + \text{ARE}_t)/n + (r - 1)
\]

\[
+ (\text{ARE}_t - (r - 1))(\text{CSE}_t + \text{AD}_t)/\text{GDA}_t
\]

\[
- \text{ARE}_t(1 - S)g/(n(g - 1)) + \text{ARE}_t(1 - S)/(g^n - 1)
\]

\[
+ [S(g - 1)/(g^n - 1)].
\]

Thus, we have

\[
0 = [\text{NI}_t - \text{PD}_t - (r - 1)\text{CSE}_t] + [1/n - (r - 1)/n(g - 1)] + \\
(r - 1)/(g^n - 1) + (r - 1) - (1 + \text{ARE}_t)/n - (r - 1) - \\
(\text{ARE}_t - (r - 1))(\text{CSE}_t + \text{AD}_t)/\text{GDA}_t + \text{ARE}_t(g/(n(g - 1)))
\]

\[
- \text{ARE}_t/(g^n - 1)]\text{GDA}_t + [-1/n + (g - 1)/(g^n - 1)]
\]

\[
+ (r - 1)/n(g - 1) - (r - 1)/(g^n - 1) + (1 + \text{ARE}_t)/n
\]

\[
- \text{ARE}_t(g/(n(g - 1))) + \text{ARE}_t/(g^n - 1) - (g - 1)/(g^n - 1)]S(\text{GDA}_t).
\]

When \( r = 1 + \text{ARE}_t \) it can be seen that the terms in the three brackets equal 0.

\( \square \)

**Appendix C. The cash flow observability problem**

In this appendix, we want to show how our (reformulated) model provides a solution to the cash-flow observability problem in Stark (1987). Stark (1987,
pp. 100–102) showed that application of the Salamon (1982) approach to a firm with current assets (more generally non-depreciable assets) can lead to an incorrect calculation of the internal rate of return. We begin with Stark’s particular example and subsequently consider the general case. Note that in this example total assets equal equity so that the example applies equally well to the return on total assets case as to the return on equity case.

C.1. The Stark (1987) example

As in Stark (1987, pp. 101–102) let $n$, the project horizon, equal 2, $C_0^0 = 1,000$, $C_1^0 = 774.4643$, and $C_2^0 = 387$. Therefore, the internal rate of return, $r - 1$, equals 0.12. This example assumes that investment consists of 700 of depreciable assets and 300 of inventory. The source of investment is entirely equity. The inventory balance falls to 150 at date 1 and then to 0 at date 2. Therefore 150 of both $C_1^0$ and $C_2^0$ is due to the inventory balance drop, and it is this part of cash flow which can be unobservable since the balance sheet inventory number is a net number and does not show increases for some projects and decreases for other projects. Stark assumes no growth so that $g = 1$.

Stark (1987) applied Salamon (1982) calculations to this example from the point of view of an actual application where observable values must be used. The calculation of a firm’s cash recovery rate, $\rho$, is based on the equation $\rho = \frac{(\text{observable}) \text{ cash flow}}{\text{gross investment}}$. From date 1 onward 1400 is invested in depreciable assets, 450 is invested in inventory, so that 1850 is the total gross investment, all from equity investment. Observable cash flow from date 1 onward is $624.4643 + 237 = 861.4643$. Thus, the observed cash recovery rate from period 1 onward is $861.4643/1850 = 0.4654$. Stark (1987, p. 102) assumed that the parameter $b$, which in general is unobservable, is known. The parameter $b$ equals $237/624.4643 = 0.3795$. Since $g = 1$ we can use the formula

$$\rho = \frac{(1 - b^x)(r - b)}{n(1 - b)(r^n - b^n)}$$

from Salamon (1982, p. 297) to solve for $r$ and the answer is $r - 1 = -0.05$ rather than the correct value of 0.12. This concludes Stark (1987, pp. 101–102) important example which shows how current assets present a problem for the cash-based rate of return calculation based on Salamon (1982).

Now consider our (reformulated) model applied to the same example. As above, the calculation of the firm’s cash recovery rate must be based on observable values. Our model sets $\rho = \frac{(\text{observable cash flow - imputed capital charge})}{\text{gross depreciable assets}}$ - recall Eqs. (4) and (5). As above, from date 1 onward 1400 is invested in depreciable assets, 450 is invested in inventory, so that 1850 is the total (equity) investment. By Eq. (4), cash flow less the imputed capital charge from date 1 onward is $861.4643 - (r - 1) 450$ since the firm’s
equity investment is 1850 and the firm’s investment in gross depreciable assets is 1400. Therefore, the cash recovery rate is $$(861.4643 - (r - 1)450)/1400$$. Following Stark (1987, p. 105), we assume that the parameter $$b$$, which in general is unobservable, is known. The value of $$b$$ is $$(237 - 225(r - 1)) / (624.4643 - 225(r - 1))$$. We are in a position to solve for $$r$$ and the answer is $$r - 1 = 0.120746$$, which is quite close to the true value of 0.12. The small discrepancy between 0.12 and 0.120746 is explained by the fact that the Stark (1987, p. 103) example does not satisfy all the assumptions of our model since inventory, the non-depreciable asset, is partly redeployed before the end of the project.

C.2. The cash flow observability problem – general case

When non-depreciable assets cannot be redeployed until the end of the project then Theorem 1 proves that our model calculates the correct rate of return and hence solves the cash flow observability problem. The generality that Stark (1987, p. 103) allows over the model of our paper is that non-depreciable assets can be redeployed before the end of the project. Therefore, let $$C_{NA0}/C_0^0$$ be the fixed fraction of investment in non-depreciable assets. Also let $$C_{NAi}, i = 1, \ldots, n$$, be the investment (relative to $$C_{NA0}$$) in non-depreciable assets of dates after an investment. (For the Stark (1987, pp. 101–102) example $$C_0^0 = 1000, C_{NA0} = 300, C_{NA1} = 150$$ and $$C_{NA2} = 0.$$.) Following Stark (1987, p. 103) we assume that $$C_{NAi+1} > C_{NAi}$$ for $$i = 1, \ldots, n$$, and that $$C_{NA_n} = 0$$. The return on equity under the assumptions of the Stark model is the $$r$$ which satisfies:

$$-C_{0E} + \sum_{i=1}^{n} (C_i^0 + (C_{NAi-1} - C_{NAi}))r^{-i} - (C_{0D} + C_{0N})r^{-n} = 0. \quad (C.1)$$

Note that if $$C_{NAi} = C_{NAi-1}$$ for $$i = 1, \ldots, n$$, then (C.1) reduces Eq. (2). (In the Stark (1987, pp. 101–102) example $$C_{0E} = 1000, C_{0D} = C_{0N} = 0, C_0^1 = 624.4643$$ and $$C_0^2 = 237$$).

The model of our paper assumes no redeployment of non-depreciable assets before the end of the project. Let $$C_{NA}$$ (S for Stark) be the value of $$C_{NA}$$ which results in the same steady-state fraction of non-depreciable assets as do $$C_{NA0}, C_{NA1}, \ldots, C_{NA_{n-1}}$$, when the growth rate is $$g$$. $$C_{NA} = (g^{n-1}C_{NA0} + g^{n-2}C_{NA1} + \cdots + C_{NA_{n-1}})/(g^{n-1} + g^{n-2} + \cdots + 1)$$. (For the Stark (1987, p. 101) example $$g = 1$$ so that $$C_{NA} = 225$$). Applying the model of this paper for the Stark (1987) formulation means that it is assumed that $$C_{NA}$$ is set equal to $$C_{NA}$$.

(Referring to the Stark (1987) example, 0.120746 is the true return when the investment consists of 700 in depreciable assets and 225 in non-depreciable assets which cannot be redeployed until the end of the project.)

The calculated rate of return using the approach of our paper for the Stark (1987) formulation is the $$r$$ which satisfies
\[-C_{0E}^r + \sum_{i=1}^{n} C_i^0 r^{-i} + (C_{NA}^S - C_{0D} - C_{0N})r^{-n} = 0.\]  
(C.2)

The term \(-C_{0E}^r = -C_{0E} + (C_{NA0} - C_{NA}^S)\) where \(C_{0E}\) is the same term as in (C.1) and \((C_{NA0} - C_{NA}^S)\) is the reduction of initial investment when \(C_{NA}^S\) non-depreciable assets are employed rather than \(C_{NA0}\). (In the Stark (1987, pp. 101–102) example \(C_{0E} = 1000, C_{0E}^r = 925\) and \((C_{NA0} - C_{NA}^S) = 300 – 225 = 75\).

Theorem 3 assumes that \(g \leq r\). This is the more likely situation. It can be shown that case \(g > r\) implies that the firm’s net cash flow is negative at each date.

**Theorem 3.** Assume that \(g \leq r\). The calculated return on equity using the model of this paper (Eq. (C.2)) is always greater than or equal to the true return on equity when non-depreciable assets can be redeployed before the end of the project (Eq. (C.1)).

Proof of Theorem 3 uses the following

**Lemma 3.** Assume that \(g \leq r\). For \(1 \leq i \leq n – 1\),

\[(1 + g + \ldots + g^{n-1})(1 - r^{-i}) \geq (g^{n-i} + \ldots + g^{n-1})(r^{-i} - r^{-n}).\]

**Proof.** Since \((1 - r^{-i}) = (1 + r^{-1} + \ldots + r^{-(i-1)})(1 - r^{-1})\), and \((r^{-i} - r^{-n}) = (r^{-i} + r^{-(i+1)} + \ldots + r^{-(n-1)})(1 - r^{-1})\), proving Lemma 3 is equivalent to showing that \((1 + r^{-1} + \ldots + r^{-(i-1)})(g^{n-i-1} + g^{n-i-2} + \ldots + 1) \geq (g^{n-1} + g^{n-2} + \ldots + g^{n-i})(r^{-i} - r^{-(i+1)} + \ldots + r^{-(n-1)})\). It suffices to show that the jth term of \((1 + r^{-1} + \ldots + r^{-(i-1)}), l \leq j \leq i\), times the kth term of \((g^{n-i-1} + g^{n-i-2} + \ldots + 1), l \leq k \leq n - i\), is greater than or equal to the jth term of \((g^{n-1} + g^{n-2} + \ldots + g^{n-i})(r^{-i} + g^{n-i+2} + \ldots + g^{n-i} + \ldots + r^{-(n-1)})\).

That is, it suffices to show that \(r^{-j-1}g^{n-i-k} \geq g^{n-j}r^{-(i+k-1)}\). Since \(r^{-j-1}g^{n-i-k} = (rg^{-1})^{i+j-1} g^{n-j}r^{-(i+k-1)}\), the result follows using \(r \geq g, k \geq 1\) and \(i \geq j\).

**Proof of Theorem 3.** Comparing (C.2) with (C.1) it suffices to show that for any \(r\),

\[(C_{NA0} - C_{NA}^S) - \sum_{i=1}^{n} (C_{NAi-1} - C_{NAi})r^{-i} + C_{NA}^S r^{-n} \geq 0.\]  
(C.3)

Let \(A_i \equiv C_{NAi-1} - C_{NAi} \geq 0, i = 1, 2, \ldots, n\), and \(G \equiv 1 + g + \ldots + g^{n-1}\). Then, since \(C_{NA0} = 0, C_{NA0} = \sum_{i=1}^{n} A_i\). Similarly \(C_{NA}^S = A_n + ((g + g^2 + \ldots + g^{n-1})/G) \Delta_{n-1} + ((g^2 + \ldots + g^{n-1})/G) \Delta_{n-2} + \ldots + ((g^{n-i} + \ldots + g^{n-1})/G) \Delta_{i} + \ldots + (g^{n-1})/G) \Delta_{1}\), using the substitutions \(C_{NA} = \sum_{j=i+1}^{n} A_j\).
It follows that \( C_{NA0} - C_{NA}^S = \sum_{i=1}^{n-1} (A_i/G)(1 + g + \cdots + g^{n-1}) \), \( \sum_{i=1}^{n} (C_{NA_{i-1}} - C_{NA_i})r^{i-1} = \Delta_n r^{-n} + \sum_{i=1}^{n-1} (A_i/G)(1 + g + \cdots + g^{n-1})r^{-i} \), and \( C_{NA}^S r^{-n} = \Delta_n r^{-n} + \sum_{i=1}^{n-1} (A_i/G)(g^{n-1} + \cdots + g^{n-1})r^{-i} \). Therefore, (C.3) can be rewritten as

\[
\sum_{i=1}^{n-1} (A_i/G) \left[ (1 + g + \cdots + g^{n-1})(1 - r^{-i}) - (g^{n-1} + \cdots + g^{n-1})(r^{-i} - r^{-n}) \right] \geq 0.
\]

Now apply Lemma 3. \( \square \)

Recall that in the Stark (1987, pp. 101–102) example the upper bound is 0.120746 and the true value is 0.12. Stark (1987, p. 106) finds his example to be an extreme case of the cash observability problem compared to more realistic parameters. Therefore, the closeness of 0.120746 to 0.12 encourages one to speculate that the upper bound in Theorem 3 will be close to the true return in most cases of interest.

References


