Remarks on a generalized beta function

Allen R. Miller
1616 Eighteenth Street NW, Washington, DC 20009, USA
Received 20 March 1998

Abstract

We show that a certain generalized beta function \( B(x, y; b) \) which reduces to Euler’s beta functions \( B(x, y) \) when its variable \( b \) vanishes and preserves symmetry in its parameters may be represented in terms of a finite number of well known higher transcendental functions except (possibly) in the case when one of its parameters is an integer and the other is not. In the latter case \( B(x, y; b) \) may be represented as an infinite series of either Wittaker functions or Laguerre polynomials. As a byproduct of this investigation we deduce representations for several infinite series containing Wittaker functions, Laguerre polynomials, and products of both. © 1998 Elsevier Science B.V. All rights reserved.

AMS classification: 33B15; 33B99; 33C15; 33C45

Keywords: Euler’s beta function and its generalizations; Sums containing Wittaker functions and Laguerre polynomials

1. Introduction

Recently, Chaudhry et al. introduced a generalized beta function defined by the Euler-type integral,

\[
B(x, y; b) \equiv \int_0^1 t^{x-1}(1-t)^{y-1}e^{-b/(1-t)} \, dt, \quad (1.1)
\]

where \( \text{Re}(b) > 0 \) and the parameters \( x \) and \( y \) are arbitrary complex numbers. When the variable \( b \) vanishes, it is clear that for \( \text{Re}(x) > 0 \) and \( \text{Re}(y) > 0 \) the generalized function reduces to the ordinary beta function \( B(x, y) \) of classical analysis. The rationale and justification for defining this extension of the beta function are given in [1] where it is studied in some detail and a statistical application is given.

One of the purposes of the present investigation is to deduce further useful results for \( B(x, y; b) \) that provide for example some of its known properties in a more economical way than given previously. To this end we note here the integral representations (see [1, Eqs. (2.8) and (2.9)]) for
\( \text{Re}(b) > 0: \)

\[
B(x, y; b) = e^{-2b} \int_0^\infty (1 + t)^{-x-y} t^{s-1} e^{-b(t+1)/t} \, dt, \tag{1.2}
\]

\[
B(x, y; b) = 2^{1-x-y} \int_{-1}^1 (1 + t)^{s-1} (1 - t)^{-s} e^{-4b(1-t^2)} \, dt. \tag{1.3}
\]

By making the transformation \( t \to 1 - t \) in Eq. (1.1), it is not difficult to see that

\[
B(x, y; b) = B(y, x; b), \tag{1.4}
\]

where \( \text{Re}(b) > 0. \)

The Mellin transform of \( B(x, y; b) \) is easily computed (cf. [1, Eq. (5.2)]) and may be written as

\[
M_b(s) = \sqrt{\pi} 2^{1-x-y-2s} \frac{\Gamma(s)\Gamma(x+s)\Gamma(y+s)}{\Gamma\left(\frac{x+y+s}{2}\right)},
\]

where \( \text{Re}(s) > 0, \text{Re}(x) > 0, \text{Re}(y) > 0. \) Now taking the inverse transform of \( M_b(s) \) gives

\[
B(x, y; b) = \sqrt{\pi} 2^{1-x-y} \int_{-1}^1 \frac{\Gamma(s)\Gamma(x+s)\Gamma(y+s)}{\Gamma((x+y)/2+s)\Gamma((x+y+1)/2+s)} (4b)^{-s} \, ds,
\]

where \( \text{Re}(b) > 0 \) and an appropriate contour is chosen so that the integral exists. The latter result may be written simply in terms of Meijer’s \( G \)-function (see e.g. [5] for an introduction to the \( G \)-function and its properties)

\[
B(x, y; b) = \sqrt{\pi} 2^{1-x-y} G^{3,0}_{2,3} \left( 4b \left| \begin{array}{c} (x+y)/2, (x+y+1)/2 \\ 0, x, y \end{array} \right. \right), \tag{1.5}
\]

where \( \text{Re}(b) > 0. \) This result specialized with \( y = 1 \) has already been noted by Glasser [2], but apparently contains a typographical error.

If \( x = y, \) Eq. (1.5) immediately reduces to

\[
B(x, x; b) = \sqrt{\pi} 2^{1-2x} G^{2,0}_{1,2} \left( 4b \left| \begin{array}{c} x + \frac{1}{2} \\ 0, x \end{array} \right. \right),
\]

where the latter \( G \)-function can be identified with a Whittaker function (see e.g. [5, p. 129]) so that for \( \text{Re}(b) > 0 \)

\[
B(x, x; b) = \sqrt{\pi} 2^{-x} e^{-b(x-1)/2} W_{-x/2,x/2}(4b).
\]

This result and several other specializations of \( B(x, y; b) \) have already been noted in [1].

When no two members of the set \( \{0, x, y\} \) differ by an integer (positive, negative or zero), the \( G \)-function in Eq. (1.5) may be written in terms of three generalized hypergeometric functions (see e.g. [5, p. 131]). Thus, we have the important result

\[
B(x, y; \frac{1}{4}b) = \sqrt{\pi} 2^{1-x-y} \left( \frac{\Gamma(x)\Gamma(y)}{\Gamma\left(\frac{x+y}{2}\right)\Gamma\left(\frac{x+y+1}{2}\right)} \right) {}_2F_2 \left[ \begin{array}{c} 2-x-y, 1-x-y \\ 2, 2 \end{array} \right| \begin{array}{c} -b \\ 1-x, 1-y \end{array}; \right]
\]
\[ + \frac{\Gamma(-x)\Gamma(y-x)}{\Gamma(y-x)} b_2^{\gamma} F_2 \left[ \begin{array}{c} 2 + x - y, 1 + x - y \\ 2 \\ 1 + x, 1 + x - y \end{array} ; -b \right] \\
+ \frac{\Gamma(-y)\Gamma(x-y)}{\Gamma(x-y)} b_2^{\gamma} F_2 \left[ \begin{array}{c} 2 + y - x, 1 + y - x \\ 2 \\ 1 + y, 1 + y - x \end{array} ; -b \right], \tag{1.6} \]

where \( \text{Re}(b) > 0 \) and no two members of \( \{0, x, y\} \) differ by an integer.

Therefore, the two sections that follow shall be devoted for the most part to the cases where Eq. (1.6) is not valid. Thus, we shall see that \( B(x, y; b) \) may be expressed in terms of well-known higher transcendental functions except when exactly one of the parameters \( x \) and \( y \) is an integer. In the latter exceptional case \( B(x, y; b) \) may be expressed in various ways as either an infinite series of Whittaker functions or Laguerre polynomials (or both). Furthermore, in the exceptional case it is unlikely that a representation for \( B(x, y; b) \) by a function (or finite number of known functions) other than that given by Meijer’s \( G \)-function in Eq. (1.5) is possible.

2. Representations in terms of Whittaker functions

From Eq. (1.3) it is easily seen that

\[ B(x, y; b) = 2^{1-x-y} \int_{-1}^{1} (1 + t)^{y-1}(1 - t^2)^{y-1} e^{-4b(1-t^2)} \, dt. \]

Thus expanding the first term in the integrand as a convergent series in \( t \) whose radius of convergence is not greater than one we have

\[ B(x, y; b) = 2^{1-x-y} \sum_{k=0}^{\infty} (y - x)_k (-1)^k \frac{1}{k!} \int_{-1}^{1} t^k(1 - t^2)^{y-1} e^{-4b(1-t^2)} \, dt, \]

where the order of summation and integration have been interchanged. When the summation index is odd the latter integral vanishes since its integrand is an odd function so that clearly

\[ B(x, y; b) = 2^{2-x-y} \sum_{k=0}^{\infty} \frac{(y - x)_{2k}}{(1)_{2k}} \int_{0}^{1} t^{2k}(1 - t^2)^{y-1} e^{-4b(1-t^2)} \, dt. \]

An obvious transformation in the variable of integration then gives

\[ B(x, y; b) = 2^{1-x-y} \sum_{k=0}^{\infty} \frac{(y - x)_{2k}}{(1)_{2k}} \int_{0}^{1} t^{y-1}(1 - t)^{k-1/2} e^{-4b/t} \, dt. \tag{2.1} \]

Finally, for \( \text{Re}(\nu) > 0 \) and \( \text{Re}(\beta) > 0 \) noting that

\[ \int_{0}^{1} t^{\mu-1}(1 - t)^{\nu-1} e^{-\beta t} \, dt = \Gamma(\nu)\beta^{\mu-1/2} e^{-\beta/2} W_{(1-\mu-2\nu)/2,\beta/2}(\beta) \tag{2.2} \]
(see e.g. [3, Section 3.471(2)]), and using the Legendre duplication formula for \( \zeta \) we obtain for \( \Re(b) > 0 \)

\[
B(x, y; b) = \sqrt{\pi} e^{-2b} \sum_{k=0}^{\infty} \frac{((y-x)/2)_k((1+y-x)/2)_k}{k!} W_{-y/2-k,y/2}(4b). \tag{2.3a}
\]

Since the parameters \( x \) and \( y \) on either side of Eq. (2.3a) may be interchanged by invoking Eq. (1.4), we see also that for \( \Re(b) > 0 \)

\[
B(x, y; b) = \sqrt{\pi} e^{-2b} \sum_{k=0}^{\infty} \frac{((x-y)/2)_k((1+x-y)/2)_k}{k!} W_{-x/2-k,x/2}(4b). \tag{2.3b}
\]

Now when \( y = x + n(n = 0, 1, 2, \ldots) \) in Eq. (2.3b), the series of Wittaker functions terminates thus giving for \( \Re(b) > 0 \) and nonnegative integers \( n \)

\[
B(x, x + n; b) = \sqrt{\pi} e^{-2b} \sum_{k=0}^{\infty} \frac{(-n/2)_k((1-n/2)_k}{k!} W_{-(x+n)/2-k,(x+n)/2}(4b). \tag{2.4a}
\]

And by replacing \( x \) by \( x - n \) in the latter result we have for \( \Re(b) > 0 \) and nonnegative integers \( n \)

\[
B(x, x - n; b) = \sqrt{\pi} e^{-2b} \sum_{k=0}^{\infty} \frac{(-n/2)_k((1-n/2)_k}{k!} W_{-(x-n)/2-k,(x-n)/2}(4b). \tag{2.4b}
\]

For integers \( n \) (positive or negative) other representations for \( B(x, x + n; b) \) in terms of \( [1 + \frac{1}{2}|n|] \)

Wittaker functions may be derived. To obtain these we first observe that by interchanging the order of summation and integration in Eq. (2.1) and using the previously mentioned Legendre duplication formula, we deduce for \( \Re(b) > 0 \) the following integral representation:

\[
B(x, y; b) = 2^{1-x-y} \int_0^1 e^{-ab/2} F_1 \left[ \frac{y-x}{2}, \frac{1+y-x}{2}; \frac{1}{2} ; 1-t \right] t^{y-1}(1-t)^{-1/2} dt. \tag{2.5a}
\]

By invoking the symmetry of the parameters \( x \) and \( y \), Eq. (2.5a) may be written as

\[
B(x, y; b) = 2^{1-x-y} \int_0^1 e^{-ab/2} F_1 \left[ \frac{x-y}{2}, \frac{1+x-y}{2}; \frac{1}{2} ; 1-t \right] t^{x-1}(1-t)^{-1/2} dt, \tag{2.5b}
\]

where \( \Re(b) > 0 \).

Now setting \( y = x + n(n = 0, 1, 2, \ldots) \) in Eq. (2.5b) gives for \( \Re(b) > 0 \)

\[
B(x, x + n; b) = 2^{1-n-2x} \int_0^1 e^{-ab/2} F_1 \left[ \frac{n}{2}, \frac{1-n}{2}; \frac{1}{2} ; 1-t \right] t^{n-1}(1-t)^{-1/2} dt. \tag{2.6a}
\]

And setting \( y = x - n \) in Eq. (2.5a) yields for \( \Re(b) > 0 \)

\[
B(x, x - n; b) = 2^{1+n-2x} \int_0^1 e^{-ab/2} F_1 \left[ \frac{n}{2}, \frac{1-n}{2}; \frac{1}{2} ; 1-t \right] t^{n-1}(1-t)^{-1/2} dt. \tag{2.6b}
\]

Since (see [6, Vol. 3, Section 7.3.1 Eqs. (207) and (209)]) for \( n = 1, 2, 3, \ldots \)

\[
F_1 \left[ \frac{-n}{2}, \frac{1-n}{2}; \frac{1}{2} ; 1-t \right] = 2^{n-1} F_1 \left[ \frac{-n}{2}, \frac{1-n}{2}; \frac{1}{2} ; 1-t \right]
\]
(which is essentially an identity involving Chebyshev polynomials of the first kind) we have from Eqs. (2.6a) and (2.6b), respectively,

\[
B(x, x + n; b) = 2^{-2x} \int_0^1 e^{-4b/t} F_1 \left[ -\frac{n}{2}, -\frac{1 - n}{2}; 1 - n; t \right] t^{x-1}(1 - t)^{-1/2} dt \]  

(2.7a)

and

\[
B(x, x - n; b) = 2^{2n-2x} \int_0^1 e^{-4b/t} F_1 \left[ -\frac{n}{2}, -\frac{1 - n}{2}; 1 - n; t \right] t^{x-n-1}(1 - t)^{-1/2} dt, \]  

(2.7b)

where \( n \) is a positive integer and \( \text{Re}(b) > 0 \).

The Gaussian function in Eqs. (2.7a) and (2.7b) is a polynomial in \( t \) of degree \( [n/2] \), so that clearly we may write, respectively,

\[
B(x, x + n; b) = 2^{-2x} \sum_{k=0}^{[n/2]} \left( -\frac{n}{2} \right)_k \left( \frac{1 - n}{2} \right)_k \int_0^1 t^{x+k-1}(1 - t)^{-1/2} e^{-4b/t} dt \]  

(2.8a)

and

\[
B(x, x - n; b) = 2^{2n-2x} \sum_{k=0}^{[n/2]} \left( -\frac{n}{2} \right)_k \left( \frac{1 - n}{2} \right)_k \int_0^1 t^{x-n+k-1}(1 - t)^{-1/2} e^{-4b/t} dt, \]  

(2.8b)

where \( n \) is a positive integer and \( \text{Re}(b) > 0 \).

The integrals in Eqs. (2.8) may be evaluated by again utilizing Eq. (2.2); thus we deduce for positive integers \( n \) and \( \text{Re}(b) > 0 \)

\[
B(x, x + n; b) = \sqrt{\pi} e^{-2b} 2^{-x-1} b^{(x-1)/2} \sum_{k=0}^{[n/2]} \left( -\frac{n}{2} \right)_k ((1 - n)/2)_k \frac{(2\sqrt{b})^k}{k!} W_{-(x+k)/2,(x+k)/2}(4b) \]  

(2.9a)

and

\[
B(x, x - n; b) = \sqrt{\pi} e^{-2b} 2^{n-x-1} b^{(n-x)/2} \sum_{k=0}^{[n/2]} \left( -\frac{n}{2} \right)_k \left( \frac{1 - n}{2} \right)_k \frac{(2\sqrt{b})^k}{k!} W_{-(x+k-n)/2,(x+k-n)/2}(4b). \]  

(2.9b)

Since

\[
\frac{(-n/2)_k ((1 - n)/2)_k}{(1 - n)_k k!} \frac{(-1)^k 2^{2k}}{k!} = \frac{n}{k} \left( \begin{array}{c} n - k \\ k \end{array} \right),
\]

Eqs. (2.9a) and (2.9b) may be written, respectively, as

\[
B(x, x + n; b) = \frac{1}{2} n \sqrt{\pi} e^{-2b} 2^{-x} b^{(x-1)/2} \sum_{k=0}^{[n/2]} \left( -\frac{1}{2} \sqrt{b} \right)_k \left( \begin{array}{c} n - k \\ k \end{array} \right) W_{-(x+k)/2,(x+k)/2}(4b) \]  

(2.10a)
and
\[
B(x, x - n; b) = \frac{1}{2} n \sqrt{\pi} e^{-b} 2^{n-1} \frac{1}{b} (x-\frac{n}{2})^{n-1/2} \\
\times \sum_{k=0}^{\left\lfloor n/2 \right\rfloor} \left( \frac{-\frac{\sqrt{b}}{2}}{n - k} \right)^k \left( \frac{n - k}{k} \right) W_{-(n+k-x)/2} \left( \frac{1}{2} - \frac{k}{x} \right) (4b),
\]

(2.10b)

where \( n \) is a positive integer and \( \text{Re}(b) > 0 \). Eq. (2.10a) has been obtained heretofore by a method which uses recurrences and induction (cf. [1, Theorem 4.4]).

Since
\[
\left( \frac{x - y}{2k} \right) = \left( \frac{(y - x)/2}{k} \right) (1 + y - x)/2_k \frac{1}{(k)} k!
\]

we see that Eqs. (2.3a) and (2.3b) may be written, respectively, as
\[
B(x, y; b) = e^{-b} 2^{-x+y} \sum_{k=0}^{\infty} \left( \frac{x - y}{2k} \right) \Gamma\left( \frac{1}{2} + k \right) W_{-(y/2-k,y)/2} (4b)
\]

(2.11a)

and
\[
B(x, y; b) = e^{-b} 2^{-y} \sum_{k=0}^{\infty} \left( \frac{y - x}{2k} \right) \Gamma\left( \frac{1}{2} + k \right) W_{-(x/2-k,x)/2} (4b)
\]

(2.11b)

where \( \text{Re}(b) > 0 \). When \( y = 1 \), Eqs. (2.11) yield respectively the specializations
\[
B(x, 1; b) = e^{-b} 2^{-x} \sum_{k=0}^{\infty} \left( \frac{x - 1}{2k} \right) \Gamma\left( \frac{1}{2} + k \right) W_{-(1/2-k,1)/2} (4b)
\]

(2.12a)

and
\[
B(x, 1; b) = \frac{1}{2} e^{-b} 2^{-y} \sum_{k=0}^{\infty} \left( \frac{1 - x}{2k} \right) \Gamma\left( \frac{1}{2} + k \right) W_{-(x/2-k,x)/2} (4b)
\]

(2.12b)

where \( \text{Re}(b) > 0 \).

Now letting \( x = n + 1 \) in the former result and \( x = 1 - n \) in the latter, we have for nonnegative integers \( n \) and \( \text{Re}(b) > 0 \)
\[
B(1 + n, 1; b) = e^{-b} 2^{-n-1} \sum_{k=0}^{\left\lfloor n/2 \right\rfloor} \left( \frac{n}{2k} \right) \Gamma\left( \frac{1}{2} + k \right) W_{-(1/2-k,1)/2} (4b)
\]

(2.13a)

and
\[
B(1 - n, 1; b) = \frac{1}{2} e^{-b} 2^{-n} \sum_{k=0}^{\left\lfloor n/2 \right\rfloor} \left( \frac{n}{2k} \right) \Gamma\left( \frac{1}{2} + k \right) W_{-(1-n)/2-k(1-n)/2} (4b).
\]

(2.13b)

Glasser [2] has obtained a result for \( B(x, 1; b) \) which does not agree with either of Eqs. (2.12) and appears to be erroneous.
3. Additional representations for $B(x, y; b)$

It has previously been noted that for nonnegative integers $n$ the specialization $B(x, -x - n; b)$ may be represented in terms of $n + 1$ Macdonald functions (see [1, Theorem 4.3]):

$$B(x, -x - n; b) = 2e^{-2b} \sum_{k=0}^{n} \binom{n}{k} K_{x+k}(2b)$$

(3.1)

where Re$(b) > 0$. As in the case of Eq. (2.10a) the latter result was obtained in [1] mutatis mutandis by using recurrences and induction. However, Eq. (3.1) may be obtained quickly by setting $y = -x - n$ in Eq. (1.2), and expanding $(1 + t)^n$ by means of the binomial theorem thereby yielding

$$B(x, -x - n; b) = e^{-2b} \sum_{k=0}^{n} \binom{n}{k} \int_{0}^{\infty} t^{x+k-1} e^{-b(1+t)/t} \, dt.$$ 

The latter integral is evaluated by using for example [6, Vol. 1, Section 2.3.16(1)] thus immediately giving Eq. (3.1). Note that when $y = -x$ in Eq. (1.5), we obtain

$$B(x, y; b) = 2\sqrt{\pi} G_{2,0}^{1,2} \left( 4b \left| \begin{array}{c} 1/2 \\ x, -x \end{array} \right. \right)$$

where the $G$-function can be identified with a Macdonald function (see e.g. [5, p. 149, Example 3.9]) thus giving Eq. (3.1) for $n = 0$.

Other representations for $B(x, y; b)$ are possible. Indeed, we shall show for Re$(b) > 0$, Re$(x) > -1$, Re$(y) > -1$ that

$$B(x, y; b) = e^{-2b} \sum_{m,n=0}^{\infty} B(x + m + 1, y + n + 1) L_m(b) L_n(b)$$

(3.2)

where the simple Laguerre polynomials $L_n(b) \equiv L_n^{(0)}(b)$ ($n = 0, 1, 2, \ldots$) are given by the generating relation (see e.g. [7, p. 202, Eq. (4)])

$$e^{-bt/(1-t)} = (1 - t)^{1+z} \sum_{n=0}^{\infty} L_n^{(0)}(b)t^n, \quad |t| < 1.$$ 

(3.3)

If $t$ is replaced by $1 - t$ and $z$ set to zero in Eq. (3.3), then clearly

$$e^{-b/(1-t)} = e^{-b} \sum_{n=0}^{\infty} L_n(b) (1 - t)^n$$

(3.4a)

and again replacing $t$ by $1 - t$ in the latter gives

$$e^{-b/(1-t)} = e^{-b} (1 - t) \sum_{m=0}^{\infty} L_m(b)t^m.$$ 

(3.4b)

Since

$$e^{-b/(1-t)} e^{-1/t} = e^{-b/(1-t)}$$

(3.4c)
it is easy to see by multiplying the respective sides of Eqs. (3.4a) and (3.4b) that
\[
e^{-\frac{b}{m}} = e^{-2b} \sum_{m,n=0}^{\infty} L_m(b)L_n(b)t^{m+1}(1-t)^{n+1}.
\]
(3.5)

Now using the latter together with Eq. (1.1) upon integrating term by term we obtain for \(\text{Re}(b) > 0\)
\[
B(x, y; b) = e^{-2b} \sum_{m,n=0}^{\infty} L_m(b)L_n(b) \int_0^1 t^{m+n}(1-t)^{y+n} \, dt.
\]
(3.6)

Since for \(\text{Re}(x) > -1, \text{Re}(y) > -1\) the integral in Eq. (3.6) is equal to \(B(x + m + 1, y + n + 1)\),
the result given by Eq. (3.2) follows.

The double sum on the right-hand side of Eq. (3.2) may be reduced to a single sum (see Eq. (3.11)) by employing the following summation formula for Laguerre polynomials. For \(x > 0\) and
\[-1 < \text{Re}(z) < 2 \text{Re}(c - a) - \frac{1}{2} \]
\[
\sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} L_n^{(c)}(x) = \frac{\Gamma(c)}{\Gamma(c-a)} x^{(c-a-z-2)/2} e^{-x/2} W_{(z+2-a-c)/2(c-a-z-1)/2}(x).
\]
(3.7)

To prove this we first note the integral representation for Laguerre polynomials
\[
L_n^{(c)}(x) = \frac{x^{-z/2}e^x}{n!} \int_0^\infty e^{-t} t^{n+z/2} J_a(2\sqrt{xt}) \, dt,
\]
(3.8)

where \(n\) is a nonnegative integer and \(\text{Re}(z) > -1\) (cf. [4, p. 243] and [6, Vol. 2, Section 2.12.9 (3)]). Now multiplying both sides of Eq. (3.8) by \((a)_n/(c)_n\), summing the result over the nonnegative integers \(n\), and then interchanging the order of summation and integration we have
\[
\sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} L_n^{(c)}(x) = x^{-z/2} e^x \int_0^\infty e^{-t} t^{n+z/2} _1F_1[a; c; t] J_a(2\sqrt{xt}) \, dt,
\]
(3.9)

where \(\text{Re}(z) > -1\).

After applying Kummer’s first transformation formula to the latter confluent function, and then making the change of variable of integration \(t = x^2\), the right side of Eq. (3.9) may be written as
\[
2x^{-z/2} e^x \int_0^{\infty} s^{z+1} _1F_1[c - a; c; -s^2] J_a(2\sqrt{xs}) \, ds.
\]

Finally, employing [3, Section 7.663(6)] to evaluate the latter integral we obtain Eq. (3.7).

If the Wittaker function in Eq. (3.7) is written in terms of two confluent functions \(_1F_1[x]\) (see e.g. [8, Eq. (1.9.10)]), a known result is obtained which, since it is given by Prudnikov et al. [6, Vol. 2, Section 5.11.1(8)] with a typographical error, we now record. Thus, we have
\[
\sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} L_n^{(c)}(x) = \frac{\Gamma(c)}{\Gamma(a)} \Gamma(a - c + 1) x^{c-a-z-1} \frac{1}{1F_1\left[\begin{array}{c}c - a - 1; \\
\end{array}x\right]} \left[\begin{array}{c}c - a - z; \\
\end{array}x\right] \\
+ \frac{\Gamma(c)}{\Gamma(c - a)} \frac{\Gamma(c - a - \alpha - 1)}{\Gamma(c - \alpha - 1)} \frac{1}{1F_1\left[\begin{array}{c}c - a - \alpha - 1; \\
\end{array}x\right]} \left[\begin{array}{c}a - c + \alpha + 2; \\
\end{array}x\right],
\]
(3.10)
where $x > 0$ and $-1 < \text{Re}(x) < 2\text{Re}(c - a) - \frac{1}{2}$.

After a brief computation it is easy to see that Eq. (3.2) may be written as

$$B(x, y; b) = e^{-\frac{1}{2}b} \frac{\Gamma(x + 1)\Gamma(y + 1)}{\Gamma(x + y + 2)} \sum_{m=0}^{\infty} \frac{(x + 1)_m}{(x + y + 2)_m} L_m(b)$$

$$\times \sum_{n=0}^{\infty} \frac{(y + 1)_n}{(x + y + 2 + m)_n} L_n(b),$$

where $\text{Re}(b) > 0$. The $n$-summation in the latter result may now be computed when $b > 0$, $\text{Re}(x) > -3/4$ (and by symmetry $\text{Re}(y) > -3/4$) by setting $a = y + 1, c = x + y + 2 + m, z = 0$ in Eq. (3.7) thus giving

$$B(x, y; b) = e^{-\frac{1}{2}b} \frac{\Gamma(y + 1)b^{(1-1/2)}e^{-(3/2)b}}{\Gamma(x + y + 2)} \sum_{m=0}^{\infty} b^{m/2} L_m(b) W_{-(y+1/2)-(x+m)/2}(x+y+m/2).$$

(3.11)

As we shall see shortly the latter result may be obtained much more quickly. Moreover, it holds under the weaker conditional inequalities $\text{Re}(x) > -1, \text{Re}(y) > -1,$ and $\text{Re}(b) > 0$. The circuitous derivation just given was employed to deduce Eq. (3.11) because it provided an opportunity to derive Eqs. (3.7) and (3.10) which evidently are of independent interest.

Eqs. (3.4b) and (3.4c) give for $|t| < 1$

$$e^{-b/(1-t)} = e^{-b(1+1/t)}(1-t) \sum_{m=0}^{\infty} L_m(b)t^m.$$  

Now using this in Eq. (1.1) we have for $\text{Re}(b) > 0$

$$B(x, y; b) = e^{-b} \sum_{m=0}^{\infty} L_m(b) \int_0^1 t^{x+m-1}(1-t)^y e^{-b/t} dt,$$  

(3.12)

where the summation and integral have been interchanged. Thus once again employing Eq. (2.2) we obtain Eq. (3.11) where $\text{Re}(y) > -1$ and by symmetry $\text{Re}(x) > -1$.

4. Infinite series containing Wittaker functions and Laguerre polynomials

From Eqs. (2.3b), (3.2), and (3.11), respectively, we have

$$\sum_{m=0}^{\infty} \frac{(x - y)/2)_m(1 + x - y)/2)_m}{m!} W_{-(y+1/2)-(x+m)/2}(b) = \frac{2^{x+y-1}}{\sqrt{\pi}} b^{(1-x)/2} e^{b/2} B(x, y; (1/4)b),$$

where $\text{Re}(b) > 0$ and

$$\sum_{m,n=0}^{\infty} B(x + m + 1, y + n + 1)L_m(b)L_n(b) = e^{2b} B(x, y; b)$$

$$\sum_{m=0}^{\infty} b^2 L_m(b) W_{-(y+1/2)-(x+m)/2}(x+y+m/2) = \frac{b^{(1-x)/2} e^{(3/2)b}}{\Gamma(y + 1)} B(x, y; b),$$
where \( \text{Re}(b) > 0, \text{Re}(x) > -1, \text{Re}(y) > -1 \). Thus, we may use any of the other results for the generalized function \( B(x, y; b) \) obtained in the previous sections to obtain evaluations for the infinite series in the latter three equations. The most general of these is given by Eq. (1.5) so that we have

\[
\sum_{m=0}^{\infty} \frac{(x - y)/2)_m(1 + x - y)/2)_m}{m!} W_{-x/2-m,y/2}(b) = b^{1-x/2} e^{b/2} G_{2,3}^{3,0} \left( b \left| \frac{(x + y)/2, (x + y + 1)/2}{0, x, y} \right. \right) \tag{4.1}
\]

where \( b > 0 \) and

\[
\sum_{m,n=0}^{\infty} B(x + m + 1, y + n + 1)L_m(b)L_n(b) = e^{2b} G_{2,3}^{3,0} \left( 4b \left| \frac{(x + y)/2, (x + y + 1)/2}{0, x, y} \right. \right) \tag{4.2}
\]

\[
\sum_{m=0}^{\infty} b^{2m} L_m(b) W_{-(y+1/2)-(x+m)/2,(x+m)/2}(b) = \sqrt{\pi} 2^{1-x-y} e^{2b} G_{2,3}^{3,0} \left( 4b \left| \frac{(x + y)/2, (x + y + 1)/2}{0, x, y} \right. \right) \tag{4.3}
\]

where \( b > 0, \text{Re}(x) > -1, \text{Re}(y) > -1 \). Eqs. (4.1)–(4.3) may also be viewed as giving series expansions in terms of either Wittaker functions or Laguerre polynomials for a specialization of Meijer’s \( G \)-function \( G_{2,3}^{3,0} \).

References