Sobolev orthogonality for the Gegenbauer polynomials \( \{C_n^{(-N+1/2)}\}_{n \geq 0} \)

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Abstract

In this work, we obtain the property of Sobolev orthogonality for the Gegenbauer polynomials \( \{C_n^{(-N+1/2)}\}_{n \geq 0} \), with \( N \geq 1 \) a given nonnegative integer, that is, we show that they are orthogonal with respect to some inner product involving derivatives. The Sobolev orthogonality can be used to recover properties of these Gegenbauer polynomials. For instance, we can obtain linear relations with the classical Gegenbauer polynomials.

Keywords: Gegenbauer polynomials; Nondiagonal Sobolev inner product; Sobolev orthogonal polynomials

1. Introduction

For any value of the parameter \( \lambda > -\frac{1}{2} \) monic Gegenbauer polynomials \( \{C_n^{(\lambda)}\}_{n \geq 0} \) can be defined by their explicit representation (see [7, p. 84])

\[
C_n^{(\lambda)}(x) = \frac{n!}{2^n \Gamma(n + \lambda)} \sum_{m=0}^{[n/2]} (-1)^m \frac{\Gamma(n - m + \lambda)}{m!(n - 2m)!} (2x)^{n-2m}, \quad \lambda > -\frac{1}{2}.
\]

Simplifying the values of the Gamma function in this expression we get

\[
C_n^{(\lambda)}(x) = \frac{n!}{2^n} \sum_{m=0}^{[n/2]} \frac{(-1)^m}{(n + \lambda - m)_m m!} (2x)^{n-2m},
\]

(1)

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where \((n + \lambda - m)_m\) denotes the usual Pochhammer symbol defined by
\[
(b)_0 = 1, \quad (b)_n = b(b + 1) \cdots (b + n - 1), \quad b \in \mathbb{R}, \quad \forall n \geq 1.
\]
In this way, we must notice that expression (1) is valid for every real value of the parameter \(\lambda \neq -l\), with \(l = 0, 1, 2, \ldots\) and, therefore, it can be used to define Gegenbauer polynomials for \(\lambda \neq 0, -1, -2, \ldots\).

From the explicit representation (1) we can deduce that the monic Gegenbauer polynomials \(\{C_n^{(\lambda)}\}_{n \geq 0}\) satisfy, for every real value of \(\lambda \neq 0, -1, -2, \ldots\), the three-term recurrence relation
\[
xC_n^{(\lambda)}(x) = C_{n+1}^{(\lambda)}(x) + \gamma_n^{(\lambda)} C_{n-1}^{(\lambda)}(x), \quad n \geq 0,
\]
where
\[
\gamma_n^{(\lambda)} = \frac{n(2n + 2\lambda - 1)}{4(n + \lambda)(n + \lambda - 1)}.
\]
Whenever \(\lambda \neq -\frac{1}{2}, -\frac{3}{2}, \ldots\), we have \(\gamma_n^{(\lambda)} \neq 0\) for all \(n \geq 1\), and Favard’s Theorem (see [2, p. 21]) ensures that the sequence \(\{C_n^{(\lambda)}\}_{n \geq 0}\) is orthogonal with respect to a quasi-definite linear functional. Besides, if \(-\frac{1}{2} < \lambda\) the functional is positive definite and the polynomials are orthogonal with respect to the weight \((1 - x^2)^{\lambda-1/2}\) on the interval \([-1, 1]\). For \(\lambda = -\frac{1}{2}, -\frac{3}{2}, \ldots\), since \(\gamma_n^{(\lambda)}\) vanishes for some value of \(n\), no orthogonality results can be deduced from Favard’s Theorem.

The main objective of this paper is to obtain orthogonality properties for the Gegenbauer polynomials \(\{C_n^{(-N+1/2)}\}_{n \geq 0}\). In fact, we are going to show that they are orthogonal with respect to an inner product involving derivatives, that is, a Sobolev inner product.

Similar results for different families of classical polynomials have been the subject of an increasing number of papers. For instance, Kwon and Littlejohn, in [3], established the orthogonality of the generalized Laguerre polynomials \(\{L_n^{(-k)}\}_{n \geq 0}, \ k \geq 1\), with respect to a Sobolev inner product of the form
\[
\langle f, g \rangle = (f(0), f'(0), \ldots, f^{(k-1)}(0)) A \left( \begin{array}{c} g(0) \\ g'(0) \\ \vdots \\ g^{(k-1)}(0) \end{array} \right) + \int_0^{+\infty} f^{(k)}(x) g^{(k)}(x) e^{-x} \, dx
\]
with \(A\) a symmetric \(k \times k\) real matrix. In [4], the same authors showed that Jacobi polynomials \(\{P_n^{(-1, -1)}\}_{n \geq 0}\), are orthogonal with respect to the inner product
\[
(f, g)_1 = d_1 f(1)g(1) + d_2 f(-1)g(-1) + \int_{-1}^{1} f'(x)g'(x) \, dx,
\]
where \(d_1\) and \(d_2\) are real numbers.
Later, in [5], Pérez and Piñar gave a unified approach to the orthogonality of the generalized Laguerre polynomials \( \{L_n^{(\alpha)}\}_{n \geq 0} \), for any real value of the parameter \( \alpha \) by proving their orthogonality with respect to a Sobolev nondiagonal inner product. In [6], they showed how to use this orthogonality to obtain different properties of the generalized Laguerre polynomials.

Alfaro et al., in [1], studied sequences of polynomials which are orthogonal with respect to a Sobolev bilinear form defined by

\[
B_s^{(N)}(f, g) = (f(c), f'(c), \ldots, f^{(N-1)}(c))A \begin{pmatrix} g(c) \\ g'(c) \\ \vdots \\ g^{(N-1)}(c) \end{pmatrix} + \langle u, f^{(N)} g^{(N)} \rangle,
\]

where \( u \) is a quasi-definite linear functional on the linear space \( \mathbb{P} \) of real polynomials, \( c \) is a real number, \( N \) is a positive integer number, and \( A \) is a symmetric \( N \times N \) real matrix such that each of its principal submatrices is regular. In particular, they deduced that the Jacobi polynomials \( \{P_n^{(\alpha, \beta)}\}_{n \geq 0} \), for \( \beta + N \) not a negative integer are orthogonal with respect to (3). In this case, \( I = [-1, 1] \), \( u \) is the Jacobi functional defined from the weight function \( u^{(0, \beta+N)}(x) = (1 + x)^{\beta+N} \), \( c = 1 \) and \( A = Q^{-1} D (Q^{-1})^T \), where \( D \) is a regular diagonal matrix and \( Q \) is the matrix of the derivatives of the Jacobi polynomials at the point 1.

Now, we are going to describe the structure of the paper. In Section 2, from the explicit representation of the monic Gegenbauer polynomials, \( \{C_n^{(\lambda)}\}_{n \geq 0} \), for \( \lambda \in \mathbb{R}, \lambda \neq 0, -1, -2, \ldots \) we deduce some of their usual properties, namely the three-term recurrence relation, the differentiation property, the second order differential equation, \ldots .

Next, in Section 3, we show that the Gegenbauer polynomials \( \{C_n^{(-N+1/2)}\}_{n \geq 0} \), are orthogonal with respect to the Sobolev inner product defined by

\[
(f, g)_s = (f(1), f'(1), \ldots, f^{(N-1)}(1), f(-1), f'(-1), \ldots, f^{(N-1)}(-1))A \begin{pmatrix} g(1) \\ g'(1) \\ \vdots \\ g^{(N-1)}(1) \\ g(-1) \\ g'(-1) \\ \vdots \\ g^{(N-1)}(-1) \end{pmatrix} + \int_{-1}^{1} f^{(2N)}(x)g^{(2N)}(x)(1 - x^2)^N \, dx,
\]

where \( f \) and \( g \) are two arbitrary polynomials and \( N \geq 1 \) is a positive integer number.

The last section of the paper is devoted to the study of a linear differential operator, \( \mathcal{F} \), which is defined on the linear space of the real polynomials \( \mathbb{P} \), and is symmetric with respect to the Sobolev inner product (4). By using this operator, we can establish several linear relations between Gegenbauer polynomials \( \{C_n^{(-N+1/2)}\}_{n \geq 0} \) and the classical Gegenbauer polynomials \( \{C_n^{(-N+1/2)}\}_{n \geq 0} \).
2. The Gegenbauer polynomials

For a given real number \( \lambda > -\frac{1}{2} \), the explicit representation of the \( n \)th monic Gegenbauer polynomial is given by

\[
C_n^{(\lambda)}(x) = \frac{n!}{2^n \Gamma(n + \lambda)} \sum_{m=0}^{[n/2]} \frac{(-1)^m \Gamma(n - m + \lambda)}{m!(n - 2m)!} (2x)^{n-2m}, \quad n \geq 0,
\]

(see [7, p. 84]). As it is well known, for \( \lambda > -\frac{1}{2} \) the sequence \( \{C_n^{(\lambda)}\}_{n \geq 0} \) constitutes a monic orthogonal polynomial sequence associated with the weight function \( \rho(x) = (1 - x^2)^{\lambda-1/2} \) on the interval \([-1, 1]\).

After simplification, expression (5) reads

\[
C_n^{(\lambda)}(x) = \frac{n!}{2^n} \sum_{m=0}^{[n/2]} \frac{(-1)^m}{\lambda + n - m} m!(n - 2m)! (2x)^{n-2m}, \quad n \geq 0.
\]

We can observe that for every value of the parameter \( \lambda \neq -l, \ l = 0, 1, 2, \ldots \), expression (6) defines a monic polynomial of exact degree \( n \). In this way, for \( \lambda \neq -l, \ l = 0, 1, 2, \ldots \) we can define a family of monic polynomials \( \{C_n^{(\lambda)}\}_{n \geq 0} \) which is a basis of the linear space of the real polynomials. From now on, these polynomials will be called generalized Gegenbauer polynomials.

Very simple manipulations of the explicit representation show that the main part of the algebraic properties of the classical Gegenbauer polynomials holds for every value of the parameter \( \lambda \neq -l, \ l = 0, 1, 2, \ldots \), as we show in the following proposition.

**Proposition 1.** Let \( \lambda \) be an arbitrary real number with \( \lambda \neq -l, \ l = 0, 1, 2, \ldots \). Then, the Gegenbauer polynomials \( \{C_n^{(\lambda)}\}_{n \geq 0} \) satisfy the following properties:

(i) **Symmetry**

\[
C_n^{(\lambda)}(-x) = (-1)^n C_n^{(\lambda)}(x).
\]

(ii) **Three-term recurrence relation**

\[
C_{n-1}^{(\lambda)}(x) = 0, \quad C_0^{(\lambda)}(x) = 1,
\]

\[
x C_n^{(\lambda)}(x) = C_{n+1}^{(\lambda)}(x) + \gamma_n^{(\lambda)} C_{n-1}^{(\lambda)}(x), \quad n \geq 0,
\]

where

\[
\gamma_n^{(\lambda)} = \frac{n(n + 2\lambda - 1)}{4(n + \lambda)(n + \lambda - 1)}.
\]

(iii) **Differentiation property**

\[
D C_n^{(\lambda)}(x) = n C_n^{(\lambda+1)}(x),
\]

where \( D = d/dx \).
(iv) Second-order differential equation

\[(1 - x^2)y'' - (2\lambda + 1)xy' + n(n + 2\lambda)y = 0,\]

where \(y = C_n^{(\lambda)}(x)\).

We can observe that the parameter \(\gamma_n^{(\lambda)}\) in the three-term recurrence relation does not vanish for every value of \(\lambda \neq -N + \frac{1}{2},\) for \(N = 1, 2, 3, \ldots\). Thus, from Favard’s Theorem (see [2, p. 21]) we deduce that the generalized Gegenbauer polynomials \(\{C_n^{(\lambda)}\}_{n \geq 0}\), \(\lambda \neq -N + \frac{1}{2}\) for \(N = 1, 2, 3, \ldots\), are orthogonal with respect to a certain moment functional. For \(\lambda > -\frac{1}{2}\) this moment functional is positive definite. However, since \(\gamma_n^{(-N+1/2)} = 0\), no orthogonality properties can be deduced from Favard’s Theorem for the sequence of polynomials \(\{C_n^{(-N+1/2)}\}_{n \geq 0}\) for \(N = 1, 2, 3, \ldots\).

In order to obtain orthogonality properties for the Gegenbauer polynomials \(\{C_n^{(-N+1/2)}\}_{n \geq 0}\), for \(N = 1, 2, 3, \ldots\), we have to emphasize some of their interesting properties.

**Proposition 2.** Let \(N\) be a given positive integer number, then the monic Gegenbauer polynomials \(\{C_n^{(-N+1/2)}\}_{n \geq 0}\) satisfy

(i) \(C_n^{(-N+1/2)}(-1) = (-1)^n C_n^{(-N+1/2)}(1),\)

(ii) \(C_n^{(-N+1/2)}(1) = \left(\frac{-2N + n}{-2N + 2n - 1}\right) C_{n-1}^{(-N+1/2)}(1), \quad n \geq 1,\)

(iii) \(C_n^{(-N+1/2)}(1) = \frac{(-2N + 1)_n}{2^n(-N + \frac{1}{2})_n}, \quad n \geq 0,\)

(iv) \(C_n^{(-N+1/2)}(1) = C_n^{(-N+1/2)}(-1) = 0 \quad \text{for} \ n \geq 2N,\)

(v) \(D^k C_n^{(-N+1/2)}(1) = \frac{n!}{(n-k)!} \frac{(-2N + 2k + 1)_{n-k}}{2^{n-k}(-N + k + \frac{1}{2})_{n-k}} \quad \text{for} \ n \geq k,\)

(vi) \(D^k C_n^{(-N+1/2)}(1) = D^k C_n^{(-N+1/2)}(-1) = 0 \quad \text{for} \ 0 \leq k \leq N, \ n \geq 2N,\)

(vii) \(C_{2N}^{(-N+1/2)}(x) = (x^2 - 1)^N\) and

(viii) for \(n \geq 2N,\) the following identity holds:

\(C_n^{(-N+1/2)}(x) = (x^2 - 1)^N C_{n-2N}^{(N+1/2)}(x).\)
3. Sobolev orthogonality for \( \{C_n^{(-N+1/2)}\}_{n \geq 0} \)

Let \( N \geq 1 \) be a given integer number. Let \( A \) be a symmetric and positive definite matrix of order \( 2N \). The expression

\[
(f, g)_S = (F(1)|F(-1))A(G(1)|G(-1))^T + \int_{-1}^{1} f^{(2N)}(x)g^{(2N)}(x)(1 - x^2)^N\, dx,
\]

where

\[
(F(1)|F(-1)) = (f(1), f'(1), \ldots, f^{(N-1)}(1), f(-1), f'(-1), \ldots, f^{(N-1)}(-1)),
\]

defines an inner product on the linear space of polynomials \( \mathbb{P} \). Our aim in this section is to show that it is possible to find a matrix \( A \) such that the sequence of Gegenbauer polynomials \( \{C_n^{(-N+1/2)}\}_{n \geq 0} \) is orthogonal with respect to the Sobolev inner product (11).

**Proposition 3.** Let \( N \geq 1 \) be a given integer number. There exists a symmetric and positive-definite matrix \( A \) such that the sequence \( \{C_n^{(-N+1/2)}\}_{n \geq 0} \) is orthogonal with respect to the Sobolev inner product defined by (11).

**Proof.** From Proposition 2(iv), we have

\[
C_n^{(-N+1/2)}(1) = C_n^{(-N+1/2)}(-1) = 0
\]

for \( n \geq 2N \). On the other hand, from the differentiation property (9), we deduce that

\[
D^{2N}C_n^{(-N+1/2)}(x) = \frac{n!}{(n - 2N)!} C_n^{(N+1/2)}(x)
\]

for \( n \geq 2N \). Therefore, the polynomial \( C_n^{(-N+1/2)} \), for \( n \geq 2N \), is orthogonal to the linear space \( \mathbb{P}_{n-1} \) with respect to a Sobolev inner product defined by means of (11), for every symmetric and positive-definite matrix \( A \).

In this way, it only remains to construct a symmetric and positive definite matrix \( A \) providing orthogonality of the first \( 2N \) Gegenbauer polynomials \( \{C_n^{(-N+1/2)}\}_{n=0, \ldots, 2N-1} \). With this aim we are going to construct a regular matrix of order \( 2N \), that we will denote by \( Q \), whose elements are obtained from the values of the derivatives of the Gegenbauer polynomials at the points \(-1\) and \(1\). Namely,

\[
Q = (Q(1)|Q(-1))_{2N \times 2N},
\]

where

\[
Q(1) = (D^kC_n^{(-N+1/2)}(1))_{n=0, \ldots, 2N-1, \ldots, 0, \ldots N},
\]

\[
Q(-1) = (D^kC_n^{(-N+1/2)}(-1))_{n=0, \ldots, 2N-1, \ldots, 0, \ldots N}.
\]
The matrix $Q$ is regular since it can be expressed as a product of two regular matrices. Indeed, let us expand the polynomials $\{C_n^{(-N+1/2)}\}_{n \geq 0}$ in powers of $x$

$$C_n^{(-N+1/2)}(x) = \sum_{j=0}^{n} p_{n,j} x^j, \quad p_{n,n} \neq 0, \quad \forall n \geq 0.$$  

(12)

Let $P$ be the regular matrix constructed with the coefficients $p_{n,j}$, for $0 \leq j \leq n$ and $0 \leq n \leq 2N - 1$

$$P = \begin{pmatrix} p_{0,0} & 0 & \ldots & 0 \\ p_{1,0} & p_{1,1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p_{2N-1,0} & p_{2N-1,1} & \ldots & p_{2N-1,2N-1} \end{pmatrix}.$$  

Let us denote by $H(x)$ the matrix obtained from the first $N$ derivatives of the elements of the Hamel basis of the linear space $\mathbb{P}_{2N-1}$, that is

$$H(x) = (D^k(x^j))_{j=0,\ldots,2N-1, \atop k=0,\ldots,N-1},$$

and denote by $V$ the square matrix obtained from $H(x)$ as follows:

$$V = (H(1)|H(-1)).$$

$V$ is a generalized Vandermonde matrix and, therefore, regular. From Eq. (12) we deduce that $Q = P V$, and therefore we conclude the regularity of $Q$.

For a given diagonal matrix $D$, of order $2N$ and positive elements in the diagonal, we define the matrix $A$ of the inner product (11) as follows:

$$A = Q^{-1} D (Q^{-1})^T.$$  

As we can easily check, $A$ is a positive definite and symmetric matrix and the sequence of polynomials $\{C_n^{(-N+1/2)}\}_{n \geq 0}$ is orthogonal with respect to the inner product (11).

4. The linear operator $\mathcal{F}$

In this section, we will consider a linear operator denoted by $\mathcal{F}$, defined on the space of real polynomials $\mathbb{P}$, which is symmetric with respect to the Sobolev inner product defined in (11). From this operator, we can deduce the existence of several relationships between the sequences of Gegenbauer polynomials $\{C_n^{(-N+1/2)}\}_{n \geq 0}$ and $\{C_n^{(N+1/2)}\}_{n \geq 0}$.

We define the linear operator $\mathcal{F}$ by means of the following expression:

$$\mathcal{F} = (1 - x^2)^N D^{2N}[(1 - x^2)^N D^{2N}],$$

(13)

where $D$ denotes the derivative operator. In this way, $\mathcal{F}$ is a $4N$th-order differential operator that vanishes for every polynomial of degree less than or equal to $2N - 1$. 

Our next result shows that operator $F$ allows us to obtain a representation of the nondiagonal Sobolev inner product (11) in terms of the inner product associated with the weight function $\rho^{(N)}(x) = (1 - x^2)^N$.

**Proposition 4.** Let $f$ and $g$ be two arbitrary polynomials, then

$$((1 - x^2)^{2N} f, g)_S = \int_{-1}^{1} f(x) F g(x) (1 - x^2)^N \, dx.$$ 

**Proof.** Since 1 and $-1$ are zeros of multiplicity $N$ of the polynomial $(1 - x^2)^N$, we have

$$((1 - x^2)^{2N} f, g)_S = \int_{-1}^{1} D^{2N}[(1 - x^2)^{2N} f(x)] D^{2N} g(x) (1 - x^2)^N \, dx.$$ 

Integrating by parts $2N$ times, we deduce

$$((1 - x^2)^{2N} f, g)_S = \int_{-1}^{1} (1 - x^2)^{2N} f(1 - x^2)^N \, dx.$$ 

Theorem 5. The linear operator $F$ is symmetric with respect to the Sobolev inner product (11), i.e.,

$$(F f, g)_S = (f, F g)_S.$$ 

**Proof.** From the definition of the Sobolev inner product we get

$$(F f, g)_S = \int_{-1}^{1} D^{2N} [F f(x)] D^{2N} g(x) (1 - x^2)^N \, dx$$

$$= \int_{-1}^{1} D^{2N} [(1 - x^2)^N D^{2N} f(x)] D^{2N} g(x) (1 - x^2)^N \, dx.$$ 

Integrating by parts $2N$ times, we obtain

$$(F f, g)_S = \int_{-1}^{1} (1 - x^2)^N D^{2N} [(1 - x^2)^N D^{2N} f(x)] D^{2N} g(x) (1 - x^2)^N \, dx.$$ 

As we can see, this last expression is symmetric in $f$ and $g$, and the result follows interchanging their roles. □

A direct computation shows that the linear operator $F$ preserves the degree of the polynomials, in fact we have
Proposition 6. For \( n \geq 2N \)

\[
\mathcal{F} x^n = \left( \frac{n!}{(n-2N)!} \right)^2 x^n + \text{lower degree terms.}
\]

As a consequence of Proposition 6, we can deduce that the monic Gegenbauer polynomials \( C_n^{(-N+1/2)} \) are the eigenfunctions of the linear operator \( \mathcal{F} \).

Proposition 7. For \( n \geq 2N \), we have

\[
\mathcal{F} C_n^{(-N+1/2)}(x) = \left( \frac{n!}{(n-2N)!} \right)^2 C_n^{(-N+1/2)}(x).
\]

Proof. Since \( \mathcal{F} C_n^{(-N+1/2)} \) is a polynomial of exact degree \( n \), it can be expressed as a linear combination of the polynomials \( \{ C_i^{(-N+1/2)} \}_{i=0}^{n} \), that means

\[
\mathcal{F} C_n^{(-N+1/2)}(x) = \sum_{i=0}^{n} \gamma_{n,i} C_i^{(-N+1/2)}(x). \tag{14}
\]

The coefficients \( \gamma_{n,i} \) can be obtained from Theorem 5 in the following way:

\[
\gamma_{n,i} = \frac{\mathcal{F} C_n^{(-N+1/2)}(x), C_i^{(-N+1/2)}}{C_i^{(-N+1/2)}, C_i^{(-N+1/2)}} = \frac{C_n^{(-N+1/2)}(x), \mathcal{F} C_i^{(-N+1/2)}}{C_i^{(-N+1/2)}, C_i^{(-N+1/2)}}
\]

Finally, from the orthogonality of the polynomial \( C_n^{(-N+1/2)} \), we deduce that \( \gamma_{n,i} = 0 \) for \( i < n \), and the result follows by simple inspection of the leading coefficient on both sides of (14). \( \square \)

Next, we can deduce some interesting relations involving Gegenbauer polynomials.

Proposition 8. The following relations hold:

(i) \((1 - x^2)^{2N} C_n^{(N+1/2)}(x) = \sum_{i=0}^{n+4N} \alpha_{n,i} C_i^{(-N+1/2)}(x), \quad n \geq 0, \tag{15}\)

(ii) \(\mathcal{F} C_n^{(-N+1/2)}(x) = \sum_{i=n-4N}^{n} \beta_{n,i} C_i^{(N+1/2)}(x), \quad n \geq 4N. \tag{16}\)

Proof. (i) If we expand the polynomial \((1 - x^2)^{2N} C_n^{(N+1/2)}(x)\) in terms of the Sobolev orthogonal polynomials \( \{ C_i^{(-N+1/2)} \}_{i \geq 0} \), we get

\[
(1 - x^2)^{2N} C_n^{(N+1/2)}(x) = \sum_{i=0}^{n+4N} \alpha_{n,i} C_i^{(-N+1/2)}(x).
\]
From Proposition 4, we can deduce the value of the coefficients \( \alpha_{n,i} \)

\[
\alpha_{n,i} = \frac{((1 - x^2)^{2N} C^{(N+1/2)}_n, C^{(-N+1/2)}_i)}{(C^{(-N+1/2)}_i, C^{(-N+1/2)}_i)_S},
\]

and from the orthogonality of the polynomial \( C^{(N+1/2)}_n \), we conclude that \( \alpha_{n,i} = 0 \), for \( 0 \leq i \leq n - 1 \).

(ii) If we expand the polynomial \( F C^{(-N+1/2)}_n \) in terms of the Gegenbauer polynomials \( \{C^{(N+1/2)}_i\}_{i \in \mathbb{N}} \), we get

\[
F C^{(-N+1/2)}_n(x) = \sum_{i=0}^{n} \beta_{n,i} C^{(N+1/2)}_i(x).
\]

Again, the coefficients \( \beta_{n,i} \) can be computed from Proposition 4 as follows:

\[
\beta_{n,i} = \frac{\int_{-1}^{1} C^{(N+1/2)}_i(x) F C^{(-N+1/2)}_n(x)(1 - x^2)^N \, dx}{\int_{-1}^{1} C^{(N+1/2)}_i(x) C^{(-N+1/2)}_n(x)(1 - x^2)^N \, dx}
\]

and from the orthogonality of the polynomial \( C^{(-N+1/2)}_n \), with respect to the Sobolev inner product, we conclude that \( \beta_{n,i} = 0 \) for \( 0 \leq i < n - 4N \). \( \square \)

References