Numerical solution of stochastic differential-algebraic equations with applications to transient noise simulation of microelectronic circuits

O. Schein\textsuperscript{a,*}, G. Denk\textsuperscript{b}

\textsuperscript{a} Technische \textit{Universität} Darmstadt, Fachbereich Mathematik, AG Stochastik und Operations Research, Schloßgartenstraße 7, D-64289 Darmstadt, Germany
\textsuperscript{b} Siemens AG, Corporate Technology, D-81730 München, Germany

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Abstract

The transient simulation of noise in electronic circuits leads to differential-algebraic equations, additively disturbed by white noise. For these systems, we present a mathematical model based on the theory of stochastic differential equations, along with an implicit two-step method for their numerical treatment. This numerical scheme works directly on the given structure of the equations which makes very efficient implementations possible. The order of convergence is preserved. The theoretical results are verified by numerical noise simulations of benchmark circuits. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

The theory of stochastic differential equations (SDEs) has proven to be the proper approach for modeling and solving ordinary differential equations (ODEs) disturbed by a white noise process (see Section 2.2). However, in various fields of applications including mechanical multibody systems, control theory, chemical engineering or network simulation, one is not confronted with ODEs, but rather with differential equations on manifolds, also called differential-algebraic equations or DAEs (see Section 2.1). In the deterministic setting, much research has been devoted to these descriptor systems [4, 10]. Yet modeling or numerically solving SDAEs, their stochastically disturbed
counterparts, remains an open problem, even though they are of great importance whenever DAEs arise in engineering science.

Arguably the most striking example is given by the computer-aided design of electronic circuits. Their numerical simulation with packages like SPICE [17] or TITAN [8] is well established [5]. The increasing scale of integration and the decrease of the supply voltages leads to high gain coupled with high signal-to-noise ratio. It becomes necessary to include the nondeterministic nature of charge conduction in the simulation [1, 16]. The classical noise analysis in the frequency domain exhibits some limitations, only a simulation in the time domain gives sufficient information to the designer [6]. As shown in [20], the time-domain simulation of thermal and shot noise, using the well-known modified nodal analysis (MNA) [5, 11] together with suitable stochastic noise models for the respective circuit elements, yields additively disturbed, quasilinear-implicit DAEs of the general form

$$C(x(t)) \cdot \dot{x}(t) + f(x(t)) + s(t) + B(t,x(t)) \cdot v(\omega, t) = 0,$$

where $v$ is an $m$-dimensional vector of white noise and $f: \mathbb{R}^d \to \mathbb{R}^d$ and $s: \mathbb{R} \to \mathbb{R}^d$ are nonlinear functions, respectively. Furthermore, for all real numbers $t$, the intensity matrix $B(t,x(t))$ is an element of $\text{Hom}(\mathbb{R}^m, \mathbb{R}^d)$, where $m$ denotes the number of (uncorrelated) noise sources (see Section 4.1). The capacitance matrix $C(x(t)) \in L(\mathbb{R}^d)$ has rank $r \leq d$. It is assumed that the rank of $C(x(t))$ is constant for the simulation interval. If $r < d$ holds, the corresponding deterministic system has an index $\mu > 0$.

For the index-0 case, the $d$-dimensional solution vector $x(t)$ could be modeled as an Itô-process with respect to classic integration theory of SDEs [3, 18]. However, even in this simple setting, the standard numerical solution techniques will fail, since they are merely constructed for explicit SDEs [14]. Demir [6] has proposed an approach for disturbed DAEs, i.e., for the case $\mu > 0$, which does, unfortunately, neglect the important algebraic noise component (see Section 2.3). Moreover, his method is based on manual index reduction and thus prohibitive for general CAD purposes.

For transferable linear-implicit DAEs with additive noise (see Section 2.3), we will propose a mathematical model based on the theory of SDEs and generalized stochastic processes, by decoupling the system with canonical projectors [15, 21]. The two-step scheme derived in Section 3 directly exploits the given implicit system structure, while obtaining the same (strong) order of convergence for the differential component of the solution as in the explicit case. The simulation results of our algorithm (see Section 4) confirm its efficiency and indicate that it can be of great use in the non-linear case, too.

2. Mathematical foundation

2.1. Linear differential-algebraic equations and canonical projectors

Differential-algebraic equations, i.e., implicit ODE systems $F(\dot{x}(t), x(t), t) = 0$, where $F_x$ is singular and of constant rank for all argument values, differ from implicit ODEs in many ways [4, 10]. Especially the determination of hidden algebraic constraints, which require consistent initial values to lie on solution manifolds, has been investigated intensively (see the references in [4]). Canonical
projectors have been proposed for decoupling transferable DAEs [15, 21]. We will illustrate this method by considering the best-known class, namely linear-implicit DAEs of the form

\[ C \cdot \dot{x}(t) + G \cdot x(t) + s(t) = 0, \tag{1} \]

with constant coefficients \( C, G \in L(\mathbb{R}^d) \) and \( N := \ker(C) \neq \{0\} \). The DAE (1) is called normal, since the nullspace \( N \) of \( C \) is constant.

Let \( Q \) be a projector function \( Q \in L(\mathbb{R}^d) \) such that \( Q^2 = Q \) and \( \text{im}(Q) = N \) are fulfilled. Introducing the projector \( P := I - Q \), the relations \( QP = PQ \equiv 0 \) and \( CP \equiv C \) hold. In the following, we shall assume that the matrix \( D \) defined by \( D := C + GQ \) has full rank, which means that (1) has tractability index \( \mu = 1 \) [21]. Recall that this is equivalent to a regular matrix pencil \( \{C,G\} \) of index 1 or to the condition \( N \cap S = \{0\} \), with \( S \) defined by \( S := \{y \in \mathbb{R}^d : G \cdot y \in \text{im}(C)\} \). In this case, the subspace \( S \) is filled by solutions of the homogeneous system

\[ C \cdot \dot{x}(t) + G \cdot x(t) = 0. \]

Multiplying (1) by \( PD^{-1} \) and \( QD^{-1} \), respectively, the system is transformed into the explicit ODE

\[ P \cdot \dot{x}(t) + PD^{-1}GP \cdot x(t) + PD^{-1} \cdot s(t) = 0 \tag{2} \]

for the differential solution component \( P \cdot x(\cdot) \) and the constraint system

\[ Q \cdot x(t) + QD^{-1}GP \cdot x(t) + QD^{-1} \cdot s(t) = 0, \tag{3} \]

since, obviously, the properties \( D^{-1}C = P \) and \( D^{-1}G = Q + D^{-1}GP \) hold. It is easily checked that \( \hat{Q} := QD^{-1}G \) is again a projector onto \( N \), but with \( \ker(\hat{Q}) = S \). Along with \( \hat{P} := I - \hat{Q} \), these so-called canonical index-1 projectors completely decouple (1), because (3) now simplifies to

\[ \hat{Q} \cdot x(t) + \hat{Q}D^{-1} \cdot s(t) = 0. \tag{4} \]

Moreover, since it is now obvious that only the differential component \( \hat{P} \cdot x(\cdot) \) of the solution vector has to be differentiable, Eq. (1) should be stated more precisely in the form

\[ C \cdot \frac{d}{dt}(\hat{P} \cdot x(t)) + G \cdot x(t) + s(t) = 0, \]

in which the minimal smoothness requirements become obvious: the solution vector has to belong to \( \mathcal{C}^1_N := \{x(\cdot) \in \mathcal{C} : \hat{P} \cdot x(\cdot) \in \mathcal{C}^1\} \). Consequently, (2) should then read

\[ \frac{d}{dt}(\hat{P} \cdot x(t)) + \hat{P}D^{-1}GP \cdot x(t) + \hat{P}D^{-1} \cdot s(t) = 0. \]

Note that only the differential component of a consistent initial value for (1) can be chosen arbitrarily in \( \text{im}(\hat{P}) \), since the algebraic part has to fulfill (4). For \( \mu > 1 \), DAEs represent inherent differentiation problems, which also can be extracted by using special projector chains [15]. The difficulties expected when solving a DAE by numerical integration are closely related to its index.

2.2. Stochastic differential equations

The theory of SDEs [3, 19] models disturbed ODEs of the type

\[ \dot{x}(t) = f(t, x(t)) + g(t, x(t)) \cdot \nu(\omega, t), \]
where the driving process \( \{v_t; \, t \geq t_0\} \) is generalized white noise [9], as the Itô-process

\[
X_t : \Omega \to \mathbb{R}, \quad \omega \mapsto \int_{[t_0,t]} f(u,X_u(\omega)) \, d\lambda(u) + \left( \int_{t_0}^t g(u,X_u) \, dW_u \right)(\omega), \quad (5)
\]

or, symbolically,

\[
dX_t = f(t,X_t) \, dt + g(t,X_t) \, dW_t.
\]

This mathematical model is based upon a stochastic integration theory with regard to a Wiener–Hopf process, namely the Itô-calculus first introduced in [13]. In analogy to the Riemann–Stieltjes integral, the Itô-integral is defined in a straightforward way for so-called elementary processes as a random variable. This concept is then extended to a larger class of processes as a limit in the \( \mathcal{L}^2 \)-sense [19]. Existence and (path-wise stochastic) uniqueness of solutions can be established, if a Lipschitz property is fulfilled. Stochastic Taylor formulae, adequate convergence concepts and numerical methods for the efficient solution of (5) have been derived, giving rise to applications in all areas of science [14]. A family of two-step methods for strong approximation of vector valued SDEs is reviewed in Section 3.2

2.3. Stochastic differential-algebraic equations

Efficient numerical methods have been derived for the solution of explicit SDEs with additive noise (see, for instance, [7]). We will present a model for the important class of linear-implicit DAEs with additive noise, arising for example in the simulation of linear circuits under the influence of thermal noise [20]. We will restrict ourselves to the index-1 case (transferable DAEs) and use the projector method reviewed in Section 2.1 for their decoupling.

**Definition 1.** A disturbed linear-implicit DAE is said to have additive noise, if it is of the type

\[
C \cdot \frac{d}{dt}(\hat{P} \cdot x)(t) + G \cdot x(t) + s(t) + B(t) \cdot v(\omega,t) = 0, \quad (6)
\]

with \( C, \ G \) and \( B \) belonging to \( L(\mathbb{R}^d) \) and \( \mathcal{C}(\mathbb{R}, \text{Hom}(\mathbb{R}^m, \mathbb{R}^d)) \), respectively, \( \hat{P} \) being the canonical index-1 projector and \( v \) denoting an \( m \)-dimensional vector of white noise. It is called transferable, if its deterministic subsystem

\[
C \cdot \frac{d}{dt}(\hat{P} \cdot x)(t) + G \cdot x(t) + s(t) = 0
\]

has tractability index 1.

Let us drop the argument \( t \) for the sake of readability in the next argument. If (6) is transferable, it can completely be decoupled in analogy to Section 2.1 into

\[
\frac{d}{dt}(\hat{P} \cdot x) + \hat{P}D^{-1}G\hat{P} \cdot x + \hat{P}D^{-1} \cdot s + \hat{P}D^{-1}B \cdot v(\omega,t) = 0 \quad (7)
\]

and

\[
\hat{Q} \cdot x + \hat{Q}D^{-1} \cdot s + \hat{Q}D^{-1}B \cdot v(\omega,t) = 0. \quad (8)
\]
Under the slight assumption of measurability, by (7) we can interpret the differential solution component $y(t) := \tilde{P} \cdot x(t)$ of the disturbed DAE as the process solving the underlying stochastic ordinary differential equation

$$dY_t = -\{\tilde{P}D^{-1}G \cdot Y_t + \tilde{P}D^{-1} \cdot s(t)\} \, dt - \tilde{P}D^{-1}B(t) \, dW_t$$

in the Itô sense (cf. [20]). For a given initial value, the solution process $\{Y_t\}$ exists, can be given explicitly and is (stochastically path-wise) unique, since, obviously, linear SDEs do fulfill the Lipschitz property [14].

Let $C^\infty_c$ denote the set of all test functions in the sense of distribution theory [9]. White noise is canonically modeled as a generalized Gaussian stochastic process $\{\Phi_\phi; \, \phi \in C^\infty_c\}$ with zero mean functional and covariance functional

$$C^\prime : C^\infty_c \times C^\infty_c \to \mathbb{R}; \quad C^\prime(\phi, \psi) = \int_{-\infty}^{\infty} \phi(t) \psi(t) \, d\lambda(t)$$

(see, for instance, [3]). Consequently modeling $z(\cdot)$ as a generalized stochastic process $\{Z_\phi; \, \phi \in C^\infty_c\}$, denoting the finite support of $\phi$ by $[t_0, t_1]$ and using (8), we can establish an approximate representation for the algebraic component $z(\cdot) := \tilde{Q} \cdot x(\cdot)$ [20], namely given by

$$Z_\phi : \Omega \to \mathbb{R}, \quad \omega \mapsto \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} -\phi(t)\{\tilde{Q}D^{-1} \cdot s(t) + \tilde{Q}D^{-1}B(t) \cdot \Phi_\phi(\omega)\} \, d\lambda(t).$$

The basic idea behind our model is similar in spirit to the deterministic setting, where the solution components have to fulfill different smoothness requirements (cf. Section 2.1). The differential component is modeled as an Itô-process, the algebraic part, however, which must not be neglected, is interpreted as a generalized stochastic process in the sense of distribution theory.

3. Derivation and analysis of numerical solution method

3.1. Numerical treatment of differential-algebraic equations

The numerical simulation of technical systems often requires the numerical solution of DAEs. Though it is theoretically possible to extract the underlying ODE of transferable DAEs (cf. Section 2.1), it is not advisable to do so in practice. This transformation would introduce numerical instability and would destroy the inherent structure of the DAE. Therefore, it is necessary to construct numerical schemes which are built directly on the given form and reflect the technical background of the DAE. But the transformation can be used for the development and the theoretical investigation of numerical schemes.

Up to now, no standard scheme exists for all different types of DAEs. This is partly due to the index of the DAE which indicates the difficulties of the numerical solution. Depending on the index and with structural assumptions about the system, it is possible to derive specialized numerical schemes. In many cases the modeling of technical systems leads to index-1 DAEs and for this type both theory and schemes are well established. Similar to the numerical analysis of ODEs, multistep
3.2. Numerical treatment of stochastic differential equations

Let a general explicit $d$-dimensional SDE
\[ dX_t = f(s,X_t) \, ds + g(s,X_t) \, dW_t, \quad s \in [0, T] \] (9)
be given, together with an $m$-dimensional Wiener-process \{\(W_t\)\} and the two measurable functions \(f : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d\) and \(g : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times m}\). The standard method for the numerical treatment of (9) is the path-wise simulation of a discrete approximation process \(\{\tilde{X}_{\tau_i}; i = 0, \ldots, N\}\), where
\[ 0 = \tau_0 < \tau_1 < \cdots < \tau_n < \cdots < \tau_N = T \]
is an equidistant discretization of \([0, T]\) into parts of length \(h := T/N\). In [14], a family of implicit two-step schemes is suggested, for which the \(k\)th component of \(\tilde{X}_{\tau_{n+1}}\) for \(n = 0, \ldots, N - 1\) is given by
\[
\tilde{X}^k_{\tau_{n+1}} = (1 - \gamma_k)\tilde{X}^k_{\tau_n} + \gamma_k \tilde{X}^k_{\tau_{n-1}} + h\left[\alpha_{2,k} f^k_{n+1} + (\gamma_k \alpha_{1,k} + (1 - \alpha_{2,k})) f^k_n + \gamma_k (1 - \alpha_{1,k}) f^k_{n-1}\right]
+ V^k_n + \gamma_k V^k_{n-1},
\] (10)
where \(f^k_l\) stands for the \(k\)th component of \(f(\tau_l, \tilde{X}_l)\), and \(\alpha_{1,k}, \alpha_{2,k}\) and \(\gamma_k\) are real parameters from \([0, 1]\). Furthermore, \(V^k_n\) is given by
\[
V^k_n = \sum_{j=1}^{m} g^{k,j} \Delta W^j_n + \sum_{j_1,j_2=1}^{m} L^{j_1} g^{k,j_1,j_2} I_{(j_1,j_2), \tau_n, \tau_{n+1}}
\] with \(g^{k,j}\) denoting the \((k, j)\) component of \(g(\tau_n, X_n)\), the Wiener increments \(W^j_n\) and the operator \(L^{j}\) defined by
\[
\Delta W^j_n := W^j_{\tau_{n+1}} - W^j_{\tau_n} \quad \text{and} \quad L^{j} = \sum_{k=1}^{d} g^{k,j} \frac{\partial}{\partial x^k}, \quad j = 1, \ldots, m.
\]
The multiple Itô-integral \(I_{(j_1,j_2), \tau_n, \tau_{n+1}}\) is given by
\[
I_{(j_1,j_2), \tau_n, \tau_{n+1}} = \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{\tau_{n+1}} dW^j_{s_1} \, dW^j_{s_2}.
\]
For additive noise \(g(t,x) \equiv g(t)\) and constant parameters \(\alpha_{1,k}, \alpha_{2,k}\) and \(\gamma_k\) for all \(k\), scheme (10) in vector form degenerates to
\[
\tilde{X}^k_{\tau_{n+1}} = (1 - \gamma)\tilde{X}^k_{\tau_n} + \gamma \tilde{X}^k_{\tau_{n-1}} + h\left[\alpha_{2} f^k_{n+1} + (\gamma \alpha_1 + (1 - \alpha_2)) f^k_n + \gamma (1 - \alpha_1) f^k_{n-1}\right]
+ g \cdot \Delta W^k_n + \gamma g \cdot \Delta W^k_{n-1},
\] (11)
The cumbersome approximation of the multiple Itô-integrals can thus be avoided [14]. The components of \(\Delta W^k_n\) are normally distributed with expectation 0 and variance \(h\) and are simulated with pseudo-random numbers.
Remark 2. For $\gamma = 0$ we have a family of Runge–Kutta type methods for SDEs, $\varepsilon_2 \neq 0$ yields implicit schemes. Implicit methods require solving an algebraic equation at each time step, for example with a damped Newton method. This increases stability significantly [14]. If $\gamma = 0$ and $g \equiv 0$, (11) degenerates to a family of deterministic Euler-schemes.

The concept of strong convergence is based upon the minimization of the absolute error of the path-wise difference in the mean-square sense.

Definition 3 (Order of strong convergence). A method which assigns an approximation process $\{\tilde{X}_n\}$ to a given natural number $N$ is said to converge strongly to $\{X_\tau\}$ with order $p \in (0, \infty]$, if there exists a constant $K$, such that

$$E(X_T - \tilde{X}_\tau) \leq Kh^p$$

holds, where $h := T/N$ denotes the step size.

Proposition 4. Methods (10) and (11) have strong order of convergence 1.

Proof. By application of the stochastic Taylor expansion (see [14]). □

3.3. A numerical method for stochastic differential-algebraic equations

We will now adapt the method reviewed in the previous section for linear-implicit DAE with additive noise, namely equations of the type

$$C \cdot \frac{d}{dt}(\tilde{P} \cdot x)(t) + G \cdot x(t) + s(t) + B(t) \cdot \nu(\omega, t) = 0. \tag{12}$$

Keeping the general assumptions of Section 2 with regard to transferability and measurability, let us assume first that the deterministic descriptor system

$$C \cdot \frac{d}{dt}(\tilde{P} \cdot x)(t) + G \cdot x(t) + s(t) = 0$$

is an implicit ODE, i.e., has tractability index $\mu = 0$. Since in this case $\tilde{P} = \text{id}$ holds and the matrix $C^{-1}$ exists, we can interpret (12) as the SDE

$$dX_t = -C^{-1} \cdot (G \cdot X_t + s(t)) \, dt - C^{-1} B(t) \, dW_t.$$

The application of Method (11) yields

$$\tilde{X}_{t_{n+1}} = (1 - \gamma)\tilde{X}_{t_n} + \gamma \tilde{X}_{t_{n-1}} - h[\varepsilon_2 C^{-1} \cdot (G \cdot \tilde{X}_{t_{n+1}} + s(\tau_{n+1}))$$

$$+ (\gamma \varepsilon_1 + (1 - \varepsilon_2)) C^{-1} \cdot (G \cdot \tilde{X}_{t_n} + s(\tau_n)) + \gamma(1 - \varepsilon_1) C^{-1} \cdot (G \cdot \tilde{X}_{t_{n-1}} + s(\tau_{n-1}))]$$

$$- C^{-1} B(\tau_n) \cdot \Delta W_n - \gamma C^{-1} B(\tau_n) \cdot \Delta W_{n-1},$$
Multiplying both sides by \(-C\) and regrouping some terms, we obtain

\[
- (C + h\varepsilon_2 G) \cdot \tilde{X}_{t_n} = \left[ (\gamma - 1)C + h(\gamma \varepsilon_1 + (1 - \varepsilon_2))G \right] \cdot \tilde{X}_{t_n} \\
+ \left[ -\gamma C + h(\gamma(1 - \varepsilon_1))G \right] \cdot \tilde{X}_{t_{n-1}} \\
+ h\varepsilon_2 s(\tau_{n+1}) + h(\gamma \varepsilon_1 + (1 - \varepsilon_2))s(\tau_n) \\
+ h(\gamma(1 - \varepsilon_1))s(\tau_n + 1) + B(\tau_n) \cdot (A W_n + \gamma A W_{n-1}),
\]

(13)

where for a given \(\varepsilon\), the regularity of the matrix pencil \(\{C, G\}\) ensures the regularity of the matrix \(-(C + h\varepsilon_2 G)\) for almost all \(h\). Therefore, (13) is an implementable method for the index 0 case, which works directly with the implicit system structure and reaches the same strong order of convergence 1.

We will now show that (13) is well-adapted for transferable DAEs, also:

**Theorem 5.** Suppose \(\mu = 1\). For the differential solution component of (12) (i.e., for the underlying SDE), the strong order of convergence of Method (13) is still 1.

**Proof.** By decoupling (12) with canonical projectors, the differential solution component of the system can be interpreted as the solution of the SDE

\[
dY_t = -(\hat{P}^{-1}G \cdot Y_t + \hat{P}^{-1} \cdot s(t)) \, dt - \hat{P}^{-1}B(t) \, dW_t
\]

(see Section 2.3). Application of (11) yields

\[
\begin{align*}
\dot{Y}_{t_n} = (1 - \gamma) \dot{Y}_{t_n} + h(\gamma \varepsilon_1 + (1 - \varepsilon_2))\hat{P}^{-1} \cdot (G \cdot \dot{Y}_{t_{n+1}} + s(\tau_{n+1})) \\
+ (\gamma \varepsilon_1 + (1 - \varepsilon_2))\hat{P}^{-1} \cdot (G \cdot \dot{Y}_{t_n} + s(\tau_n)) + \gamma(1 - \varepsilon_1)\hat{P}^{-1} \cdot (G \cdot \dot{Y}_{t_{n-1}} + s(\tau_n - 1)) \\
- \hat{P}^{-1}B(\tau_n) \cdot A W_n - \gamma \hat{P}^{-1}B(\tau_n) \cdot A W_{n-1}.
\end{align*}
\]

(14)

Conversely, denoting \(\hat{P} \cdot \dot{X}_t\) by \(\dot{Y}_t\) and multiplying (13) by \(\hat{P}^{-1}\) and \(\hat{Q}D^{-1}\), respectively, we can directly decouple the method, obtaining

\[
- (I + h\varepsilon_2 \hat{P}^{-1}G) \cdot \dot{Y}_{t_n} = \left[ (\gamma - 1)I + h(\gamma \varepsilon_1 + (1 - \varepsilon_2))\hat{P}^{-1}G \right] \cdot \dot{Y}_{t_n} \\
+ \left[ -\gamma I + h(\gamma(1 - \varepsilon_1))\hat{P}^{-1}G \right] \cdot \dot{Y}_{t_{n-1}} \\
+ h\varepsilon_2 \hat{P}^{-1} \cdot s(\tau_{n+1}) + h(\gamma \varepsilon_1 + (1 - \varepsilon_2))\hat{P}^{-1} \cdot s(\tau_n) \\
+ h(\gamma(1 - \varepsilon_1))\hat{P}^{-1} \cdot s(\tau_n + 1) + \hat{P}^{-1}B(\tau_n) \cdot (A W_n + \gamma A W_{n-1})
\]

(15)

and

\[
- h\varepsilon_2 \hat{Q} \cdot \dot{X}_{t_n} = h(\gamma \varepsilon_1 + (1 - \varepsilon_2))\hat{Q} \cdot \dot{X}_{t_n} + h(\gamma(1 - \varepsilon_1))\hat{Q} \cdot \dot{X}_{t_{n+1}} + h\varepsilon_2 \hat{Q}D^{-1} \cdot s(\tau_{n+1}) \\
+ h(\gamma \varepsilon_1 + (1 - \varepsilon_2))\hat{Q}D^{-1} \cdot s(\tau_n) + h(\gamma(1 - \varepsilon_1))\hat{Q}D^{-1} \cdot s(\tau_n + 1) \\
+ \hat{Q}D^{-1}B(\tau_n) \cdot (A W_n + \gamma A W_{n-1}).
\]

We have (14) = (15), which completes the proof. \(\square\)
We have thus found a numerical method working directly with the given implicit system, allowing to exploit sparseness and structure of $C$ and $G$, while still preserving the same (strong) order of convergence for the underlying SDE. Since the algebraic component has to be modeled as a generalized stochastic process, we cannot establish a general convergence result here. However, the so-called direct approach we have taken is well-known in dealing with singular perturbation problems and allows a nice heuristic justification in analogy to the deterministic context (for more details, see [12, 20]).

**Remark 6.** The computational results obtained in the next section suggest that the method works also well for disturbed non-linear DAEs of the general type

$$M \cdot \dot{x}(t) + f(x(t), t) + B(t) \cdot v(\omega, t) = 0,$$

which play an important role in multibody dynamics [12].

### 4. Numerical results

#### 4.1. Noise model and generation of equations

In order to simulate electronic circuits, the network of such a circuit has to be described in mathematical equations. In most simulators this is done with the modified nodal analysis (MNA) [5]. Kirchhoff’s laws are used to construct the equations: Kirchhoff’s current law is applied to every node of the circuit while directly expressing the element branch currents as a function of their branch voltages. Only the branch currents having no characteristic equation in admittance form remain in the vector of unknowns. The branch voltages are expressed with nodal voltages due to Kirchhoff’s voltage law. Therefore, the vector $x$ of the unknown quantities consists of nodal voltages and some of the branch currents. MNA leads to equations of the type

$$C(x(t)) \cdot \dot{x}(t) + f(x(t)) + s(t) = 0.$$

The “capacitance” matrix $C(x(t))$ mainly holds the capacitances and inductances, in $f(x(t))$ the conductances and nonlinear elements are assembled. The function $s(t)$ denotes the independent voltage and current sources.

The main problem in constructing the equations for the simulation is the modeling of the currents. Especially the currents for the nonlinear devices such as MOSFETs are rather difficult to formulate. This is also true for the modeling of many noise sources. One of the simpler noise sources is the thermal noise of a resistor with resistance $R$ which is modeled as a shunt current source with a stochastically disturbed current $\Delta I_R$, see Fig. 1:

$$\Delta I_R = \sqrt{\frac{4kT}{R}} \Delta f \cdot v(\omega, t).$$

Here $k$ denotes Boltzmann’s constant, $T$ is the temperature, $\Delta f$ is the band width, and $v(\omega, t)$ represents a white noise source. Similar equations hold for other noise sources such as shot noise or flicker noise, cf. [16].
As the noise sources introduce a different type of quantities, namely stochastically disturbed ones, they appear as an extra term in the resulting equation:

\[ C(x(t)) \cdot \dot{x}(t) + f(x(t)) + s(t) + B(t,x(t)) \cdot v(\omega, t) = 0. \]

4.2. A test problem (test circuit)

In order to numerically test the scheme (13) we have constructed an academic test circuit shown in Fig. 2 with three resistors with noise effects. MNA gives

\[
\begin{pmatrix}
    C_0 & -C_0 & 0 & 0 \\
    -C_0 & C_0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
    \dot{u}_1 \\
    \dot{u}_2 \\
    \dot{u}_3 \\
    \dot{I}_Q
\end{pmatrix}
+ \begin{pmatrix}
    \frac{1}{R_1} & \frac{1}{R_1} & 0 & 1 \\
    -\frac{1}{R_1} & \frac{1}{R_1} + \frac{1}{R_2} & -\frac{1}{R_1} & 0 \\
    0 & -\frac{1}{R_2} & \frac{1}{R_2} + \frac{1}{R_1} & 0 \\
    1 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
    u_1 \\
    u_2 \\
    u_3 \\
    I_Q
\end{pmatrix}
= \begin{pmatrix}
    \sqrt{\frac{4kT}{R_1}} Af \\
    -\sqrt{\frac{4kT}{R_1}} Af \\
    0 \\
    0
\end{pmatrix}
\begin{pmatrix}
    v_1(t, \omega) \\
    v_2(t, \omega) \\
    v_3(t, \omega) \\
    V(t)
\end{pmatrix}. \]
Fig. 3. Test circuit: simulation results: differential variable $u_2(t)$ (left), algebraic variable $u_3(t)$ (right).

The components $v_1$, $v_2$, and $v_3$ denote uncorrelated white noise processes. The parameters of the circuit have been chosen as $R_1 = 3 \cdot 10^3$, $R_2 = 4 \cdot 10^3$, $R_3 = 5 \cdot 10^3$, $C_0 = 1.3 \cdot 10^{-11}$, $V(t) = 2 \sin(2t \cdot 10^3)$. The test circuit is constructed in such a way that the nodal voltage $u_1(t)$ at node $N_1$ is deterministic, $u_2(t)$ at node $N_2$ and hence $u_1(t) - u_2(t)$ is modeled by an Itô-process (differential variable), whereas $u_3(t)$ at $N_3$ is a generalized stochastic process (algebraic variable). These different properties are reflected in the simulated waveforms. In Fig. 3, the differential variable $u_2(t)$ shows a typical behavior of an Itô-process, whereas $u_3(t)$ has a much larger variation reflecting the distributional aspect of the generalized stochastic process.

In Fig. 4, the error $\text{err}(u_1)$ of $u_1(t)$, $\text{err}(u_{1,2})$ of $u_1(t) - u_2(t)$, and $\text{err}(u_3)$ of $u_3(t)$, resp., are plotted versus the step size. The error is averaged over 100 paths and 100 time points. As expected, the error of the algebraic variable $u_3(t)$ does not decrease significantly with decreasing step size. This behavior is well-known from numerical schemes for DAEs. The error of the other variables decreases linearly with the step size, which confirms the order of convergence of 1. Though the error of the stochastical differential variable is larger than the error of the deterministic one, the order of convergence is the same.

4.3. A linear problem (differentiator circuit)

As a first test example we have investigated a simple differentiator circuit. It consists of an operational amplifier, a capacitance and a resistor, see Fig. 7. The purpose of the circuit is to differentiate the input voltage $V(t)$. We have supposed a noisy resistor $R$, whereas all other elements shall not exhibit noise. Applying MNA results in

$$
\begin{pmatrix}
C_0 & -C_0 & 0 & 0 & 0 \\
-C_0 & C_0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\dot{u}_1 \\
\dot{u}_2 \\
\dot{u}_3 \\
I_Q \\
I_{\text{ampl}}
\end{pmatrix}
+ 
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & \frac{1}{R} & -\frac{1}{R} & 0 & 0 \\
0 & -\frac{1}{R} & \frac{1}{R} & 0 & -1 \\
0 & A & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2 \\
u_3 \\
I_Q \\
I_{\text{ampl}}
\end{pmatrix}
$$
The parameters of the circuit were chosen as $C_0 = 10^{-12}$, $R = 10^4$, amplification factor $A = 300$, the band width is $\Delta f = 1$. The input voltage $V(t)$ is shown in Fig. 6. Simulating the differentiator circuit with scheme (13) using $\xi_2 = 0.9$, $\gamma = 0$ and a step size $\Delta = 2.5 \cdot 10^{-11}$ gives the nodal voltage $u_{3,\text{noi}}$ at node $N_3$ as shown in Fig. 6. In addition, the simulation result $u_{3,\text{det}}$ without noise is plotted. It shows that the differentiator circuit is rather sensitive regarding noise. This is due to the differentiating behavior of the circuit and the nondifferentiable noise current $\Delta I_R$.

4.4. A non-linear problem (ring oscillator)

The next example is a ring-oscillator circuit which is part of many integrated circuits. It consists of several inverter blocks (Fig. 7) each consisting of one MOSFET, one resistor and two capacitors. At node $N_{k+1}$, the block shows the inverted signal of node $N_k$. Combining several inverter blocks gives a ring-oscillator circuit (Fig. 8), where each output is the input of the next block and the last output signal is fed back as the input of the first block.
In this example, we have several noise sources. Each inverter block has a resistor with thermal noise. In addition, the noise of the MOSFET has been taken into account. The noise current $\Delta I_{DS}$ is modeled as

$$\Delta I_{DS} = \sqrt{\frac{8}{3}} kT g_m(u_G, u_S, u_D) \Delta f \cdot \nu(t, \omega),$$

with a transfer conductance $g_m$ depending on the gate voltage $u_G$, the source voltage $u_S$, and the drain voltage $u_D$, for details see [2]. Applying MNA yields

$$I_{op} + \frac{1}{R} \sum_{i=2}^{n+1} u_{1,i} + \sum_{i=2}^{n+1} \Delta I_{R_i} = 0,$$

$$\frac{1}{R} u_{2,1} + C_p (\dot{u}_{2,3} - \dot{u}_{n+1,2}) + C_0 \dot{u}_2 + I_{DS_2} + \Delta I_{DS_2} - \Delta I_{R_2} = 0,$$

For $j = 3, \ldots, n$:

$$\frac{1}{R} u_{j,1} + C_p (\dot{u}_{j,j+1} - \dot{u}_{j-1,1}) + C_0 \dot{u}_j + I_{DS_j} + \Delta I_{DS_j} - \Delta I_{R_j} = 0,$$

$$\frac{1}{R} u_{n+1,1} + C_p (\dot{u}_{n+1,2} - \dot{u}_{n,n+1}) + C_0 \dot{u}_{n+1} + I_{DS_{n+1}} + \Delta I_{DS_{n+1}} - \Delta I_{R_{n+1}} = 0,$$

$$u_1 - V_{op} = 0,$$

$$I_{DS_2} - f(u_{n+1}, u_2, 0) = 0,$$

For $j = 3, \ldots, n+1$:

$$I_{DS_j} - f(u_{j-1}, u_j, 0) = 0$$

with $u_{i,j} = u_i - u_j$ and $f(u_i, u_j, u_k)$ describing the nonlinear current $I_{DS}$ according to the level-1 model (see [2]) from drain to source through the MOSFET.

The simulation was carried out with a ring oscillator consisting of 5 inverters. The waveforms of the nodal voltage at nodes $N_4$ and $N_6$ are given in Fig. 9. Here the nodal voltages $u_{4,\text{noi}}$ and $u_{6,\text{noi}}$...
Fig. 6. Differentiator circuit: simulation results: (-----) $u_{3, noi}$ with noise source, (- - -) $u_{3, det}$ without noise source, (· · · · ·) $V(t)$ input voltage.

Fig. 7. Single inverter: block $k$. 
of the circuit with noise source are compared with the voltages $u_{4,\text{det}}$ and $u_{6,\text{det}}$ of the deterministic circuit. Though the deterministic and the realistic voltages differ, the results seem to be acceptable. If, however, smaller MOSFET designs or a larger band width is used, it is possible that the difference between deterministic and noisy signals becomes too large, a re-design may be necessary. For circuits with a varying operation point like oscillators or mixers, this information is only available with the transient noise simulation presented in this paper.
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