Invertibly convergent infinite products of matrices

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Abstract

The standard definition of convergence of an infinite product of scalars is extended to the infinite product $P = \prod_{n=1}^{\infty} B_n$ of $k \times k$ matrices; that is, $P$ is convergent according to the definition given here if and only if there is an integer $N$ such that $B_n$ is invertible for $n \geq N$ and $P = \lim_{n \to \infty} \prod_{m=1}^{n} B_m$ is invertible. A family of sufficient conditions for this kind of convergence is given, along with examples showing that they have nontrivial applications. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

A scalar infinite product $p = \prod_{n=1}^{\infty} b_n$ of complex numbers is said to converge if $b_n$ is nonzero for $n$ sufficiently large, say $n \geq N$, and $q = \lim_{n \to \infty} \prod_{m=1}^{n} b_n$ exists and is nonzero. If this is so then $p$ is defined to be $p = q \prod_{n=1}^{N-1} b_n$. With this definition, a convergent infinite product vanishes if and only if one of its factors vanishes.

If $\{B_n\}$ are $k \times k$ complex matrices we define

$$\prod_{n=r}^{s} B_{j} = \begin{cases} B_{s}B_{s-1}\cdots B_{r} & \text{if } r \leq s, \\ I & \text{if } r > s; \end{cases}$$

thus, successive terms multiply on the left. The standard definition of convergence of an infinite product $\prod_{n=1}^{\infty} B_n$ requires only that $P = \lim_{n \to \infty} \prod_{m=1}^{n} B_m$ exist. With this definition, $P$ may be singular even if $B_n$ is nonsingular for all $n \geq 1$.

We gave the following definition in [1].

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**Definition 1.** An infinite product $\prod_{n=1}^{\infty} B_n$ of $k \times k$ matrices converges invertibly if there is an integer $N$ such that $B_n$ is invertible for $n \geq N$ and

$$Q = \lim_{n \to \infty} \prod_{m=N}^{n} B_m$$

exists and is invertible. In this case we define $\prod_{n=1}^{\infty} B_n = Q \prod_{n=1}^{N-1} B_n$.

Trgo [3] defines convergence of an infinite product of matrices as in Definition 1, without the adverb “invertibly”.

Definition 1 has the following obvious consequence.

**Theorem 2.** An invertibly convergent infinite product is singular if and only if at least one of its factors is singular.

As discussed in [1], the motivation for Definition 1 stems from a question about linear systems of difference equations: Under what conditions on $\{B_n\}_{n=1}^{\infty}$ does the solution $\{x_n\}_{n=0}^{\infty}$ of the system $x_n = B_n x_{n-1}, \ n = 1, 2, \ldots$, approach a finite nonzero limit whenever $x_0 \neq 0$? A system with this property is said to have linear asymptotic equilibrium. It is easy to show that the system has linear asymptotic equilibrium if and only if $B_n$ is invertible for every $n \geq 1$ and $\prod_{n=1}^{\infty} B_n$ converges invertibly.

The following theorem was proved in [1].

**Theorem 3.** If $\prod_{n=1}^{\infty} B_n$ converges invertibly then $\lim_{n \to \infty} B_n = I$.

Because of Theorem 3 we consider only infinite products of the form $\prod_{n=1}^{\infty} (I + A_n)$ where $\lim_{n \to \infty} A_n = 0$. (Since invertible convergence of an infinite is independent of the first finitely many factors, we will usually not specify the lower limit of the product when discussing conditions for invertible convergence.)

In [1] we gave some sufficient conditions for invertible convergence of infinite products of $k \times k$ matrices. In this paper, we present a succession of invertible convergence tests, along with examples showing that they have nontrivial application. These results generalize results obtained in [2] for conditional convergence of scalar infinite products.

2. Sufficient conditions for invertible convergence

Throughout this paper, if $A$ is a $k \times k$ matrix then $|A|$ is some $p$-norm of $A$. The following theorem is proved in [1]; however, for convenience we include a shorter and more direct proof here.

**Theorem 4.** The product $\prod_{n=1}^{\infty} (I + A_n)$ converges invertibly if $\sum_{n=1}^{\infty} |A_n| < \infty$.

**Proof.** Let $N$ be an integer such that $\sum_{n=N}^{\infty} |A_n| < 1$, and let $\mathcal{B}_N$ be the Banach space of all bounded sequences $\mathcal{X} = \{X_n\}_{n=1}^{\infty}$ of $k \times k$ matrices, with norm $\|\mathcal{X}\| = \sup_{n \geq N} |X_n|$. Then the transformation
\( \Psi = T \Psi \) defined by

\[
Y_n = I - \sum_{m=n}^{\infty} A_m X_m, \quad n \geq N,
\]

is a contraction mapping of \( \Psi_N \) into itself. Let \( \hat{X} \) be the fixed point of this mapping; thus,

\[
\hat{X}_n = I - \sum_{m=n}^{\infty} A_m \hat{X}_m, \quad n \geq N, \tag{2}
\]

so \( \hat{X}_{n+1} = (I + A_n) \hat{X}_n, \ n \geq N, \) and therefore

\[
\hat{X}_n = \left( \prod_{m=N}^{n-1} (I + A_m) \right) \hat{X}_N, \quad n \geq N. \tag{3}
\]

From (2), \( \lim_{n \to \infty} \hat{X}_n = I. \) Therefore (3) implies that \( I = (\prod_{m=N}^{\infty} (I + A_m)) \hat{X}_N, \) so \( \prod_{m=N}^{\infty} (I + A_m) = \hat{X}_N^{-1}. \)

The following theorem is a weaker sufficient condition for invertible convergence of \( \prod_{m=N}^{\infty} (I + A_m). \)

**Theorem 5.** If there is a sequence \( \{R_n\} \) of \( k \times k \) matrices such that

\[
\lim_{n \to \infty} R_n = I
\]

and

\[
\sum_{n=0}^{\infty} |(I + A_n)R_n - R_{n+1}| < \infty \tag{5}
\]

then \( \prod_{m=N}^{\infty} (I + A_m) \) converges invertibly.

**Proof.** Let \( G_n = (I + A_n)R_n - R_{n+1}. \) Then \( \sum_{n=0}^{\infty} |G_n| < \infty \) from (5), so \( \lim_{n \to \infty} G_n = 0 \) and therefore \( \lim_{n \to \infty} A_n = 0 \) by (4). Choose \( N \) so that \( R_n, I + A_n \) and \( I + R_{n+1}^{-1} G_n \) are invertible if \( n \geq N. \) Now define \( P_{N-1} = I \) and \( P_n = \prod_{m=N}^{n} (I + A_n), \ n \geq N. \) If \( n \geq N \) then \( I + A_n = P_{n+1}^{-1} R_n - R_{n+1}, \) and therefore \( P_n = R_{n+1} (I + R_{n+1}^{-1} G_n) R_n^{-1} P_{n-1}, \) which implies that

\[
P_n = R_{n+1} \left[ \prod_{m=N}^{n} (I + R_{m+1}^{-1} G_m) \right] R_n^{-1}. \tag{6}
\]

Since (4) and (5) imply that \( \sum_{n=0}^{\infty} |R_n^{-1} G_n| < \infty, \) Theorem 4 implies that the infinite product \( Q = \prod_{m=N}^{\infty} (I + R_{m+1}^{-1} G_m) \) converges invertibly; moreover \( Q \) is invertible because \( I + R_{m+1}^{-1} G_m \) is invertible if \( m \geq N. \) Now (4) and (6) imply that \( \lim_{n \to \infty} P_n = QR_N^{-1} \) is finite and invertible. \( \Box \)

To apply this theorem nontrivially we must exhibit sequences \( \{R_n\} \) that yield results even if \( \sum_{n=0}^{\infty} |A_n| = \infty. \) The following theorem provides ways to do this.
Theorem 6. Suppose that for some positive integer \( q \) the sequences
\[
A_n^{(k)} = \sum_{m=n}^{\infty} A_m A_m^{(k-1)}, \quad k = 1, \ldots, q \text{ (with } A_n^{(0)} = I),
\]
are all defined, and
\[
\sum_{n=1}^{\infty} |A_n A_n^{(q)}| < \infty.
\tag{7}
\]
Then \( \prod_{n=1}^{\infty} (I + A_n) \) converges invertibly.

Proof. Define
\[
R_n^{(l)} = I + \sum_{j=1}^{l} (-1)^j A_n^{(j)}, \quad 1 \leq l \leq q.
\]
We show by finite induction on \( l \) that
\[
(I + A_n) R_n^{(l)} - R_n^{(l+1)} = (-1)^l A_n^{(l)} A_n^{(l)}
\tag{8}
\]
for \( 1 \leq l \leq q \). Since \( \lim_{n \to \infty} R_n^{(q)} = I \) we can then set \( l = q \) in (8) and conclude from (7) and Theorem 5 with \( R_n = R_n^{(q)} \) that \( \prod_{n=1}^{\infty} (I + A_n) \) converges invertibly.

Since \( R_n^{(1)} = I - A_n^{(1)} \) the left-hand side of (8) with \( l = 1 \) is
\[
(I + A_n)(I - A_n^{(1)}) - (I - A_n^{(1)}) = A_n - A_n^{(1)} - A_n A_n^{(1)} + A_n^{(1)} = -A_n A_n^{(1)},
\]
since \( A_n^{(1)} + A_n = A_n^{(1)} \). This proves (8) for \( l = 1 \).

Now suppose that (8) holds if \( 1 \leq l < q - 1 \). Since
\[
R_n^{(l)} = R_n^{(l+1)} + (-1)^l A_n^{(l+1)},
\]
(8) implies that
\[
(I + A_n) (R_n^{(l+1)} + (-1)^l A_n^{(l+1)}) - R_n^{(l+1)} = (-1)^l A_n A_n^{(l)} A_n^{(l)}
\]
Therefore,
\[
(I + A_n) R_n^{(l+1)} - R_n^{(l+1)} = (-1)^l(A_n A_n^{(l)} - A_n^{(l+1)} - A_n A_n^{(l+1)} + A_n^{(l+1)})
\]
\[
= (-1)^{l+1} A_n A_n^{(l+1)},
\]
since
\[
A_n^{(l+1)} + A_n A_n^{(l)} = A_n^{(l+1)}.
\]
This completes the induction.
3. Preparation for applications of Theorem 6

We now prepare for specific applications of Theorem 6. Henceforth let

\[ E(t) = \text{diag}(e^{i\phi_1 t}, e^{i\phi_2 t}, \ldots, e^{i\phi_k t}) \]

where \( \phi_1, \phi_2, \ldots, \phi_k \) and \( t \) are real numbers. Let \( \Delta \) be the forward difference operator; thus, \( \Delta G_m = G_{m+1} - G_m \).

**Lemma 7.** Suppose that \( t \) is not an integral multiple of any of the numbers \( 2\pi/\phi_1, \ldots, 2\pi/\phi_k \), and \( \{G_m\}_{m=0}^{\infty} \) is a sequence of complex \( k \times k \) matrices such that \( \lim_{m \to \infty} G_m = 0 \) and

\[
\sum_{m=0}^{\infty} |\Delta^v G_m| < \infty
\]

for some positive integer \( v \). Then \( \sum_{m=0}^{\infty} E(mt)G_m \) converges and

\[
\sum_{m=0}^{\infty} E(mt)G_m = (I - E(t))^{-v} \left[ \sum_{s=0}^{v-1} Q_s G_s + \sum_{m=0}^{\infty} E(mt)(\Delta^v G_m) \right],
\]

where

\[
Q_s = \sum_{m=s}^{v-1} (-1)^{v-s} \binom{v}{m-s} E(mt), \quad 0 \leq s \leq v - 1.
\]

**Proof.** Our assumptions imply that \( I - E(t) \) is invertible. Suppose that \( M > 2v \) and let

\[
S_M = (I - E(t))^v \sum_{m=0}^{M} E(mt)G_m.
\]

Since

\[
(I - E(t))^v E(mt) = \sum_{r=0}^{v} (-1)^r \binom{v}{r} E((m + r)t),
\]

we have

\[
S_M = \sum_{m=0}^{M} \left( \sum_{r=0}^{v} (-1)^r \binom{v}{r} E((m + r)t) \right) G_m = \sum_{r=0}^{v} (-1)^r \binom{v}{r} \sum_{m=0}^{M} E((m + r)t)G_m
\]

\[
= \sum_{r=0}^{v} (-1)^r \binom{v}{r} \sum_{m=0}^{M} E(mt)G_{m-r}
\]

\[
= \sum_{m=0}^{v} E(mt) \left( \sum_{r=0}^{v} (-1)^r \binom{v}{r} G_{m-r} \right) + \sum_{m=v}^{M} E(mt) \left( \sum_{r=0}^{v} (-1)^r \binom{v}{r} G_{m-r} \right)
\]

\[
+ \sum_{m=M+1}^{M+v} E(mt) \left( \sum_{r=m-M}^{v} (-1)^r \binom{v}{r} G_{m-r} \right).
\]
Since \( \lim_{m \to \infty} G_m = 0 \) the last sum on the right converges to 0 as \( M \to \infty \). The second sum on the right is
\[
\sum_{m=0}^{M} E(mt)(\Delta^m G_{m+1}) = E(vt) \sum_{m=0}^{M-1} E(mt)(\Delta^m G_m),
\]
which converges as \( M \to \infty \) because of (9). Therefore,
\[
\lim_{M \to \infty} S_M = S = \sum_{m=0}^{\infty} E(mt) \left( \sum_{r=0}^{m} (-1)^r \binom{m}{r} G_{m-r} \right) + E(vt) \sum_{m=0}^{\infty} E(mt)(\Delta^m G_m),
\]
which can also be written as
\[
S = \sum_{s=0}^{v-1} Q_s G_s + E(vt) \sum_{m=0}^{\infty} E(mt)(\Delta^m G_m),
\]
with \( Q_s \) as in (11). This and (12) imply (10). \( \square \)

**Definition 8.** If \( \alpha > 0 \) let \( \mathcal{F}_z \) be the set of infinitely differentiable \( k \times k \) matrix functions \( F = (F_{rs})_{r,s=1}^{k} \) on \([1, \infty)\) such that
\[
F_{rs}^{(v)}(x) = O(x^{-\alpha-v}), \quad 1 \leq r, s \leq n, \quad v = 0, 1, \ldots
\]

The following lemma provides an infinite set of examples of functions belonging to \( \mathcal{F}_z \).

**Lemma 9.** For \( r, s = 1, \ldots, n \) let \( E_{rs} = u_{rs}^{\gamma_{rs}} \), where \( \gamma_{rs} > 0 \) and \( u_{rs} \) is a rational function with positive values on \([1, \infty)\) and a zero of order \( p_{rs} > 0 \) at \( \infty \). Then \( F = (E_{rs})_{r,s=1}^{k} \) is in \( \mathcal{F}_z \), with \( z = \min \{ p_{rs} \gamma_{rs} \}_{r,s=1}^{k} \).

**Proof.** Applying the formula of Faa di Bruno [4] for the derivatives of a composite function to \( f(u) = u^v \) where \( u = u(x) \) yields
\[
\frac{d^v}{dx^v} f(u(x)) = \sum_{r_1 + r_2 + \cdots + r_v = r} \frac{r!}{r_1! \cdots r_v!} \left( \frac{u'(x)}{1!} \right)^{r_1} \left( \frac{u''(x)}{2!} \right)^{r_2} \cdots \left( \frac{u^{(v)}(x)}{v!} \right)^{r_v},
\]
where \( \sum_r \) is over all partitions of \( r \) as a sum of nonnegative integers,
\[
r_1 + r_2 + \cdots + r_v = r
\]
such that
\[
r_1 + 2r_2 + \cdots + vr_v = v
\]
and \( \gamma^{(v)} = \gamma(\gamma - 1) \cdots (\gamma - r + 1) \). Since \( u^{(j)}(x) = O(x^{-\alpha-j}) \), it follows that
\[
u^{\gamma r}(x)(u(x))^{(v)}(u''(x))^{(v-2)} \cdots (u^{(v)}(x))^{(v)} = O(x^{-\alpha-\gamma r}),
\]
since
\[ p(\gamma - r) + (p + 1)r_1 + (p + 2)r_2 + \cdots + (p + v)r_v = p\gamma + v \]
because of \((13)\) and \((14)\). Applying this argument with \(f = E_s, r, s = 1, \ldots, k\) yields the conclusion. 

Note that \(\mathcal{F}_s\) is a vector space over the complex numbers. Moreover, if \(F_i \in \mathcal{F}_s, i = 1, 2, \) then \(F_1F_2 \in \mathcal{F}_{z_1 + z_2}\).

Henceforth, if \(F\) is a \(k \times k\) matrix function then \(\Delta F(x) = F(x+1) - F(x)\). We write \(F(x) = O(x^{-\beta})\) to indicate that \(|F(x)| = O(x^{-\beta})\).

**Lemma 10.** If \(F \in \mathcal{F}_s\) then
\[ \Delta F(x) = O(x^{-\beta}), \quad v = 0, 1, 2, \ldots. \]

**Proof.** Taylor’s theorem shows that
\[ |\Delta F(x)| \leq K \max_{x \leq \xi \leq x+v} |F^{(v)}(\xi)|, \]
where \(K\) is a constant independent of \(F\). Since \(F^{(v)}(x) = O(x^{-\beta})\), this implies the conclusion. 

**Lemma 11.** Suppose that \(F \in \mathcal{F}_s\). Let \(v\) be a fixed positive integer and \(t\) be a real number, not an integral multiple of any of the numbers \(2\pi/\phi_1, \ldots, 2\pi/\phi_k\). Then
\[ \sum_{m=n}^{\infty} E(mt)F(m) = E(nt)T(n) + O(n^{-\beta+1}), \]
where \(T \in \mathcal{F}_s\) (and \(T\) depends upon \(v\)).

**Proof.** We write
\[ \sum_{m=n}^{\infty} E(mt)F(m) = E(nt)\sum_{m=0}^{\infty} E(mt)F(n + m). \tag{15} \]
From Lemma 10, \(\Delta F(n + m) = O((n + m)^{-\beta})\); therefore,
\[ \sum_{m=0}^{\infty} |\Delta F(n + m)| = O(n^{-\beta+1}). \]
Applying Lemma 7 (specifically, \((10)\)) with \(G_m = F(n + m)\) and \(n\) fixed shows that
\[ \sum_{m=0}^{\infty} E(mt)F(n + m) = T(n) + O(n^{-\beta+1}), \]
with
\[ T(x) = (I - E(t))^{-\gamma} \sum_{s=0}^{v-1} Q_s(t)F(x + s), \]
so \(T \in \mathcal{F}_s\). Now \((15)\) implies the conclusion. 

4. Applications of Theorem 6

The following theorem shows that Theorem 6 has nontrivial applications for every positive integer \( q \).

**Theorem 12.** Suppose that

\[
A_n = E(n\theta)F(n), \quad n = 1, 2, 3, \ldots,
\]

where \( \theta \) is real and \( F \in \mathcal{F}_\gamma \) for some \( \gamma \in (0, 1] \). Let \( q \) be the smallest integer such that

\[
(q + 1)\gamma > 1,
\]

and define

\[
\mathcal{N}_q = \{(2\mu\pi)/(p\phi_i) \mid \mu = \text{integer}, \quad p = 1, \ldots, q, \ i = 1, \ldots, k\}.
\]

Then the infinite product \( \prod_1^\infty (1 + A_n) \) converges invertibly if \( \theta \notin \mathcal{N}_q \).

**Proof.** We show by finite induction on \( p \) that if \( p = 1, \ldots, q \) then

\[
A_nA_n^{(p)} = E((p + 1)n\theta)F_p(n) + O(n^{-(p+1)\gamma-q+1}),
\]

where \( F_p \in \mathcal{F}_{(p+1)\gamma} \). In particular, (18) with \( p = q \) implies that \( A_nA_n^{(q)} = O(n^{-(q+1)\gamma}) \), so (17) implies (7) and \( P \) converges invertibly, by Theorem 6.

From (16) and Lemma 11 with \( t = \theta, \alpha = \gamma, \) and \( v = q \),

\[
A_n^{(1)}(m) = \sum_{m=n}^\infty E(m\theta)F(m) = E((n\theta))T_1(n) + O(n^{-\gamma-q+1}),
\]

with \( T_1 \in \mathcal{F}_\gamma \). Therefore

\[
A_nA_n^{(1)} = E(n\theta)F(n)(E(n\theta)T_1(n) + O(n^{-\gamma-q+1})).
\]

However, \( F(n)E(n\theta) = E(n\theta)\hat{F}(n) \), where \( \hat{F}_s(n) = e^{i(\phi_s - \phi_1)n\theta}F_s(n) \) for \( 1 \leq r, s \leq k \), so \( \hat{F} \in \mathcal{F}_\gamma \). Therefore (19) can be rewritten as

\[
A_nA_n^{(1)} = E((2n\theta)F_1(n) + O(n^{-2\gamma-q+1}), \text{with } F_1 = \hat{F}T_1 \in \mathcal{F}_\gamma. \text{ This establishes (18) with } p = 1, \text{ so we are finished if } q = 1.
\]

Now suppose that \( q > 1 \) and (18) holds if \( 1 \leq p \leq q \). Since \( \theta \notin \mathcal{N}_q \), Lemma 11 with \( t = (p+1)\theta, \alpha = (p+1)\gamma, \) and \( v = q - p \) implies that

\[
\sum_{m=n}^\infty E((p+1)m\theta)F_p(m) = E((p+1)n\theta)T_p(n) + O(n^{-(p+1)\gamma-q+p+1}),
\]

where \( T_p \in \mathcal{F}_{(p+1)\gamma} \). This and (18) imply that

\[
A_n^{(p+1)} = \sum_{m=n}^\infty A_mA_m^{(p)} = E((p+1)n\theta)T_p(n) + O(n^{-(p+1)\gamma-q+p+1}),
\]

so

\[
A_nA_n^{(p+1)} = E(n\theta)F(n)(E((p+1)n\theta)T_p(n) + O(n^{-(p+1)\gamma-q+p+1})).
\]
However, $F(n)E((p + 1)n\theta) = E((p + 1)n\theta)\tilde{F}(n)$, where $\tilde{F}_r(n) = e^{i(\phi_r - \phi)(p+1)n}\tilde{F}_r(n)$ for $1 \leq r, s \leq k$, so $\tilde{F} \in \mathcal{F}$. Therefore (20) can be rewritten as

$$A_n^{(p+1)} = E((p + 2)n\theta)F_{p+1}(n) + O(n^{-(p+1)} - q + (p+1)),$$

with $F_{p+1} = \tilde{F}T_p \in \mathcal{F}_{(p+2)\gamma}$. This completes the induction. □

In the following corollaries $\mathcal{N}_\infty = \{(2\rho\pi)/\phi_i \mid \rho \text{ rational}, i = 1, \ldots, k\}$.

**Corollary 13.** If $\{A_n\}$ is as defined in Theorem 12 then $\prod_\infty (I + A_n)$ converges invertibly if $\theta \notin \mathcal{N}_\infty$.

**Corollary 14.** Let $F = (u_{rs}^\gamma)_{r,s=1}^k$ where, for $1 \leq r, s \leq k$, $\gamma_r > 0$ and $u_{rs}$ is a rational function with positive values on $[1, \infty)$ and a zero of positive order at $\infty$. Then the infinite product $\prod_\infty (I + E(n\theta)F(n))$ converges invertibly if $\theta \notin \mathcal{N}_\infty$.

**Corollary 15.** If $F = (n^{-\gamma_n})_{r,s=1}^k$ where, $\gamma_n > 0$ for $1 \leq r, s \leq n$, then the infinite product $\prod_\infty (I + E(n\theta)F(n))$ converges invertibly if $\theta \notin \mathcal{N}_\infty$.

**References**


