Letter to the Editor

Comments on “The parallel version of the successive approximation method for quasilinear boundary-value problem” by Scheiber Ernö

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Received 29 July 1998

In this note we want to show that the parallel version of the successive approximation method (SAM) especially designed by Ernö for quasilinear boundary-value problems (BVPs) with large overall integration interval [1] may not converge.

Consider the following quasilinear two-point BVP:

\[ \dot{x} = Q(t)x + f(t,x), \quad 0 > t > T, \]  \tag{1}

\[ Ax(0) + Bx(T) = c \]  \tag{2}

with the same notations as in [1]. In what follows, we suppose that the function \( f(t,x) \) is Lipschitz continuous in \( x \) with the Lipschitz constant \( L \) in the domain \( D = \{(x,t): t \in [0,T]; |x| \leq r \} \) and the shooting matrix \( A + BH(T,0) \), where \( H(t,s) = X(t)X^{-1}(s) \) and \( X(t) \) is a fundamental matrix of linear homogeneous system \( \dot{x} = Q(t)x \), is nonsingular.

It is well known that if the length of integration interval \( T \) is sufficiently small, then problem (1) and (2) possesses a unique solution and the SAM can be implemented for finding the solution of (1) and (2).

If \( T \) is not small, Ernö proposed a parallel version of the SAM. The main idea of his method is to divide the overall integration interval into \( m \) parts with sufficiently small \( h = T/m \) and to apply the shooting method to an enlarged BVP with a small integration interval \([0,h]\). It has been proved [1] that the shooting matrix \( R + S\mathcal{H}(h,0) \) of the enlarged system is also nonsingular and its inverse can be effectively calculated. Moreover, his approach allows a complete distribution of the computation on the components of the enlarged system.

However, a closer examination of convergence theorem 3.1 [1] shows that if \( T \) is not small then the main requirements \( ||[R + S\mathcal{H}(h,0)]^{-1}|| \ll \gamma \) and \( \rho Lwh < 1 \), where \( \rho \geq \max \{|H(t,s)|: 0 \leq t, s \leq T\} \), \( \sigma = \max \{|A|, |B|, 1\} \) and \( w = 1 + \gamma \rho \sigma \) may not hold. Indeed, let us consider a simple scalar problem (1) and (2) with \( n = 1 \); \( Q(t) \equiv q > 0 \); \( A = 1 \); \( B = -1 \); \( c = 0 \).

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PII: S0377-0427(98)00227-1
Putting $Z := 1 - e^{\sigma T}; \quad \rho = e^{\sigma T}; \quad \sigma = 1$, we have (see [1])

$$[R + S \mathcal{H}(h, 0)]^{-1} = \begin{pmatrix} Z & Z e^{(m-1)hq} & \cdots & Z e^{hq} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix}.$$ 

Thus, $\gamma \geq \|[R + S \mathcal{H}(h, 0)]^{-1}\|_{\infty} \geq |Z|(1 + e^{(m-1)hq} + \cdots + e^{hq}) > m|Z| = ((e^{\sigma T} - 1)T)/h$.

Now let $T \geq (Lq)^{-1/2}$, then

$$\rho Lwh = e^{\sigma T} Lh(1 + \gamma e^{\sigma T}) > LTe^{2\sigma T}(e^{\sigma T} - 1) > LqT^2 \geq 1.$$ 

Consequently, the conditions of Theorem 3.1 are not fulfilled. It is worth noting that there are a lot of examples, when $\|[R + S \mathcal{H}(h, 0)]^{-1}\|_{\infty} \to \infty (h \to 0)$ and the inequality $\rho Lwh < 1$ holds only if $T$ is small enough.

Reference