On the stability of large matrices

A.N. Malyshev\textsuperscript{a}, M. Sadkane\textsuperscript{b,*}

\textsuperscript{a} Department of Informatics, University of Bergen, N-5020 Bergen, Norway
\textsuperscript{b} Département de Mathématiques, Université de Bretagne Occidentale, 6, avenue Le Gorgeu, B.P. 809, 29285 Brest Cedex, France

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Abstract

The distance $r_{\text{stab}}(A)$ of a stable matrix $A$ to the set of unstable matrices and the norm of the exponential of matrices constitute two important topics in stability theory. We treat in this note the case of large matrices. The method proposed partitions the matrix into two blocks: a small block in which the stability is studied and a large block whose field of values is located in the complex plane. Using the information on the blocks and some results on perturbation theory, we give sufficient conditions for the stability of the original matrix, a lower bound of $r_{\text{stab}}(A)$ and an upper bound on the norm of the exponential of $A$. We illustrate these theoretical bounds on a practical test problem. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let us begin with the following classical example of the bidiagonal matrix $A$ of order $n = 20$ with $-1$ on the diagonal and $10$ on the subdiagonal. This matrix is theoretically stable (in the sense of Hurwitz) since its eigenvalues, which are equal to $-1$, belong to the left-half part of the complex plane. Now if the zero element $A(1,20)$ is replaced by $\varepsilon$, then it is easy to see that the eigenvalues $\lambda$ of $A$ satisfy the equation $(1 + \lambda)^{20} - 10^{19}\varepsilon = 0$. If we take $\varepsilon = 10^{-18}$, then one of the eigenvalues becomes $\lambda = 2^{\sqrt{10}} - 1 > 0$.

This example shows that the stability analysis based only on the numerical computation of eigenvalues may be misleading.

Another important issue on stability theory is the stability radius denoted hereafter by $r_{\text{stab}}(A) = \min_{\Re z = 0} \sigma_{\min}(A - zI)$, where $\sigma_{\min}$ stands for the smallest singular value and $\Re z$ is the real part of $z$. The quantity $r_{\text{stab}}(A)$ measures the distance between a stable matrix $A$ and the set of unstable matrices.

* Corresponding author. E-mail: sadkane@univ-brest.fr.
matrices [17, 2]. Note that if $A$ is stable, then $\min_{\text{Re } z = 0} \sigma_{\min}(A - zI) = \min_{\text{Re } z \geq 0} \sigma_{\min}(A - zI)$, and if $A$ is a matrix such that $\min_{\text{Re } z \geq 0} \sigma_{\min}(A - zI) > 0$ then $A$ is stable. For a stable matrix $A$, the quantity $\text{rstab}(A)$ measures the smallest perturbation $\Delta$ such that $A + \Delta$ has an eigenvalue on the imaginary axis.

Finally, another point in stability theory is the norm of the exponential of a stable matrix $A$: it is important to give estimate in the form $\|e^{tA}\| \leq M e^{-\omega t}$ with $\omega > 0$, $t \geq 0$ and $M \geq 1$, where the symbol $\| \|$ denotes, throughout this note, the Euclidean norm or its induced matrix norm.

This work is concerned with the study of the stability radius and the norm of the exponential of large matrices. The idea is to reproduce the behavior of large problems by combining the standard stability techniques with Krylov type methods. More precisely, the method proposed partitions the large matrix $A$ in the following way:

$$A = [V_1, V_2] \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} [V_1, V_2]^* + R \quad \text{with } A_{11} \in \mathbb{C}^r \times r, \quad V_1 \in \mathbb{C}^n \times r, \quad r \ll n,$$

the other matrices $A_{12}, A_{22}$ and $V_2$ are of conforming size. The matrices $A_{11}, V_1$ and the quantity $\|R\|$ are explicitly given by Krylov-type methods.

Since $r \ll n$, reliable methods for computing $\text{rstab}(A_{11})$ [2, 17] and for bounding $\|e^{tA_{11}}\|$ [1] are used. The large matrices $V_2, A_{22}$ and $A_{12}$ are of course not computed, but we show that the largest eigenvalue $\mu$ of $(A_{22} + A_{22}^*)/2$ can easily be computed.

Using the information on $A_{11}$ and $\mu$ and some results on perturbation theory, we give a lower bound for $\text{rstab}(A)$ and an upper bound for $\|e^{tA}\|$. The bounds involve quantities readily computable.

This note is organized as follows: in Section 2, we briefly recall some known results on stability, stability radius and the norm of the exponential of matrices. In Section 3 we propose a variant suitable for large matrices. Section 4 is devoted to numerical results on some test problems.

Throughout this note the identity matrix in $\mathbb{C}^p \times p$ is denoted by $I_p$ or just $I$ if the order $p$ is clear from context. $C^* = C > 0$ ($< 0$) for a matrix $C$ means that $C$ is Hermitian positive (negative) definite. If $H^* = H$ and $C^* = C > 0$ then the set of eigenvalues of the Hermitian positive-definite matrix pair $(H, C)$ is defined by $\lambda(H, C) = \{\mu \in \mathbb{R} : \det(H - \mu C) = 0\}$. Since $C^* = C > 0$, we also have $\lambda(H, C) = \lambda(C^{1/2}HC^{-1/2}, I)$ where $C^{1/2}$ denotes the square root of $C$. From the Courant–Fischer minimax theorem [7, p. 411] the largest eigenvalue of the matrix pair $(H, C)$ is then characterized by $\lambda_{\text{max}}(H, C) = \max_{x \neq 0} (C^{1/2}HC^{-1/2}x, x)/(x, x) = \max_{x \neq 0} (Hx, x)/(Cx, x)$. The smallest eigenvalue is defined in a similar way.

2. On the stability of matrices

We recall a few classical results on stability, the stability radius and the norm of the exponential of matrices. Consider the Lyapunov equation

$$A^*H + HA + C = 0,$$

where $A$ and $C$ are given square matrices and where $H$ is an unknown Hermitian matrix.
Theorem 2.1. If the matrix $A$ is stable then the solution $H$ of (1) exists for all matrices $C$ and is given by the formula

\[ H = \int_0^\infty e^{tA^*} C e^{tA} \, dt. \]  

(2)

Conversely, if $C^* = C > 0$ and if Eq. (1) has a solution $H = H^* > 0$, then the matrix $A$ is stable. Moreover,

\[ \text{rstab}(A) \geq \frac{\lambda_{\text{min}}(C)}{2\|H\|}, \]  

(3)

\[ \|e^{tA}\| \leq \sqrt{\|H\|\|H^{-1}\|} e^{-t[2\lambda_{\text{max}}(H,C)]}, \quad t \geq 0. \]  

(4)

**Proof.** The first part of the theorem is known. See for example [11].

To prove (3), let $\xi_0 \in \mathbb{R}$ and $u_0 \in \mathbb{C}^n$ with $\|u_0\| = 1$ such that

\[ \frac{1}{\text{rstab}(A)} = \max_{\text{Re } z = 0} \|(A - zI)^{-1}\| = \|(A - i\xi_0 I)^{-1} u_0\| \]  

(5)

and let $x = (A - i\xi_0 I)^{-1} u_0$, then from (1), we have

\[ \lambda_{\text{min}}(C)\|x\|^2 \leq (Cx,x) = \|(A^* H + HA)x,x\| = 2|\text{Re } (Hx,u_0)| \leq 2\|H\|\|x\|. \]  

(6)

For the proof of (4), consider the differential equation $\frac{d}{dt} x(t) = Ax(t)$ whose solution is $x(t) = e^{tA}x(0)$. Then,

\[ \frac{d}{dt} (Hx(t),x(t)) = (H Ax(t),x(t)) + (Hx(t),Ax(t)) \]  

(7)

\[ = -(Cx(t),x(t)) \]  

(8)

\[ \leq -\frac{(Hx(t),x(t))}{\lambda_{\text{max}}(H,C)}, \]  

(9)

which implies

\[ (Hx(t),x(t)) \leq (Hx(0),x(0)) e^{-t/\lambda_{\text{max}}(H,C)}. \]  

(10)

Inequality (4) follows then from (10) and from the inequalities $\lambda_{\text{min}}(H)\|x(t)\|^2 \leq (H(x(t),x(t))$ and $(H(x(0),x(0)) \leq \lambda_{\text{max}}(H)\|x(0)\|^2$.  

There are, of course, sharper bounds for $\|e^{tA}\|$ [12, 16], but they are often based on the Schur or the Jordan factorization. The advantage with the bounds (3) and (4) is that, as we will see in Sections 3 and 4, they involve computable quantities even with large matrices.
An obvious, but important consequence of Theorem 2.1 is given in the following corollary.

**Corollary 2.2.** If \((A + A^*) < 0\) then

\[
\text{rstab}(A) \geq \lambda_{\min}\left(\frac{-A + A^*}{2}\right),
\]

\[
\|e^{At}\| \leq e^{-t\lambda_{\min}\left(\frac{-A + A^*}{2}\right)}, \quad t \geq 0.
\]

Inequality (12) is actually true for any matrix \(A\) [3], but it exhibits the asymptotic behavior of \(e^{At}\) only if \(A + A^* > 0\).

**Theorem 2.3.** Let \(A\) be a matrix of order \(n\).

(i) If \(A + \Delta\) is stable, then \(\text{rstab}(A + \Delta) > \text{rstab}(A) - \|\Delta\|\).

(ii) If \(\|e^{At}\| \leq Me^{-\omega t}\) with \(\omega > 0\), \(t \geq 0\), \(M > 1\), then \(\|e^{(t+\Delta)t}\| \leq Me^{(-\omega + M\|\Delta\|)t}\).

**Proof.** see [13, p. 495] for a proof of (ii).

In Theorem 2.3(ii), it is important that the quantity \(-\omega + M\|\Delta\|\) remains negative. In the case where \(-\omega + M\|\Delta\| > 0\) and \(\|\Delta\| > \text{rstab}(A)\), we have the following result.

**Proposition 2.4.** Assume that there exist \(M, \omega, \gamma > 0\) such that

\[
\|e^{At}\| \leq Me^{-\omega t}, \quad t \geq 0
\]

with

\[
\|\Delta\| > \text{rstab}(A) - \gamma \quad \text{and} \quad \gamma \leq \omega,
\]

then \(A + \Delta\) is stable and

\[
\|e^{(A+\Delta)t}\| \leq Me^{\gamma}(1 + \frac{\|\Delta\|}{\text{rstab}(A) - \|\Delta\| - \gamma})e^{-\gamma t}, \quad t \geq 0.
\]

**Proof.** Note first that condition (13) implies that \(A\) is stable, and as a consequence \(\text{rstab}(A)\) is well defined.

Let \(\zeta \in \mathbb{C}\) with \(\text{Re}\zeta \geq 0\), then \(\sigma_{\min}(\zeta I - (A + \Delta)) \geq \text{rstab}(A) - \|\Delta\| > \gamma > 0\) and hence \(A\) and \(A + \gamma I\) are stable and \((I - (A + A + \gamma I))\) is nonsingular.

From the equality \((\zeta I - (A + \Delta + \gamma I))^{-1} = (\zeta I - (A + \gamma I))^{-1}(I + \Delta(\zeta I - (A + A + \gamma I))^{-1})\) and the formula \((\zeta I - (A + \gamma I))^{-1} = \int_0^\infty e^{-\zeta t}e^{(A+\gamma I)t}dt\) we see that for all \(\zeta\) with \(\text{Re}\zeta > 0\)

\[
\|((\zeta I - (A + A + \gamma I))^{-1}\| \leq \frac{M}{\text{Re}\zeta + \omega - \gamma} \left(1 + \frac{\|\Delta\|}{\text{rstab}(A) - \|\Delta\| - \gamma}\right) \leq \frac{M}{\text{Re}\zeta} \left(1 + \frac{\|\Delta\|}{\text{rstab}(A) - \|\Delta\| - \gamma}\right).
\]
This implies (see [4, p. 227]) that

\[ \| e^{(A+\gamma I)} \| \leq 4n \left( 1 + \frac{\| A \|}{\text{rstab}(A) - \| A \| - \gamma} \right), \quad t \geq 0. \]  

(18)

### 3. Case of large matrices

If \( n \) is large, the study of the stability and estimates (3) and (4) cannot be obtained by the standard techniques because the quantities in (3) and (4) would involve large expense of storage requirements and high computational cost. We propose the use of Krylov-type methods [18] for computing an approximate invariant subspace corresponding to a few rightmost eigenvalues of \( A \).

We thus obtain two matrices \( V_1 \) and \( A_{11} \) of size \( n \times r \) and \( r \times r \), respectively, such that

\[ AV_1 - V_1 A_{11} = R, \quad \| R \| \leq \delta, \quad V_1^* V_1 = I, \quad V_1^* R = 0, \]  

(19)

where \( \delta \) is a small positive parameter and where \( r \) is an integer small compared to \( n \).

Let us now consider a matrix \( V_2 \) of size \( n \times (n - r) \) such that the matrix \( V = [V_1 V_2] \) is unitary. The matrix \( A \) can be written as

\[ A = V \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} V^* \equiv A_0 + \Delta, \]  

(20)

with

\[ A_{ij} = V_i^* A V_j, \quad i, j = 1, 2 \]  

(21)

\[ A_0 = V \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} V^* \]  

(22)

and

\[ \Delta = V \begin{pmatrix} 0 & 0 \\ A_{21} & 0 \end{pmatrix} V^*, \quad \| \Delta \| = \| R \| \leq \delta. \]  

(23)

Note that the matrices \( A_{12} \) and \( A_{22} \) can be determined in the following way.

Let \( V_1 = P_1 P_2 \ldots P_r(l_0) \) be the QR factorization of \( V_1 \) obtained by the Householder transformations \( P_i, \ i = 1, \ldots, r \) [7]. The matrix \( V_2 \) can be written as \( V_2 = P_1 P_2 \ldots P_r(l_{n-r}) \), and hence

\[ A_{12} = (0 \ I_{n-r}) P_r^* \ldots P_2^* P_1^* A^* V_1. \]  

(24)

Since the matrix \( A_{22} \) is large, only matrix–vector multiplications can be used on it. We notice that

\[ A_{22} x = (0 \ I_{n-r}) P^*_r \ldots P^*_2 P_1^* A P_1 P_2 \ldots P_r \begin{pmatrix} 0 \\ I_{n-r} \end{pmatrix} x, \]  

(25)

\[ A^*_2 x = (0 \ I_{n-r}) P^*_r \ldots P^*_2 P_1^* A^* P_1 P_2 \ldots P_r \begin{pmatrix} 0 \\ I_{n-r} \end{pmatrix} x \]  

(26)

for all vector \( x \). Thus, we can apply the Lanczos method to the Hermitian matrix \( (A_{22} + A^*_2)/2 \) and estimate its largest eigenvalue. If this eigenvalue is negative, we conclude that the field of values of \( A \) and hence the eigenvalues of \( A_{22} \) belong the left-half part of the complex plane.
3.1. Stability and stability radius

We first note that if $A_{11}$ is unstable, then so is the matrix $A_0$ and hence $\min_{\Re z > 0} \sigma_{\min}(A_0 - zI) = 0$. This means that $\min_{\Re z > 0} \sigma_{\min}(A - zI) \leq \delta$.

Let $\mu = \lambda_{\min}(-(A_{22} + A_{22}^*)/2)$. The following proposition gives a sufficient condition for the stability of $A$ and a lower bound for $\text{rstab}(A)$.

**Proposition 3.1.** Assume that $A_{11}$ is stable and $(A_{22} + A_{22}^*) < 0$. Let

\[
\begin{align*}
    r_1 &= \max \left( \frac{1}{\mu} r_{\text{stab}}(A_{11}) + \frac{\|A_{12}\|}{\mu r_{\text{stab}}(A_{11})} \right), \\
    r_2 &= \max \left( \frac{1}{r_{\text{stab}}(A_{11})}, \frac{1}{\mu} + \frac{\|A_{12}\|}{\mu r_{\text{stab}}(A_{11})} \right), \\
    r_3 &= \sqrt{\frac{1}{(r_{\text{stab}}(A_{11}))^2} + \frac{1}{\mu^2} + \frac{\|A_{12}\|^2}{\mu^2 (r_{\text{stab}}(A_{11}))^2}}.
\end{align*}
\]

If $\delta < 1/\min(\sqrt{r_1 r_2}, r_3)$ then $A$ is stable and $\text{rstab}(A) \geq 1/\min(\sqrt{r_1 r_2}, r_3) - \delta$.

**Proof.** We have

\[
\min_{\Re z > 0} \sigma_{\min}(A - zI) \geq \min_{\Re z > 0} \sigma_{\min}(A_0 - zI) - \delta = r_{\text{stab}}(A_0) - \delta,
\]

and

\[
\frac{1}{r_{\text{stab}}(A_0)} = \max_{\Re z > 0} \left\| (A_{11} - zI_r)^{-1} - (A_{11} - zI_r)^{-1} A_{12} (A_{22} - zI_{n-r})^{-1} \right\| \leq \|B\|
\]

with $B = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ and

\[
\begin{align*}
    a &= \max_{\Re z > 0} \left\| (A_{11} - zI_r)^{-1} \right\| = \frac{1}{r_{\text{stab}}(A_{11})}, \\
    c &= \max_{\Re z > 0} \left\| (A_{22} - zI_{n-r})^{-1} \right\| = \frac{1}{r_{\text{stab}}(A_{22})} \leq \frac{1}{\mu} \quad \text{(Corollary 2.1),} \\
    b &= \max_{\Re z > 0} \left\| (A_{11} - zI_r)^{-1} A_{12} (A_{22} - zI_{n-r})^{-1} \right\| \leq ac \|A_{12}\|.
\end{align*}
\]

The rest of the proposition is a direct application of the inequality $\|B\| \leq \min(\sqrt{\|B\|_1 \|B\|_\infty}, \|B\|_F)$, where $\|B\|_1$, $\|B\|_\infty$ and $\|B\|_F$ stand for the 1-norm, the infinite norm and the Frobenius norm, respectively. \qed

3.2. Norm of the exponential of matrices

If $A_{11}$ is stable and $(A_{22} + A_{22}^*) < 0$, then Theorem 2.1 and Corollary 2.2 allow us to estimate $\omega, \mu > 0, M \geq 1$ such that $\|e^{tA_{11}}\| \leq Me^{-\omega t}$ and $\|e^{-tA_{22}}\| \leq e^{-\mu t}$ for all $t \geq 0$. We then have the following result.
Proposition 3.2. Assume that $A_{11}$ is stable and $(A_{22} + A_{22}^*) < 0$. Let $\omega, \mu > 0$, $M \geq 1$ such that for all $t \geq 0$, $\|e^{tA_{11}}\| \leq Me^{-\omega t}$ and $\|e^{tA_{22}}\| \leq e^{-\mu t}$, then

$$\|e^{tA_0}\| \leq \left\| \begin{pmatrix} Me^{-\omega t} & M\|A_{12}\| & \frac{e^{-\omega t} - e^{-\mu t}}{\omega - \mu} \\ 0 & e^{-\mu t} \end{pmatrix} \right\| \leq M(1 + t\|A_{12}\|)e^{-2\nu t} \leq C e^{-\nu t}$$

and

$$\|e^{tA}\| \leq Ce^{-(\nu + C\delta)t}$$

for all $t \geq 0$ with $\nu = \frac{1}{2} \min(\omega, \mu)$ and $C = M \max(1, \|A_{12}\|/\nu)$.

Proof. We have

$$e^{tA_0} = V_1^* e^{tA_{11}} X(t) e^{tA_{22}} \left( \begin{array}{c} e^{tA_{11}} \\ 0 \end{array} \right) V.*$$

The matrix $X(t)$ satisfies the differential equation $(d/dt)X(t) = A_{11}X(t) + A_{12}e^{tA_{22}}$, $X(0) = 0$ whose solution is $X(t) = \int_0^t e^{(t-s)A_{11}}A_{12}e^{sA_{22}} ds$ with $\|X(t)\| \leq M\|A_{12}\|e^{-\omega t}\int_0^t e^{(\omega - \mu)s} ds$, from which (27) follows. Bounds (28) and (29) are obvious and bound (30) is a consequence of (29) and Theorem 2.2(ii).

Remark. Let $r_1, r_2$ and $r_3$ be the parameters defined in Proposition 3.1 and let $\gamma$ be a positive number such that $\delta < \min(\sqrt{r_1r_2}, r_3) - \gamma$ and $\gamma \leq \nu$. From Proposition 2.4 we have the following result which, unfortunately, depends on $n$

$$\|e^{tA}\| \leq Cen\left(1 + \frac{\delta}{\min(\sqrt{r_1r_2}, r_3) - \delta - \gamma} \right) e^{-\gamma t}$$

4. Numerical experiments

In this section, we outline and test an algorithm that summarizes the discussion of Section 3:

Algorithm

$\mu = 0$; $r = 0$

until ($\mu > 0$)

$r = r + 1$

Compute $V_1 \in C^{n \times r}$ and $A_{11} \in C^{r \times r}$ as in (19)

Compute $\mu = \lambda_{\min}(-(A_{22} + A_{22}^*)/2)$ as discussed in Section 3

end

The algorithm needs two methods: first, an Arnoldi-type method for computing the approximate invariant subspace span $\{V_1\}$ associated with a few rightmost eigenvalues of the large matrix $A$.
and second, the Lanczos method for computing the smallest eigenvalue of the Hermitian matrix 
\(- (A_{22} + A_{-22}^*)/2\).

It is clear that the proposed algorithm works well if the number \( r \) of the required rightmost

eigenvalues is not very large.

Let us illustrate the behavior of the algorithm on the Orr–Sommerfeld operator [14] defined by

\[
\frac{1}{\alpha R} L^2 y - i(ULy - U'y) - \lambda Ly = 0,
\]

where \( \alpha \) and \( R \) are positive parameters, \( \lambda \) is a spectral parameter number, \( U = 1 - x^2 \), \( y \) is a function defined on \([-1, +1]\) with \( y(\pm 1) = y'(\pm 1) = 0 \), \( L = d^2/dx^2 - x^2 \).

Discretizing this operator using the following approximation

\[
x_i = -1 + ih, \quad h = \frac{2}{n + 1},
\]

\[
L_h = \frac{1}{h^2} \text{Tridiag}(1, -2 - x^2 h^2, 1),
\]

\[
U_h = \text{diag}(1 - x_1^2, \ldots, 1 - x_n^2),
\]

gives rise to the eigenvalue problem

\[
Au = \lambda u \quad \text{with} \quad A = \frac{1}{\alpha R} L_h - iL_h^{-1}(U_h L_h + 2L_h).
\]

Taking \( \alpha = 1, R = 1000, n = 400 \) yields a complex non Hermitian matrix \( A \) (order \( n = 400, \|A\| = 160.80 \) whose spectrum is plotted in Fig. 1.

For comparison purpose, and since the matrix \( A \) is not so large, an application of the Matlab minimization function FMINU to the map \( \xi \in \mathbb{R} \to \| (i\xi I - A)^{-1} \| \) gives \( \text{rst}(A) \approx 1.97e-03 \). Fig. 2 shows the \( \| e^{tA} \| \) as a function of \( t \).
The stability of the Orr–Sommerfeld operator has been studied in [5]. Here we are interested in a simple version of the operator using the above discretization that leads to large matrices.

As we have already mentioned, the approximation of $V_1$ and $A_{11}$ may be obtained, for example, by Arnoldi’s method [18]. For the Orr–Sommerfeld example, the basic Arnoldi method was inefficient and we had to combine it with complex Chebyshev acceleration techniques [9, 19] which consist in restarting the Arnoldi process using Chebyshev polynomials that amplify the components of the required eigendirections while damping those in the unwanted ones. In this example, we worked with an Arnoldi basis of dimension 40 and Chebyshev polynomials of degree 20. It is clear that other techniques may also be used. We stress that it is not the intention of this note to compare the efficiency of existing sophisticated eigenvalue methods, but rather to show how $\text{rstab}(A)$ and $\|e^{tA}\|$ may be relatively cheaply approximated.

The computation of $\mu$ is done with the Hermitian Lanczos method which is very suitable for computing the extreme parts of the spectrum of Hermitian matrices [18].

Using the above algorithm, the condition “$A_{11}$ stable and $A_{22} + A_{22}^* < 0$” of Proposition 3.1 occurred for the first time when

$$r = 10 \quad \text{and} \quad \mu = 4.91e-02$$

with a tolerance

$$\delta = 1.00e-10$$

for both Arnoldi and Lanczos methods.

We applied the Matlab minimization function FMINU to the map $\xi \in \mathbb{R} \rightarrow \|(i\xi I - A_{11})^{-1}\|$ and obtained

$$\text{rstab}(A_{11}) = 2.71e-03.$$
Several “tricks” can be used to estimate $\|A_{12}\|$. The obvious one is $\|A_{12}\| \leq \|A\| = 160.80$ but we can notice from (21) or (24) that $\|A_{12}\| \leq \|P^k A\| = 1.35e+00$. Note that the exact computation of $\|A_{12}\|$ gives $\|A_{12}\| = 0.65$.

The parameters $r_1, r_2$ and $r_3$ of Proposition 3.1 are then:

$$ r_1 \approx 1.01e+04, \quad r_2 \approx 9.83e+03 \quad \text{and} \quad r_3 \approx 9.81e+03. \quad (37) $$

Therefore,

$$ \text{rstab}(A) \geq 1.01e-04 - \delta. \quad (38) $$

The Lyapunov equation $A_{11}^n H + H A_{11} + I = 0$, solved by the Matlab LYAP function gives

$$ M = \sqrt{\|H\| \|H^{-1}\|} = 2.86e+01 \quad \text{and} \quad \omega = \frac{1}{2\lambda_{\text{max}}(H)} = 2.04e-03. \quad (39) $$

Therefore, inequality (4) of Theorem 2.1 can be written as

$$ \|e^{t A_{11}}\| \leq M e^{-\omega t}, \quad t \geq 0. \quad (40) $$

From Proposition 3.2, we have

$$ \|e^{t A_2}\| \leq M (1 + t \|A_{12}\|) e^{-\omega t}, \quad t \geq 0. \quad (41) $$

and

$$ \|e^{t A}\| \leq Ce^{(\omega/2 + C\delta^\epsilon) t}, \quad t \geq 0 \quad \text{with} \quad C = 3.78e+04. \quad (42) $$

5. Conclusion

We have proposed and justified mathematically some techniques for analyzing the stability radius and the behavior of the norm of the exponential of large matrices. The proposed method uses Krylov subspace techniques to split the matrix into two blocks: a first block corresponding to the rightmost eigenvalues, explicitly given by the Krylov method, in which the stability radius and the norm of the exponential can be estimated using the standard methods, and a second block, whose field of values, estimated by the Hermitian Lanczos method, must belong to the left-half part of the complex plane. The method should work well if the block corresponding to the rightmost eigenvalues is small. In particular, this technique is suitable for elliptic operators and more generally for sectorial operators [13, p. 280], [6], i.e. operators whose field of values lies in a sector, provided that the part of the sector which lies on the right-half plane of the complex plane is small.

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