A direct one-step pressure actualization for incompressible flow with pressure Neumann condition

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Abstract

We develop a velocity-pressure algorithm, in primitive variables and finite differences, for incompressible viscous flow with a Neumann pressure boundary condition. The pressure field is initialized by least-squares and updated from the Poisson equation in one step without iteration. Simulations with the square cavity problem are made for several Reynolds numbers. We obtain the expected displacement of the central vortex and the appearance of secondary and tertiary eddies. Different geometry ratios and a 3D cavity simulation are also considered. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this work, we consider incompressible viscous flow in primitive variables by using finite differences and a Neumann pressure boundary condition, as discussed by Gresho and Sani [13]. This allows us to develop a direct one-step pressure actualization.

The discretization by difference methods of the Navier–Stokes equations on a staggered grid, as made by Casulli [8], when formulated in matrix terms, allows to identify a singular evolutive matrix system. When we derive the Poisson equation for the pressure and perform its integration, we can observe that a clear influence of the Neumann condition arises. From this we can extract a nonsingular system for determining the pressure values at the interior points. The initialization process of the pressure, by a least-squares procedure, somehow incorporates an optimal pressure as a starting point, instead of employing an arbitrary constant as it usually occurs with iterative methods. The values of the velocity at interior points can then be well determined by a forward Euler or Adams–Bashforth method. For the pressure we solve a nonsingular Poisson equation without iteration. The latter means that we incorporate the values of the pressure and velocity as soon as they are computed.

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This velocity–pressure algorithm with central differences has been tested with the cavity problem for a wide range of Reynolds numbers and geometric ratios that include square, deep and shallow cavities. For a square cavity the displacement of the central vortex to the geometrical center of the cavity was obtained by increasing the Reynolds number, as earlier established by Burggraf [5], Ghia et al. [12] and Schreiber and Keller [20], among others. Also, the apparition of secondary and tertiary vortices can be observed.

The proposed algorithm, described for 2D regions, can be appropriately modified for 3D regions. Simulations were also made for a 3D cavity.

2. The continuum equations for incompressible flow

We consider the Navier–Stokes equations

\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \nu \nabla^2 \mathbf{u}, \quad t > 0, \quad (1)
\]

\[
\nabla \cdot \mathbf{u} = 0, \quad (2)
\]

\[
\mathbf{u}(x, 0) = \mathbf{u}_0(x), \quad x \in \Omega = \Omega \oplus \Gamma, \quad (3)
\]

\[
\mathbf{u} = \mathbf{w}(x, t) \in \Gamma = \partial \Omega. \quad (4)
\]

For the pressure, we have the Poisson equation

\[
\nabla^2 p = \nabla \cdot (\nu \nabla^2 \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{u}) = \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) \quad (5)
\]

together with the Neumann condition [13]

\[
\frac{\partial p}{\partial n} = \nu \nabla^2 u_n - \left( \frac{\partial u_n}{\partial t} + \mathbf{u} \cdot \nabla u_n \right) \in \Gamma \quad \text{for } t \geq 0, \quad (6)
\]

where \( u_n = \mathbf{u} \cdot \mathbf{n} \) is the normal velocity component.

The determination of the solution of the Poisson equation with Neumann boundary conditions requires that the following compatibility relation holds:

\[
\int \int_{\Omega} -\nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) \, d\Omega = \int \mathbf{p}_n \, d\Gamma, \quad (7)
\]

where \( p_n = \mathbf{n} \cdot \nabla p \), and \( \mathbf{n} \) is an exterior normal unit vector to \( \Gamma \).

3. Discretization of the Navier–Stokes Equations

The primitive equations for a 2D incompressible viscous flow are

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \quad (8)
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right), \quad (9)
\]
\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \]  

(10)

where \( u(x, y, t) \) and \( v(x, y, t) \) denote the velocity components in \( x \) and \( y \) directions, \( p(x, y, t) \) the pressure and \( \nu \geq 0 \) the kinematic viscosity coefficient. Following Casulli [8], we use central differences for approximating the spatial derivatives and the explicit Euler method for approximating the time derivative. Thus, with reference to a staggered grid, we have

\[
\begin{align*}
    u_{i+1/2,j}^{k+1} &= F_1 u_{i+1/2,j}^k - \Delta t \frac{p_{i+1,j}^k - p_{i,j}^k}{\Delta x}, \quad i = 1, 2, \ldots, n - 1, \quad j = 1, 2, \ldots, m, \\
    F_1 u_{i+1/2,j}^k &= u_{i+1/2,j}^k - \Delta t \left[ \frac{u_{i+3/2,j}^k - u_{i-1/2,j}^k}{2\Delta x} + \frac{u_{i+1/2,j+1}^k - u_{i+1/2,j-1}^k}{2\Delta y} \right] \\
    + v \Delta t \left( \frac{u_{i+3/2,j}^k - 2u_{i+1/2,j}^k + u_{i-1/2,j}^k}{(\Delta x)^2} + \frac{u_{i+1/2,j+1}^k - 2u_{i+1/2,j}^k + u_{i+1/2,j-1}^k}{(\Delta y)^2} \right), \\
    v_{i+1/2,j}^k &= \frac{v_{i,j+1/2}^k + v_{i,j-1/2}^k + v_{i+1,j+1/2}^k + v_{i+1,j-1/2}^k}{4}.
\end{align*}
\]

(11)

and

\[
\begin{align*}
    v_{i,j+1/2}^{k+1} &= F_2 v_{i,j+1/2}^k - \Delta t \frac{p_{i,j+1}^{k+1} - p_{i,j}^k}{\Delta y}, \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, m - 1, \\
    F_2 v_{i,j+1/2}^k &= v_{i,j+1/2}^k - \Delta t \left[ \frac{v_{i+1,j+1/2}^k - v_{i-1,j+1/2}^k}{2\Delta x} + \frac{v_{i,j+3/2}^k - v_{i,j-1/2}^k}{2\Delta y} \right] \\
    + v \Delta t \left( \frac{v_{i+1,j+1/2}^k - 2v_{i,j+1/2}^k + v_{i-1,j+1/2}^k}{(\Delta x)^2} + \frac{v_{i,j+3/2}^k - 2v_{i,j+1/2}^k + v_{i,j-1/2}^k}{(\Delta y)^2} \right), \\
    \tilde{u}_{i,j+1/2}^k &= \frac{u_{i+1/2,j}^k + u_{i-1/2,j}^k + u_{i+1/2,j+1}^k + u_{i-1/2,j+1}^k}{4}.
\end{align*}
\]

(13)

(15)

(16)

4. The pressure equation discretization

The Poisson equation for the pressure is given by

\[ \Delta p = - \nabla \cdot (u \cdot \nabla u) - D_t, \]

(17)

where the dilation term

\[ D = u_x + v_y \]

is included for numerical stability purposes. We now restrict our discussion to a rectangular domain.
The following Neumann boundary conditions for the pressure are obtained from the momentum equation on a solid boundary 
\[ x = 0, 0 \leq y \leq B, \quad 0 \leq x \leq A, \quad y = B, \quad x = A, \quad 0 \leq y \leq B, \quad 0 \leq x \leq A, \quad y = 0: \]
\[ -p_x = u_t + u u_x + v u_y + v (v_{xy} - u_{yy}) \quad \text{in} \quad x = 0, \quad A, \quad (18) \]
\[ -p_y = v_t + u v_x + v v_y - v (v_{xx} - u_{xy}) \quad \text{in} \quad y = 0, \quad B. \quad (19) \]

The Poisson equation (17) and the boundary conditions (18) and (19) for the pressure are now approximated on a staggered grid with 
\[ x = y = h. \] The spatial derivatives in (17)–(19) shall be now approximated by second-order central differences for interior cells and cells adjacent to the boundary.

### 4.1. Interior cells

We consider the Poisson equation
\[ p_{xx} + p_{yy} = -\frac{\partial}{\partial x} \left( u \frac{\partial u}{\partial x} \right) - \frac{\partial}{\partial x} \left( v \frac{\partial u}{\partial y} \right) - \frac{\partial}{\partial y} \left( u \frac{\partial v}{\partial x} \right) - \frac{\partial}{\partial y} \left( v \frac{\partial v}{\partial y} \right) - D_t. \quad (20) \]

As usual, the dilatation term \( D_t \) is approximated by
\[ D_t \approx \frac{D^{k+1} - D^k}{\Delta t}, \quad (21) \]
where the superscript indexes \( k \) and \( k + 1 \) refer to the time levels \( t \) and \( t + \Delta t \). In order to satisfy the continuity equation (10), \( D^{k+1} \) is made equal to zero.

Let \((i, j)\) refer to an interior cell, that is, without common sides with the boundary. By using second-order central differences for approximating the derivatives \( p_{xx} \) and \( p_{yy} \), the Poisson equation (20) is approximated by
\[
\begin{align*}
p_{i+1,j} + p_{i-1,j} + p_{i,j+1} + p_{i,j-1} - 4p_{i,j} & = -\frac{1}{2}u_{i+1/2,j} \left(u_{i+3/2,j} - u_{i-1/2,j}\right) \\
& + \frac{1}{2}u_{i-1/2,j} \left(u_{i+1/2,j} - u_{i-3/2,j}\right) - \frac{1}{2}v_{i+1/2,j} \left(u_{i+1/2,j+1} - u_{i+1/2,j-1}\right) \\
& + \frac{1}{2}v_{i-1/2,j} \left(u_{i-1/2,j+1} - u_{i-1/2,j-1}\right) - \frac{1}{2}u_{i,j+1/2} \left(v_{i+1,j+1/2} - v_{i-1,j+1/2}\right) \\
& + \frac{1}{2}u_{i,j-1/2} \left(v_{i+1,j-1/2} - v_{i-1,j-1/2}\right) - \frac{1}{2}v_{i,j+1/2} \left(v_{i,j+3/2} - v_{i,j-1/2}\right) \\
& + \frac{1}{2}v_{i,j-1/2} \left(v_{i,j+1/2} - v_{i,j-3/2}\right) + \frac{h}{\Delta t} \left(u_{i+1/2,j} - u_{i-1/2,j} + v_{i,j+1/2} - v_{i,j-1/2}\right),
\end{align*}
\]
\[ i = 2, 3, \ldots, n - 1, \quad j = 2, 3, \ldots, m - 1. \quad (22) \]
5. Cells adjacent to the boundary

The boundary condition (18) is computed at \( u_{3/2,j} \) by using a central difference approximation. Thus

\[
p_{2,j} - p_{1,j} = -\frac{h}{\Delta t} (u_{3/2,j}^{k+1} - u_{3/2,j}^k) - \frac{1}{2} u_{3/2,j} (u_{5/2,j} - u_{1/2,j}) \\
- \frac{1}{2} \pi_{3/2,j} (u_{3/2,j+1} - u_{3/2,j-1}) - \frac{v}{h} (-v_{1,j+1/2} + v_{2,j+1/2} \\
+ v_{1,j-1/2} - v_{2,j-1/2} - u_{3/2,j+1} + 2u_{1/2,j} - u_{3/2,j-1}), \quad j = 2, 3, \ldots, m - 1. \quad (23)
\]

Similar expressions are obtained by using (18) at \( u_{n-1/2,j} \) and computing the boundary condition (19) at \( v_{i,3/2} \) and \( v_{i,m-1/2} \), that is

\[
-p_{n,j} + p_{n-1,j} = \frac{h}{\Delta t} (u_{n-1/2,j}^{k+1} - u_{n-1/2,j}^k) + \frac{1}{2} u_{n-1/2,j} (u_{n+1/2,j} - u_{n-3/2,j}) \\
+ \frac{1}{2} \pi_{n-1/2,j} (u_{n-1/2,j+1} - u_{n-1/2,j-1}) + \frac{v}{h} (-v_{n-1,j+1/2} + v_{n,j+1/2} \\
+ v_{n-1,j-1/2} - v_{n,j-1/2} - u_{n-1/2,j+1} + 2u_{n-1/2,j} - u_{n-1/2,j-1}), \quad j = 2, 3, \ldots, m - 1, \quad (24)
\]

\[
p_{i,2} - p_{i,1} = -\frac{h}{\Delta t} (v_{i,3/2}^{k+1} - v_{i,3/2}^k) - \frac{1}{2} \pi_{i,3/2} (v_{i+1,3/2} - v_{i-1,3/2}) \\
- \frac{1}{2} v_{i,3/2} (v_{i,1/2} - v_{i,1/2}) + \frac{v}{h} (v_{i+1,1/2} - 2v_{i,1/2}) \\
+ v_{i-1,1/2} + u_{i-1,1/2} - u_{i+1,1/2} - u_{i-1/2,1} + u_{i+1/2,1}), \quad i = 2, 3, \ldots, n - 1, \quad (25)
\]

\[
-p_{i,m} + p_{i,m-1} = \frac{h}{\Delta t} (v_{i,m-1/2}^{k+1} - v_{i,m-1/2}^k) + \frac{1}{2} \pi_{i,m-1/2} (v_{i+1,m-1/2} - v_{i-1,m-1/2}) \\
+ \frac{1}{2} v_{i,m-1/2} (v_{i,m+1/2} - v_{i,m-3/2}) - \frac{v}{h} (v_{i+1,m-1/2} - 2v_{i,m-1/2}) \\
+ v_{i-1,m-1/2} + u_{i-1,m-1/2} - u_{i+1,m-1/2} - u_{i-1,m-3/2} - u_{i+1,m+1/2} - u_{i+1,2,m-1}) \quad i = 2, 3, \ldots, n - 1. \quad (26)
\]

The terms \( \pi_{i,j+1/2} \) and \( u_{n+1/2,j} \) in (22)–(26) are defined by (13) and (16).

The addition of terms on both sides of (22)–(26) can be interpreted as a discrete divergence theorem [1]. In our case, both add up to zero which tell us that the compatibility equation (7) is exactly satisfied on a staggered grid.

We should observe that the viscous terms in the momentum equations (8) and (9) do not appear in the source term for the Poisson equation (17). However, they are present within the Neumann boundary conditions (18), (19). In order to satisfy the compatibility condition (7), the integral of the viscous terms over the boundary must cancel. This is obtained by writing the viscous terms in a convenient way. More precisely, by using the continuity equation (10), we can write \( u_{xx} + u_{yy} = -v_{xy} + u_{yy} \) in (18) and \( v_{xx} + v_{yy} = v_{xx} - u_{xy} \) in (19). The additional term does not ocassionate any trouble on the compatibility condition because the integral of the dilatation over the solution domain vanishes due to global continuity.
6. The velocity–pressure algorithm

We now give an algorithm for integrating the Navier–Stokes equations. First, the pressure is initialized by applying least squares to the singular system that arises from the discretization of (17) with the Neumann conditions (18) and (19). Second, the momentum Eqs. (8) and (9) are solved for the velocity field at each time step. Third, the pressure is updated from (17)–(19) by giving a special treatment for the interior points that correspond to interior cells and to the adjacent cells in such a way that the compatibility condition is verified. The pressure at interior points of interior cells are computed in a direct manner, that is, by incorporating the already known pressure values at cause points in (22).

6.1. Pressure initialization

From (22) at the time level \( k = 0 \) and discretizing the Neumann condition we obtain the matrix system

\[
A p_0 = b,
\]

where \( A \) is the singular matrix

\[
A = \begin{bmatrix}
S_1 & I & I & S_2 & I & \cdots & I & S_2 & I \\
1 & S_2 & I & I & S_2 & I & \cdots & I & S_2 & I \\
& & \ddots & & & \ddots & & & \ddots & & \\
& & & 1 & S_2 & I & I & S_2 & I & \\
& & & & & & \ddots & & & \ddots & \\
& & & & & & & 1 & S_2 & I \\
& & & & & & & & I & S_1 \\
\end{bmatrix}_{(m \times n) \times (m \times n)}
\]

with

\[
S_1 = \begin{bmatrix}
-2 & 1 & & & & & & \\
1 & -3 & 1 & & & & & \\
& & \ddots & & & & & \\
& & & 1 & -3 & 1 & & \\
& & & & 1 & -3 & 1 & \\
\end{bmatrix}_{n \times n}, \quad S_2 = \begin{bmatrix}
-3 & 1 & & & & & & \\
1 & -4 & 1 & & & & & \\
& & \ddots & & & & & \\
& & & 1 & -4 & 1 & & \\
& & & & 1 & -3 & & \\
\end{bmatrix}_{n \times n}
\]

and \( I \) is the identity matrix of order \( n \).

At time \( k = 0 \), the vector \( p_0 \) contains all associated values of the pressure at interior points, that is,

\[
p_0 = \begin{bmatrix}
p_{1,1}^0 & p_{2,1}^0 & \cdots & p_{n,1}^0 & p_{1,2}^0 & p_{2,2}^0 & \cdots & p_{n,2}^0 & \cdots & p_{1,m}^0 & p_{2,m}^0 & \cdots & p_{n,m}^0
\end{bmatrix}^T.
\]

The vector \( b \) contains all values \( u_{i+1/2,j}^0, v_{i+1/2,j}^0 \), from the right-hand side of (22)–(26), which are given initial values, and this has the particular form,

\[
b = \begin{bmatrix}
0 & \cdots & 0 & b_{mn-n+1}^0 & \cdots & 0 & b_{mn}^0 & \end{bmatrix}^T_{m \times n}.
\]
where
\[ b_{mn-n+1} = \frac{2v}{h} \quad \text{and} \quad b_{mn} = -\frac{2v}{h}. \]
Hence, \( b \) is a nonzero vector.

The singular system (27) is then solved by least squares.

6.2. Pressure equation

Once the pressure is initialized, the interior pressure values \( p_{i,j} \) at time \( t + \Delta t \) are computed with the following criteria:
1. At the interior points corresponding to adjacent boundary cells we use (23) and (36).
2. At interior points of the interior cells, we employ (22) to compute the pressure values at each time level by incorporating previous values of the velocity and pressure fields. This modification lead us to

\[
p_{i,j}^{k+1} = \frac{1}{4} \left( p_{i+1,j}^{k} + p_{i,j+1}^{k} + p_{i,j-1}^{k} + p_{i-1,j}^{k} \right) + \frac{1}{8} t_{i+1/2,j}^{k+1} \left( u_{i+3/2,j}^{k+1} - u_{i-1/2,j}^{k+1} \right)
\]
\[
- \frac{1}{8} u_{i-1/2,j}^{k+1} \left( u_{i+1/2,j}^{k} - u_{i-3/2,j}^{k} \right) + \frac{1}{8} v_{i+1/2,j}^{k+1} \left( v_{i+1/2,j+1}^{k} - v_{i+1/2,j-1}^{k} \right)
\]
\[
- \frac{1}{8} v_{i-1/2,j}^{k+1} \left( v_{i+1/2,j+1}^{k} - v_{i-1/2,j-1}^{k} \right) + \frac{1}{8} u_{i,j+1/2}^{k+1} \left( v_{i,j+3/2}^{k+1} - v_{i,j-1/2}^{k+1} \right)
\]
\[
- \frac{1}{8} u_{i,j-1/2}^{k+1} \left( v_{i,j+1/2}^{k+1} - v_{i,j-1/2}^{k+1} \right) = \frac{h}{4\Delta t} \left( u_{i+1/2,j}^{k+1} - u_{i-1/2,j}^{k+1} + v_{i,j+1/2}^{k+1} - v_{i,j-1/2}^{k+1} \right),
\]
\[ i = 2, 3, \ldots, n - 1, \ j = 2, 3, \ldots, m - 1. \]  \((28)\)

6.3. Velocity–pressure algorithm

The algorithm for solving an incompressible viscous flow with prescribed Neumann condition for the pressure is as follows.
1. Introduction of the initial velocity components \( u_{i+1/2,j}^{0}, v_{i,j+1/2}^{0} \) at time \( t_0 = 0 \), corresponding to level \( k = 0 \), and the boundary conditions for the velocity field.
2. Initialization of the pressure through least squares, that is, to solve a singular linear system of the type

\[ A p_0 = b. \]
3. Computation of the velocity field \( u_{i+1/2,j}^{k+1} \) and \( v_{i,j+1/2}^{k+1} \) by using (11)–(13) and (14)–(16).
4. Direct computation of the pressure \( p \) at time level \( k + 1 \) through (28).
5. Updating of the pressure and velocity field by setting \( p_{i,j}^{k+1} \) instead of \( p_i \) and \( \overline{u}_{i,j}^{k+1} \) for \( \overline{u}_{i,j}^{k} \).
6. To perform steps (3)–(5) for \( k = 1, 2, \ldots \).
7. End the calculations.

7. Pressure correction of a multi-step velocity integration

By using Adams–Bashforth for the time derivatives, the momentum equation can be written in discretized form

\[
\mathbf{u}^{k+1} = \mathbf{u}^k + \Delta t \sum_{l=0}^{n_p-1} \alpha_l [F(u^{k-l}) - \nabla p^{k-l}],
\]

where \( F \) only contains convective and diffusive terms.

By applying the divergence operator to (29), we have that the incompressibility condition at time level \( k + 1 \) is characterized by

\[
\nabla^2 p^k = \nabla \cdot F(u^k) + \frac{\nabla \cdot \mathbf{u}^k}{\alpha_0 \Delta t} + \frac{1}{\alpha_0} \sum_{l=1}^{n_p-1} \alpha_l [\nabla \cdot F(u^{k-l}) - \nabla^2 p^{k-l}].
\]

Following the steps of the pressure discretization for Euler integration, we obtain

\[
p_{i,j}^{k+1} = \frac{1}{4} [p_{i-1,j}^{k+1} + p_{i+1,j}^{k+1} + p_{i,j-1}^{k+1} + p_{i,j+1}^{k+1}]
\]

\[
- \frac{h^2}{4} \left[ \nabla \cdot F(u_{i,j}^k) + \frac{\nabla \cdot u_{i,j}^k}{\alpha_0 \Delta t} + \frac{1}{\alpha_0} \sum_{l=1}^{n_p-1} \alpha_l (\nabla \cdot F(u_{i,j}^{k-l}))
\]

\[
- \nabla^2 p_{i,j}^{k-1} \right].
\]

By Taylor expansion about \( p_{i,j}^{k-1} \) and replacing \( \nabla^2 p^{k+1} \) in \( \nabla \cdot \mathbf{u}^{k+1} \) turns out that

\[
\nabla \cdot \mathbf{u}^{k+1} = O\left( \frac{\Delta t^2 p_t}{h^2} \right) + O(\Delta t \Delta h^2 p_{ss}).
\]

This shows that we have an artificial compressibility of order \( O((\Delta t^2/h^2)(\partial p/\partial t)) \).

8. Simulations

Numerical simulations were carried out for the cavity problem for a broad range of Reynolds numbers. Figs. 1 and 2 show the velocity for \( Re = 400, 1000, 5000 \) and 10000 on a square grid with \( \Delta x = 0.01 \) and time steps \( \Delta t = 0.001, 0.002 \). This values meet the stability criteria \( \Delta t/h < 1 \) and \( \Delta t \leq h^2/4v \) as suggested by Roache, [18, Casulli [8], among others.

The last simulation was made with an extension of the proposed algorithm to a 3D cavity.
9. Conclusions

An algorithm has been developed for the numerical solution of the incompressible Navier–Stokes with central differences in primitive variables and the Neumann boundary condition for the pressure on a staggered grid. This algorithm was tested with the cavity problem for several Reynolds numbers in 2D and 3D regions. For the square cavity, it has been observed the apparition of the central vortex and the recirculation with secondary and tertiary eddies. As the Reynolds number increases, the central vortex moves toward the geometrical center of the cavity as shown before.
by Burggraf [5] Ghia et al. [12] and Schreiber and Keller [20], etc. The time integration can be also performed by other methods in order to increase the time step and to diminish the number of iterations. We can also consider, within Casulli’s unified formulation, up-wind and semi-Lagrangian methods for the spatial discretization.

Fig. 2. Tridimensional cavity at Re = 400 (a) perspective view; (b) y–z plane; (c) x–z plane, (d) x–y plane.
References