Discrete analogue of Fučík spectrum of the Laplacian

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Received 16 May 1997; received in revised form 26 June 1997

Abstract

The discrete analog of the Fučík spectrum for elliptic equations, namely M-matrices, is shown to have properties analogous to the continuum. In particular, the Fučík spectrum of a M-matrix contains a continuous and decreasing curve which is symmetric with respect to the diagonal. © 1999 Elsevier Science B.V. All rights reserved.

Keywords: Fučík spectrum; Discrete Laplacian; M-matrices

1. Introduction

Let $\Omega$ a be bounded domain contained in the plane $\mathbb{R}^2$ and $u$ be defined on $\Omega$ and then define

$$u^+ = \max(u, 0), \quad u^- = \min(u, 0).$$

Then the Fučík spectrum of the Laplacian with a Dirichlet boundary condition is defined as the set $\Sigma$ of those $(\alpha, \beta)$ such that

$$-\Delta u = \alpha u^+ - \beta u^- \quad \text{on } \Omega,$$
$$u = 0 \quad \text{on } \partial \Omega,$$

has a nontrivial weak solution.

In the discrete case, assume that $x \in \mathbb{R}^n$ and then define

$$x^+ = \max(x, 0), \quad x^- = \min(x, 0).$$

If $A$ is a real $n \times n$ matrix then the discrete analog of the Fučík spectrum for $A$ is the set $\Sigma(n)$ of those $(\alpha, \beta)$ such that

$$A x = \alpha x^+ - \beta x^- , \quad x \in \mathbb{R}^n ,$$

has a nontrivial solution. It is well-known that discretizations of (1) are commonly M-matrices, particularly when finite-difference methods are used for the discretization (see [4]). Actually, this
is true, more generally, for elliptic operators in divergence form. So we restrict our attention to M-matrices.

The discrete analogue $\Sigma(n)$ appears already in Espinoza [3, Proposition 2.3], where the discrete analogue of semilinear boundary value problems using variational methods are studied. De Figueiredo and Gossez [1] have obtained a variational characterization of the first nontrivial curve $\Gamma$ contained in the Fučík Spectrum of a general differential operator in divergence form, with Dirichlet boundary condition. They have proved that $\Gamma$ is a continuous and strictly decreasing curve, which is symmetric with respect to the diagonal, unbounded and asymptotic to the lines $\lambda_1 \times [\lambda_1, \infty)$ and $[\lambda_1, \infty) \times \lambda_1$, where $\lambda_1$ is the first eigenvalue of the differential operator.

2. The discrete analogue

Using the ideas described in [1], we prove that the Fučík spectrum for a $n \times n$ M-matrix $\Sigma(n)$ contains a curve $\Gamma(n)$ with the same properties, except for one, as the curve $\Gamma$ in $\Sigma$. Example 2.8 below shows that the remaining property is not true for the discrete case.

2.1. Some properties of $\Sigma(n)$

M-matrices are defined as in [4]. It is known that M-matrices are real, symmetric, irreducible, diagonally dominant and positive definite. Hence their eigenvalues are all positive and the first eigenvalue $\lambda_1$ has multiplicity one, with a unique associated norm one eigenvector $\varphi_1$ whose components are all positive. From this it is easy to see that:

1. The lines $\lambda_1 \times [\lambda_1, \infty)$ and $[\lambda_1, \infty) \times \lambda_1$ are contained in $\Sigma(n)$.

2. Because $(-x)^+ = x^-$ and $(-x)^- = x^+$, $\Sigma(n)$ is symmetric with respect to the diagonal.

Now suppose that $(x, \beta) \in \Sigma(n)$. Then there exists a vector $x \neq 0$ satisfying (2). Multiplying both sides of (2) by $\varphi_1$, and taking into account that $\varphi_1$ is the eigenvector with eigenvalue $\lambda_1$, gives

$$\langle (\lambda_1 - \alpha) x^+ - (\lambda_1 - \beta) x^-, \varphi_1 \rangle = 0,$$

where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathbb{R}^n$ and $\| \cdot \|$ is the norm given by this inner product. If $(\lambda_1 - \alpha) \neq 0$, then

$$\langle x^+ - rx^-, \varphi_1 \rangle = 0 \quad \text{where} \quad r = \frac{\lambda_1 - \alpha}{\lambda_1 - \beta} x^-,$$

which says that there exists an orthogonality relationship between the solution $x$ of (2) and the first eigenvalue $\varphi_1$ of $A$. This relationship motivates the following definitions:

**Definition 1.** For each $r > 0$ define the functions $\psi_1$ and $\psi_2$ on $\mathbb{R}^n$ by

$$\psi_1(x) = 1 - \|x^+\|^2 - r \|x^-\|^2,$$

$$\psi_2(x) = \langle x^+ - rx^-, \varphi_1 \rangle,$$
and then define the sets \( N_r \) and \( M_r \) by
\[
N_r = \{ x \in \mathbb{R}^n : \psi_1(x) = 0 \}, \\
M_r = \{ x \in \mathbb{R}^n : \psi_2(x) = 0 \}.
\]

**Remark 2.** It is easy to prove the following:
1. \( \psi_1(x) \) is a differentiable function and \( \psi_2(x) \) is a Lipschitzian and increasing function.
2. \( M_r \) is a closed set and \( N_r \) is a compact set; hence \( M_r \cap N_r \) is a compact set.
3. If \( x \in M_r \cap N_r \), then \( x^+ \neq 0 \) and \( x^- \neq 0 \). Moreover \( \langle x^-, \varphi_1 \rangle \neq 0 \) and \( \langle x^+, \varphi_1 \rangle \neq 0 \).
4. The subspace \( \langle \varphi_1 \rangle \) spanned by \( \varphi_1 \) is disjoint from \( M_r \cap N_r \).

**Definition 3.** Define the function \( \psi \) by
\[
\psi(x) = \langle Ax, x \rangle - \lambda_1 \| x \|^2.
\]

The function \( \psi \) plays a fundamental role in the study of discrete analogue \( \Sigma(n) \) of the Fučík spectrum. Since \( \psi(x) \) is a function of class \( C^\infty \) on \( \mathbb{R}^n \), there exists \( \omega \in M_r \cap N_r \), not necessarily unique, such that
\[
\psi(\omega) = \min \{ \psi(x) : x \in M_r \cap N_r \}.
\]

From Remark 2, \( \omega^+ \neq 0 \) and \( \omega^- \neq 0 \), and thus \( \omega \notin \langle \varphi_1 \rangle \). Furthermore, as the first eigenvalue \( \lambda_1 \) of \( A \) is positive and strictly less than the other eigenvalues, \( \psi(\omega) > 0 \).

As \( \psi(x) \) and \( \psi_1(x) \) are differentiable functions and \( \psi_2(x) \) is a Lipschitzian function, by the Lagrange Multiplier Rule (see [2, p. 347]), there exists nonnegative constants \( a, b, c \) and a vector \( u \) that belongs to the subgradient (see [2]) of \( \psi_2(x) \) at \( \omega \) and such that
\[
2a (Aw - \lambda_1 \omega) - 2b (\omega^+ - \omega^-) + cu = 0, \tag{3}
\]
and \( a + b + c = 1 \). Multiply both sides of the Eq. (3) by \( \varphi_1 \) to get \( c = 0 \), since \( \langle u, \varphi_1 \rangle \neq 0 \).

If \( a = 0 \) then \( b = 0 \), which contradicts the Lagrange Multiplier Rule, so \( a \neq 0 \). Letting \( \theta(r) = b/a \), we have
\[
Aw = (\lambda_1 + \theta(r)) \omega^+ - (\lambda_1 + r \theta(r)) \omega^-.
\]

Multiplying this equality by \( \omega \) gives the explicit form of the function \( \theta \):
\[
\theta(r) = \psi(\omega) = \min \{ \psi(x) : x \in M_r \cap N_r \}.
\]

Thus we have proved the following proposition.

**Proposition 4.** If \( M_r \) and \( N_r \) are as in Definition 1 then
1. The function \( \psi(x) \) given in Definition 3 attains its minimum on \( M_r \cap N_r \) at the point \( \omega \). Thus there exists a positive function
\[
\theta(r) = \min \{ \psi(x) : x \in M_r \cap N_r \} = \psi(\omega)
\]
defined for every \( r > 0 \);
2. \(\omega\) is a solution of the equation
\[ Ax = \alpha x^+ - \beta x^- \]
with
\[ \alpha(r) = \lambda_1 + \theta(r), \quad \beta(r) = \lambda_1 + r\theta(r). \]

Consequently, the curve
\[ \Gamma(n) = \{(\alpha(r), \beta(r)): r > 0\} \]
is contained in the discrete analogue of the Fučik spectrum: \(\Sigma(n)\).

Now we will prove the continuity of \(\theta(r)\) for \(r > 0\). Recall that \(\alpha(r)\) and \(\beta(r)\) are continuous functions for every \(r > 0\). So let \(r_m\) be a sequence of real positive numbers that converge to \(r > 0\). Also let \(\omega_m \in M_{r_m} \cap N_{r_m}\) and \(\omega \in M_r \cap N_r\) be points where \(\theta(r_m) = \psi(\omega_m)\) and \(\theta(r) = \psi(\omega)\). Since \(\omega_m\) is bounded, there exists a subsequence (denoted in the same way) and \(v \in \mathbb{R}^n\) such that \(\omega_m \to v\). Because of the continuity of \(\psi_1(x), \psi_2(x)\) and \(\psi(x)\) it follows that \(v \in M_r \cap N_r\) and \(\theta(r_m) = \psi(\omega_m) \to \psi(v)\).

On the other hand, it is quite easy to prove that there exists \(z_m\) and \(z_m\), positive real numbers, such that
\[ z_m = \alpha_m\omega^+ - \beta_m\omega^+ \in M_{r_m} \cap N_{r_m}. \]
Hence \(\psi(\omega_m) \leq \psi(z_m)\). Moreover, since \((x_m, \beta_m) \to (1, 1)\), and \(\psi(v) \leq \psi(\omega)\), the continuity of \(\theta(r)\) then follows. It is easy to check that \(x \in M_r \cap N_r\) if only if
\[ (-\sqrt{r}x) \in M_{1/r} \cap N_{1/r}, \]
from which it follows \(r \psi(x) = \psi(-\sqrt{r}x)\) and consequently \(r \theta(r) = \theta(1/r)\).

Finally, we conclude this section by showing that \(\Gamma(n)\) is a decreasing curve. The functions \(g\) and \(h\) are defined as

**Definition 5.**
\[
g(\varepsilon) = \| (x + \varepsilon \varphi_1)^+ \|^2 + r \| (x + \varepsilon \varphi_1)^- \|^2, \\
h(\varepsilon) = \| (x + \varepsilon \varphi_1)^+ \|^2 + (r + t) \| (x + \varepsilon \varphi_1)^- \|^2.
\]

It is easy to check that \(g(\varepsilon) > 1\) for all \(\varepsilon > 0\) and thus \(h(\varepsilon) > 1\). There exists \(\varepsilon > 0\) such that
\[ z = \omega + \varepsilon \varphi_1 \in M_{r+t} \cap N_{r+t}, \]
and then also
\[ \frac{z}{\sqrt{h(\varepsilon)}} \in M_{r+t} \cap N_{r+t}. \]
Hence
\[ \theta(r) \leq \psi \left( \frac{z}{\sqrt{h(\varepsilon)}} \right) = \frac{1}{h(\varepsilon)} \psi(z) = \frac{1}{h(\varepsilon)} \psi(\omega) < \psi(\omega) = \theta(r). \]
Thus we have proved the proposition:

**Proposition 6.** Let \( \alpha(r) \), \( \beta(r) \) and \( \Gamma(n) \) be as in Proposition 2.6. Then for every \( r > 0 \)
(a) \( \alpha(r) = \beta(1/r) \);
(b) \( \alpha(r) \) and \( \beta(r) \) are continuous functions;
(c) \( \alpha(r) \) is a decreasing function.

**Example 7.** Consider the matrix
\[
A = \begin{bmatrix}
2 & -1 \\
-1 & 2
\end{bmatrix}.
\]
In this case \( \lambda_1 = 1 \) and
\[
\Gamma(n) = \left\{ (2 + \frac{1}{r} - 2 + r) : r > 0 \right\}.
\]
This is a curve which is not asymptotic to the lines \( \lambda_1 \times [\lambda_1, \infty) \) and \( [\lambda_1, \infty) \times \lambda_1 \).
Consequently, the asymptotic property in the continuum cannot hold in the discrete case.

**References**