The completely conservative difference schemes for the nonlinear Landau–Fokker–Planck equation

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Abstract

Conservativity and complete conservativity of finite difference schemes are considered in connection with the nonlinear kinetic Landau–Fokker–Planck equation. The characteristic feature of this equation is the presence of several conservation laws. Finite difference schemes, preserving density and energy are constructed for the equation in one- and two-dimensional velocity spaces. Some general methods of constructing such schemes are formulated. The constructed difference schemes allow us to carry out the numerical solution of the relaxation problem in a large time interval without error accumulation. An illustrative example is given. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

The kinetic Landau–Fokker–Planck (LFP) equation is widely used for the description of collisional plasma processes. As an intrinsic part of physical models, both analytical and numerical, this equation has many applications in laboratory as well as in space plasma physics [5, 6, 15].

The mathematical model of any physical process has to reflect the main properties of this process, for example, the conservation laws. Since the dissipative qualities of the usual finite difference schemes for solving the space uniform LFP equation may lead to distorted results, conservative finite difference schemes are widely used [3, 14].

It should be remarked that for the nonlinear LFP equation a nontrivial situation exists: at least two conservation laws (for particle density and energy) are valid. If the difference scheme possesses only an approximate analog of the conservation laws, then this can easily lead to the accumulation of errors in the analysis of the nonstationary problems. The difference schemes which satisfy these
two laws we call the completely conservative schemes. There still does not exist a general method of constructing completely conservative difference schemes [7, 9]. The ideal situation is that where one succeeds in satisfying the conservation laws by only one approximation of an operator on the spatial mesh, without thereby sacrificing the order of approximation and retaining the free choice of the approximation of the time derivative (stability). In this formulation the problem was solved for the LFP equation in [1, 8, 11] and an entropy scheme for LFP equation was constructed in [2].

In the present work, we give a general description for the construction of completely conservative schemes for the LFP collisional operator in one- and two-dimensional cases in the velocity space. These schemes can be easily generalized for the space inhomogeneous case. The constructed difference schemes allow us to carry out numerical calculations of the relaxation problem in a large time interval without error accumulation.

The paper is organized as follows. Section 2 contains the problem statement and the equations which have been analyzed. Section 3 gives the general approach to the construction of the completely conservative difference schemes. An illustrative example of the monoenergetic beam relaxation is presented in Section 4. Section 5 is devoted to the final remarks.

2. Preliminaries

For charged particles, in the case of scattering at small angles, Boltzmann’s equation is reduced to an integro-differential equation, which was obtained in [4] in the form

$$\frac{m^2}{2\pi e^4 L} \frac{\partial f}{\partial t} = \frac{\partial}{\partial v_i} \left\{ \int d\omega \ U_{ij} \left( \frac{\partial}{\partial v_j} - \frac{\partial}{\partial w_j} \right) f(v)f(w) \right\}, \quad v, w \in R, \ t \geq 0,$$

(1)

where the symmetric kernel $U_{ij}$ is a function of the relative particle speed $u = v - w$ and $L$ is the so-called Coulomb logarithm:

$$U_{ij} = \frac{\partial^2}{\partial v_i \partial w_j} \left( u^2 \delta_{ij} - u_i u_j \right), \quad L = \ln \frac{3 T^{3/2}}{\sqrt{4\pi n e^3}}.$$

Here, $n, T, m, e$ are the number of particles, the temperature, the mass and the charge of electrons, respectively. For simplicity we consider only one class of particles.

In the Fokker–Planck form, Eq. (1) was obtained in [13]:

$$\frac{m^2}{2\pi e^4 L} \frac{\partial f}{\partial t} = -\frac{\partial}{\partial v_i} \left\{ f \frac{\partial h}{\partial v_i} + \frac{1}{2} \frac{\partial}{\partial v_j} \left( f \frac{\partial^2 g}{\partial v_i \partial v_j} \right) \right\},$$

(2)

where the functions

$$h = 8\pi \int d\omega \ f(w,t)|v - w|^{-1}, \quad g = 4\pi \int d\omega \ f(w,t)|v - w|,$$

(3)

are the so-called Rosenbluth potentials. The collisional operator written in the form (1), which is closer to the integral Boltzmann operator, contains the symmetric kernel $U_{ij}$, requiring very large computer memory even for the two-dimensional case. Hence, Eq. (2) is widely used for numerical simulations.
For the numerical calculations of the LFP integral spherical coordinates are commonly used. The plasma is assumed to be azimuthally symmetric about a magnetic (electric) field. We use dimensionless variables measuring the velocity in units of the electron thermal velocity $v_{th} = \sqrt{T/e}$, time in units of the electron–electron collisional time $t_N = v_{th}^3/m_e^2/4\pi e^4N_eA_C$, and the distribution function in units of $N_e/2\pi v_{th}^3$.

The distribution function $f(v, \mu, t)$ depends on $v = |v|$, the speed of the particles, and $-1 \leq \mu \leq 1$, the pitch angle cosine.

For the isotropic Rosenbluth potentials (3), Eqs. (1) and (2) can be written as follows [11]:

$$\frac{\partial f}{\partial t} = \frac{1}{v^2} \frac{\partial}{\partial v} \left[ \frac{1}{v} \frac{\partial W(f; v)}{\partial v} \right] + \frac{C(f; v)}{v^2} \frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial f}{\partial \mu}\right], \quad (4)$$

$$W(v; v) = \int_0^v \{F(x, t)[p(x, \mu) - p(v, \mu)] - P(x)[f(x, \mu) - f(v, \mu)]\} x^2 dx,$$

$$C(v) = (1/2v)[N(v) - R(v)/v^2 + vP(v)],$$

$$p(v, \mu) = \int_v^\infty f(x, \mu) x dx, \quad F(v) = \int_{-1}^1 f(v, \mu) d\mu, \quad N(v) = \int_0^v F(x)x^2 dx,$$

$$P(v) = \int_{-1}^1 p(v, \mu) d\mu, \quad R(v) = \int_0^v P(x)x^2 dx.$$

In many physical problems, an often-used assumption is the particle distribution symmetry about the normal direction $\mu = 0$: $\partial_\mu f(x|\mu = 0) = 0$. Under this assumption, the moment of the distribution function equals zero:

$$\int_{-1}^1 f(v, \mu, t)\mu d\mu = 0.$$

The distribution function should be bounded at $v = 0, \mu = \pm 1$, and it should tend to zero, as $v \to \infty$.

The two conservation laws, for the density and the energy, are valid:

$$n = \int dv f(v, t) = 1, \quad E = \frac{m}{2n} \int dv v^2 f(v, t) = \frac{3}{2} kT = 1. \quad (5)$$

In the absence of any sinks or sources of particles as well as of energy, Eq. (4) describes the relaxation process of the initial function $f^0$ to the equilibrium. It is worth noting that the relaxation problem can be used as a test problem in numerical simulations of any complicated collisional plasma model.

The unique equilibrium solution for the initial-value problem is the Maxwell distribution function

$$f_M(v) = n \left(\frac{2\pi T}{m}\right)^{-3/2} \exp \left[-\frac{m}{2T}v^2\right]. \quad (6)$$

3. Completely conservative schemes

We now propose a finite difference scheme for the LFP equation which preserves the conservation laws (5). For simplicity, we demonstrate the idea of constructing such a scheme by applying a finite
difference technique to the isotropic distribution function \(f(v,t)\). Let us write the kinetic equation in the symbolic form

\[
\frac{\partial f}{\partial t} = \frac{1}{v^2} \frac{\partial}{\partial v} [U(f; t)],
\]

where the operator \(U(f; t)\) represents the function with its derivatives and the integral coefficients (compare with (4)). The last equation is written in the divergence form of the local density conservation law. From this equation the conservation law (5) can be obtained as

\[
\frac{dn}{dt} = \frac{d}{dt} \int_0^\infty f v^2 \, dv = \left. U \right|_0^\infty = 0.
\]

Then, from Eq. (7) we obtain the energy conservation law

\[
\frac{dE}{dt} = \frac{d}{dt} \int_0^\infty f v^4 \, dv = v^2 \left. U \right|_0^\infty - 2 \int_0^\infty U v \, dv = 0 \quad 2 \int_0^\infty U v \, dv = 0.
\]

The first term is similar to that in (8). The elimination of the second term is provided by the symmetry properties of the exact equation.

To solve the problem under consideration we discretize differential equations (2)–(4) in time and approximate the differentiation by finite differences. To obtain the difference scheme we replace the infinite velocity interval \([0, \infty)\) by the finite segment \([0, L]\), which is chosen so as to take into account the high-energy particles. As a rule, it is sufficient to take \(L \approx (7/8)v_0\). This interval can be estimated from the Maxwell distribution (6). In the domain under consideration we introduce the space–time mesh \({v_{i+1} = v_i + h_{i+1}}; i = 1, 2, \ldots, M; v_1 = 0, v_{M+1} = L; t^{k+1} = t^k + \tau, k = 0, 1, \ldots, t^0 = 0\) and define the mesh functions \(f^k_i, U^k_i[f]\). The following notation will be used: \(f_i^{k+1} = 0.5(f_{i+1} + f_i)\). We approximate the integral by the trapezoid formula and the derivatives by central differences. The distribution function at the final point equals zero, \(f_{M+1} = 0\). Applying the integro-interpolation method to Eq. (7), we obtain the implicit difference scheme

\[
\frac{f_i^k - f_i^{k-1}}{\tau} = \frac{1}{v_i^2 h_{i+1/2}} [U^k_{i+1/2} - U^k_{i-1/2}].
\]

The difference scheme is constructed on a symmetric pattern, and has a second-order approximation with respect to the velocity space.

The numerical analogs of the density and the energy have the form

\[
n = \sum_{i=2}^{M} h_{i+1/2} v_i^2 f_i, \quad \mathcal{E} = \sum_{i=2}^{M} h_{i+1/2} v_i^4 f_i.
\]

The finite difference scheme constructed for the equation written in the form (7) satisfies the numerical analog of the density conservation law. Multiplying both sides of Eq. (7) by \(v_i^2 h_{i+1/2}\), after summation over all \(i = 2, \ldots, M + 1 (v_1 = 0)\), we obtain

\[
\frac{\Delta n}{\tau} = \sum_{i=2}^{M} \frac{f_i^k - f_i^{k-1}}{\tau} v_i^2 h_{i+1/2} = \left[ U^k_{M+1/2} - U^k_{1/2} \right].
\]
If we assume that the boundary conditions make $U^k_{M+1/2}$ and $U^k_1$ equal to zero, then we obtain the numerical analog of the conservation law (8). By analogy, we also fulfill the numerical conservation laws for the second term of the operator in Eq. (4).

Let us check the numerical analog of the energy conservation law (9). As a result of algebraic transformations, we obtain from the difference equation (7) the following:

$$
\Delta E = \frac{1}{\tau} \sum_{i=2}^M \frac{f^k_i - f^{k-1}_i}{\tau} v^k_i h_{i+1/2} = \left[ \frac{v^2_{M+1} U^k_{M+1/2}}{v_2} - \frac{v^2_2 U^k_{3/2}}{v_2} - 2 \sum_{i=2}^M U^k_{i+1/2} v_{i+1/2} h_{i+1} \right].
$$

(12)

Here the equality $v^2_{k+1} = 2v_{k+1/2} h_{k+1}$ was used. This expression is just the discrete analog of the integral in the exact formula (9). The first two terms disappear, owing to the boundary conditions mentioned above. The main problem is to make the last term equal to zero:

$$
\sum_{i=2}^M U^k_{i+1/2} v_{i+1/2} h_{i+1} = 0.
$$

(13)

This can be naturally satisfied in the special cases that we shall describe below.

If the difference scheme is only an approximate analog of the conservation laws, then this can easily lead to the accumulation of errors and therefore, the discrete analog of the equilibrium solution (6) will never be established. It is worthy to note that in the kinetic theory, generally, the form of the local conservation laws is not the characteristic form of the equations. Also, a discretization does not always allow the same manipulations as in the continuous case. The property of complete conservativity for the LFP equation consists, in particular, of the fact that the difference equation, written in the form of the density conservation law can, by an identity transformation, be reduced to a form expressing the law of the energy conservation.

The form of the exact equation employed is very important for the construction of the difference schemes. Now, we present the different forms of the LFP kinetic equation that satisfy the conservation laws in the discrete case. The first is the twice divergencey form

$$
U[t, f] = \frac{1}{v} \frac{\partial}{\partial v} W[f; t],
$$

which has the following discrete approximation (see Eq. (10)):

$$
\frac{f^k_i - f^{k-1}_i}{\tau} = \frac{1}{v^2_i} \left[ \frac{W_{i+1} - W_i}{h_{i+1}} - \frac{W_{i-1} - W_i}{h_i} \right], \quad i = 2, \ldots, M.
$$

The first term in (12) $U^k_{M+1/2}$, is proportional to $f^k_{M+1}$. If the interval under consideration, $v_{M+1}$, is large enough, and $f^k_{M+1} = 0$, then $U^k_{M+1/2} = 0$. From the structure of Eq. (4) it follows that $W_1 = 0$ by definition, and $W_2 = 0$ due to the symmetrical form of Eq. (4). Finally from equality (13) follows:

$$
\sum_{i=2}^M U^k_{i+1/2} v_{i+1/2} h_{i+1} = W^k_{M+1} - W^k_1 = 0.
$$

This form of the LFP integral works for the one- and two-dimensional equations as well as for the system of LFP equations.
The second has the form of the symmetric difference, used for the system of one-dimensional LFP equations as well as for the equations of the LFP type for power law interaction potentials [10]:

\[
U_{\text{II}}[f] = \frac{1}{v} \int_{0}^{\infty} dw \, Q(v, w) \left[ w f'(w) \frac{\partial f(v)}{\partial v} - v f'(v) \frac{\partial f(w)}{\partial w} \right],
\]

where \( Q(v, w) \) is the symmetric kernel. A detailed example for this type of equation will be given in the next section.

The third form consists of the combination of the first form and of the antisymmetric integral form:

\[
U_{\text{III}}[f] = U_{\text{I}}[f] + \frac{1}{v} W, \quad W = \int_{0}^{\infty} dw \, K(v, w) f(v) f(w),
\]

where \( K(v, w) \) is an antisymmetric kernel. The boundary terms in (12) are eliminated due to the equalities \( W_{m+1/2} = 0 \) and \( W_{3/2} = 0 \), that follow from the boundary conditions. Taking into account \( K_{i+1/2,k+1/2} = -K_{k+1/2,i+1/2} \) for equality (13), we obtain

\[
\sum_{i=2}^{M} U_{\text{III}}[f_{i+1/2}] v_{i+1/2} h_{i+1} \rightarrow \sum_{i=2}^{M} \sum_{k=2}^{M} h_{i+1} h_{k+1} K_{i+1/2,k+1/2} f_{i+1/2} f_{k+1/2} = 0.
\]

In this form, the two-dimensional LFP equations with the anisotropic Rosebluth potentials can be written [12].

The procedure for the construction of the finite difference scheme is similar for the one- and two-dimensional cases. Boundary conditions follow from the requirements for the fulfillment of the conservation laws. Obviously, the conservation property is maintained for an explicit difference scheme and for the scheme with the second order of approximation on the time step \( \tau \).

In the next section we give an illustrative example of the numerical calculation.

4. The relaxation of the monoenergetic ion beam

Eq. (7) for the power law interaction potentials \( V = \alpha/r^s \), where \( 1 \leqslant s < 4 \), is considered:

\[
\frac{\partial f}{\partial t} = \frac{1}{2v^2} \frac{\partial}{\partial v} \left\{ \frac{1}{v} \int_{0}^{\infty} dw \, Q(v, w) \left[ w f'(w) \frac{\partial f(v)}{\partial v} - v f'(v) \frac{\partial f(w)}{\partial w} \right] \right\}.
\] (14)

For the case under consideration, the symmetrical kernel \( Q(v, w) \) can be written as follows [10]:

\[
Q(v, w) = \frac{a(v, w)(v + w)^{n+4} + b(v, w)|v - w|^{n+4}}{(n + 2)(n + 4)(n + 6)},
\]

with

\[
a(v, w) = (n + 4)[vw - (v^2 + w^2)], \quad b(v, w) = (n + 4)[vw + (v^2 + w^2)], \quad n = (s - 4)/s.
\]
The negative values of $n$ correspond to the soft interaction potentials ($1 \leq s < 4$). For charged particles $s = 1$ ($n = -3$). At the initial instant, the distribution function is located in the thermal velocity region and has a $\delta$-function type: $f(v, 0) = \delta(v - 1)/v^2$. The conservation laws and the equilibrium solution are
\[
 n = \int_0^\infty dv v^2 f(v, t) = 1, \quad E = \int_0^\infty dv v^4 f(v, t) = 1, \quad f_M = \frac{4}{\pi^{1/2}} \left( \frac{3}{2} \right)^{3/2} \exp \left( -\frac{3}{2} v^2 \right).
\]

Applying the integro-interpolation method to Eq. (10), we obtain the implicit difference scheme
\[
 \frac{f^k_i - f^{k-1}_i}{\tau} = \frac{1}{v_i^2 h_{i+1/2}} \left[ \frac{S^k_{i+1/2}}{v_{i+1/2}} - \frac{S^k_{i-1/2}}{v_{i+1/2}} \right], \quad i = 2, \ldots, M,
\]
where
\[
 S^k_{i+1/2} = \sum_{m=1}^M Q_{i+1/2,m+1/2} h_{m+1} \left( \frac{f^k_{i+1} - f^k_i}{h_{i+1}} - \frac{f^k_m v_m + f^k_m v_{m+1}}{2} - \frac{f^k_{m+1} - f^k_m}{h_{m+1}} \frac{f^k_i v_i + f^k_{i+1} v_{i+1}}{2} \right).
\]

The boundary conditions are $S_{3/2} = 0$ and $f_{M+1} = 0$. The equality (13) takes the form
\[
 \sum_{i=2}^M S_{i+1/2} h_{i+1} = 0,
\]
where the property $Q_{i+1/2,m+1/2} = Q_{m+1/2,i+1/2}$ is taken into account. The initial distribution is approximated on the mesh in the usual way, that is,
\[
 f(v_i, 0) = \begin{cases} 
 2/(v_{i+1} - v_{i-1}) & \text{if } v_i = 1, \\
 0 & \text{otherwise}.
\end{cases}
\]

This approximation makes the number of particles as well as the kinetic energy equal to unity.

For the computations, the schemes are transformed to a set of nonlinear difference equations, which are solved at each time step by an iterative method, and at each iteration by a pivotal condensation. The numerical methods of variable directions and of complete splitting may be used. Since the scheme is implicit, the value of the time step $\tau$ is only defined by the desired calculation accuracy.

The density error is kept to machine error accuracy (the random error). Due to the nonlinearity of the equations, the difference scheme should be solved by an iterative procedure, and therefore the energy conservation depends on the iteration precision $\eta$. For $\eta = 10^{-3}$ the relative error in energy conservation $\Delta_e$ is equal to $10^{-2}\%$, and for $\eta = 10^{-7}$ the error in energy is $\Delta_e = 10^{-5}\%$, that is, the error in energy appears in the seventh decimal digit (Fig. 1).

The LFP equation is an equation of parabolic type, that becomes slightly pronounced in the high velocity region $v \to \infty$ (i.e., is degenerating into a hyperbolic type). The constructed difference schemes have a second-order approximation with respect to the velocity space, and they are non-monotonic. In order to take into account the fast exponential decrease of the distribution function tail, the value of the velocity mesh step $h_i = v_i - v_{i-1}$ should be sufficiently small in the high-velocity region, approximately $h_i < E/nv_i$.

Now we briefly describe the results of the numerical simulations. Very rapidly the solution acquires a quasi-equilibrium form in the thermal velocity region ($0 \leq v \leq 2$) at the instant $t_0$ that corresponds
to the so-called collision time. The characteristic time $t_0$ weakly depends on various values of the exponent $s$ in potentials $U \sim z/r^{s}$. In this region the distribution functions are close to each other throughout the entire relaxation process for different values of $s$. The main difference is observed in the region of the distribution tails for $v > 2$. For a better representation of the numerical results in the high-velocity region, we introduce the function divided by the Maxwell distribution $g(v,t) = f(v,t)/f_M(v)$. For charged particles ($s = 1$, $n = -3$, $U = z/r$), the function $g(v,t)$ is shown in Fig. 2. The solution has the character of a propagated wave with a stable profile. For the potentials with $1 \leq s \leq 2$, the tail relaxation proceeds more slowly than the relaxation of the distribution function core. For the values $2 < s < 3$ the evolution of the distribution function tail still maintains the wave character. The obtained results are in a good agreement with the asymptotic analytical results [10].

5. Conclusions

Some general methods for the construction of the completely conservative difference schemes for the nonlinear LFP kinetic equation are formulated. The suitable forms for the construction of the completely conservative difference schemes are classified. These schemes can be easily generalized for the space inhomogeneous case.

The constructed difference schemes allow us to calculate the numerical solution of the relaxation problem in a large time interval using a fairly coarse time step. In the nonstationary case, this property is extremely important for a gas with light and heavy particles, such as electrons and ions, when the characteristic time scales differ by hundred times.

The numerical results obtained coincide with the analytical results.
Fig. 2. The graph of the distribution function $g(v,t) = f(v,t)/f_M(v)$ normalized to the Maxwell distribution in the velocity region $2 < v/v_{th} < 5$ for different time instants $t$. The case corresponds to the Coulomb interaction $U = \alpha/r$.

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