A comparison of some inverse methods for estimating the initial condition of the heat equation

Wagner Barbosa Muniz\textsuperscript{a,}\textsuperscript{*}, Haroldo F. de Campos Velho\textsuperscript{b}, Fernando Manuel Ramos\textsuperscript{b}

\textsuperscript{a}Instituto de Matemática, Universidade Federal do Rio Grande do Sul, Porto Alegre, RS, Brazil
\textsuperscript{b}Laboratório Associado de Computação e Matemática Aplicada, Instituto Nacional de Pesquisas Espaciais, CP 515, 12201-970 São José dos Campos, SP, Brazil

Received 29 October 1997; received in revised form 23 December 1997

Abstract

In this work we analyze two explicit methods for the solution of an inverse heat conduction problem and we confront them with the least-squares method, using for the solution of the associated direct problem a classical finite difference method and a method based on an integral formulation. Finally, the Tikhonov regularization connected to the least-squares criterion is examined. We show that the explicit approaches to this inverse heat conduction problem will present disastrous results unless some kind of regularization is used.

Keywords: Inverse problems; Least-squares method; Spectral formulation; Tikhonov regularization

1. Introduction

Inverse problems have certainly been one of the fastest growing areas in various application fields. Science and industry are both responsible for this growth in the last years. The main difficulty in the treatment of inverse problems is the instability of their solution in the presence of noise in the observed (measured) data, that is, the \textit{ill-posed} nature of the problem in the sense of Hadamard. This problem not only defies easy solution, but has served to discourage the type of massive study that has accompanied direct or well-posed problems [10, 13]. One could generically classify inverse problems into three types (all of them based on observations of the evolution of the involved physical system): identification of physical parameters or \textit{parameter identification}; determination of the initial state of the system and; determination of the boundary conditions [3].

We will deal here with the second type of inverse problems in the field of heat transfer, that is, the determination of the initial condition from transient temperature measurements taken within the finite medium at a time $t = \tau > 0$ — the \textit{backwards heat equation} [5, 7, 11].

\textsuperscript{*} Corresponding author. E-mail: muniz@math.udel.edu.
One could say that inversion techniques may be generically divided into two categories:

- **Explicit**: Inversion methods which are obtained through an explicit inversion scheme involving the operator representing the direct problem.
- **Implicit**: Which present an iterative character that exhaustively explores the model space (solution space) until a stopping criterion is satisfied, considering, of course, the available data.

In this article we will examine some inversion techniques in order to estimate the initial temperature distribution of an inverse heat conduction problem (IHCP), from which two of them are classified as explicit inversion techniques and the other one is implicit.

2. The direct problem

The direct (forward) problem consists of a transient heat conduction problem in a slab with adiabatic boundary condition and initially at a temperature denoted by \( f(x) \).

The mathematical formulation of this problem is given by the following heat equation:

\[
\begin{align*}
\frac{\partial^2 T(x,t)}{\partial x^2} &= \frac{\partial T(x,t)}{\partial t} \quad \text{for} \quad (x,t) \in \Omega \times \mathbb{R}^+, \\
\frac{\partial T(x,t)}{\partial x} &= 0 \quad \text{for} \quad (x,t) \in \partial \Omega \times \mathbb{R}^+, \\
T(x,0) &= f(x) \quad \text{for} \quad (x,t) \in \Omega \times \{0\},
\end{align*}
\]

where \( T(x,t) \) (temperature), \( f(x) \) (initial condition), \( x \) (spatial variable) and \( t \) (time variable) are dimensionless quantities and \( \Omega = [0,1] \).

The solution of the direct problem for a given initial condition \( f(x) \) is explicitly obtained using separation of variables [2], for \( (x,t) \in \Omega \times \mathbb{R}^+ \):

\[
T(x,t) = \sum_{m=0}^{+\infty} e^{-\beta_m^2 t} \frac{1}{N(\beta_m)} X(\beta_m,x) \int_{\Omega} X(\beta_m,x') f(x') \, dx',
\]

where \( X(\beta_m,x) \) are the eigenfunctions associated to the problem, \( \beta_m \) the eigenvalues and \( N(\beta_m) \) represents the integral normalization (or the norm).

The above representation requires that the initial temperature \( f(x) \) be a bounded function satisfying Dirichlet’s conditions in the interval \( \Omega \) [2].

One must observe that the norm, \( N(\beta_m) \), is defined as

\[
N(\beta_m) = \int_{\Omega} X^2(\beta_m,x) \, dx.
\]

In particular, with the adiabatic boundary conditions, the eigenvalues and eigenfunctions for the problem defined by Eq. (1) can be expressed, respectively, as

\[
\begin{align*}
\beta_m &= m\pi, \\
X(\beta_m,x) &= \cos(\beta_m x) \quad \text{for} \quad m = 0, 1, 2, \ldots
\end{align*}
\]
3. The inverse problem

As explained previously, a transient heat conduction problem in a slab is considered, where both boundaries are kept insulated. Here the inverse problem consists of the determination of the initial temperature distribution \( f \), since the temperature distribution \( T \) is given for the time \( t = \tau > 0 \).

In order to define the discrete version of the problem we consider that the transient temperature \( T \) is available at a finite number of different locations on the medium. In actual problems, this temperature is usually found empirically and hence known only approximately. This problem is a genuinely ill-posed problem in the sense of Hadamard, as we will see below.

The present inverse problem admits an analytical solution obtained from Eq. (2), using the orthogonality property of the eigenfunctions \( X(\beta_m, x) \):

\[
\int_{\Omega} X(\beta_m, x) X(\beta_k, x) \, dx = \delta_{mk} N(\beta_k),
\]

where \( \delta_{mk} \) is the Kronecker’s delta. Therefore we have:

**Theorem 1.** If the temperature \( T(x, t) \) is known for the time \( t = \tau > 0 \) on the whole spatial domain \( \Omega \), then the initial temperature \( f(x) \) is given by

\[
f(x) = \sum_{m=0}^{\infty} e^{\beta_m^2 \tau} \frac{1}{N(\beta_m)} \int_{\Omega} X(\beta_m, x) T(x', \tau) \, dx', \quad x \in \Omega.
\]  

**Proof.** \((\tau > 0)\): We express the solution of the direct problem as

\[
T(x, \tau) = \sum_{m=0}^{\infty} c_m X(\beta_m, x) e^{-\beta_m^2 \tau}.
\]  

At this point, we multiply Eq. (4) by \( X(\beta_k, x) \) and integrate the result in \( \Omega \). So we obtain

\[
\int_{\Omega} T(x', \tau) X(\beta_k, x') \, dx' = \sum_{m=0}^{\infty} c_m e^{-\beta_m^2 \tau} \int_{\Omega} X(\beta_k, x') X(\beta_m, x') \, dx' = c_k e^{-\beta_k^2 \tau} N(\beta_k).
\]

Therefore, we have

\[
c_k = \frac{e^{\beta_k^2 \tau}}{N(\beta_k)} \int_{\Omega} T(x', \tau) X(\beta_k, x') \, dx'.
\]  

Applying the initial condition \((t = 0)\) and expression (5) we obtain

\[
f(x) = \sum_{m=0}^{\infty} c_m X(\beta_m, x)
\]

\[
= \sum_{m=0}^{\infty} e^{\beta_m^2 \tau} \frac{1}{N(\beta_m)} \int_{\Omega} X(\beta_m, x) T(x', \tau) \, dx'.
\]

\[
\square
\]
Using the result of Theorem 1, we can show the ill-posedness of the backwards heat equation, as follows:

**Theorem 2.** If we consider the problem of determining the initial condition, where \( f, T \in L_2(\Omega) \), then \( f \) does not depend continuously on the data \( T \), that is, the problem is ill-posed in the sense of Hadamard, since the stability requirement is not satisfied.

**Proof.** Consider the initial conditions \( f_1, f_2 \in L_2(\Omega) \) such that \( f_2(x) = f_1(x) + KX(\beta_n, x) \), with \( K \in \mathbb{R} \setminus \{0\} \) and \( n \in \mathbb{N} \). Assume that the corresponding transient solutions (at a fixed \( \tau > 0 \)) are, respectively, the distributions \( T_1(x, \tau), T_2(x, \tau) \). By the linearity, see Eq. (2), we have

\[
T_2(x, \tau) = T_1(x, \tau) + \sum_{m=0}^{\infty} \frac{e^{-\beta_m^2 \tau} X(\beta_m, x)}{N(\beta_m)} \int_\Omega KX(\beta_n, x')X(\beta_m, x') \, dx' = T_1(x, \tau) + Ke^{-\beta_n^2 \tau} X(\beta_n, x).
\]

Hence, the difference between \( T_1(x, \tau) \) and \( T_2(x, \tau) \) is given by

\[
\|T_2 - T_1\|_2^2 = \int_\Omega \left[ T_2(x, \tau) - T_1(x, \tau) \right]^2 \, dx = K^2 e^{-2\beta_n^2 \tau} N(\beta_n).
\]

Thus, for any number \( K \), the quantity \( \|T_2 - T_1\|_2 \) can be made arbitrarily small by choosing \( n \) sufficiently large. Similarly, if we measure the difference between \( f_1 \) and \( f_2 \) in the \( L_2 \)-metric with \( K \neq 0 \), we obtain

\[
\|f_2 - f_1\|_2^2 = \int_\Omega K^2 X^2(\beta_n, x) \, dx = K^2 N(\beta_n) = \text{constant} > 0.
\]

Hence, with arbitrarily small discrepancies between \( T_1 \) and \( T_2 \), one can choose \( n \) and \( K \) in such a way that the discrepancy between the corresponding solutions (\( f_1 \) and \( f_2 \)) can be arbitrary.

\[
\|T_1 - T_2\|_2 \to 0 \text{ but } \|f_1 - f_2\|_2 \to 0. \quad \square
\]

One can easily note that, according to the proof of Theorem 2, we have

\[
\|f_2 - f_1\|_2 = e^{\beta_n^2 \tau} \|T_2 - T_1\|_2,
\]

which means that the error in the solution of this inverse problem is amplified exponentially by the factor \( e^{\beta_n^2 \tau} \). Clearly, the error becomes worse as \( \tau \) increases, but even with a small \( \tau > 0 \) the exponential amplification of the error remains. In this context, a simple example arises if we suppose that the data measurement error \( \delta = O(e^{-10}) \) and we put \( \beta_n^2 \tau = 20 \), then we obtain an error in the solution \( \varepsilon = O(e^{10}) \).

Note that the operator representing the forward problem has strong smoothing properties. Generally, this characteristic arises when we are dealing with ill-posed problems, in the sense of instability [3].

In the following section we will present three techniques in order to solve the inverse problem here established, where the transient temperature is available in a discrete form.
4. Inversion techniques

We primarily present three approaches which would seem very natural (or appropriate) if we were dealing with a well-posed problem, for which the uniqueness, existence and stability are ensured. The first two are classified as explicit, according to the initial explanation.

4.1. The linear explicit method

We consider Eq. (2), with a fixed time $\tau > 0$, and we rewrite it as a Fredholm integral equation of the first kind

$$\int_{\Omega} A(x, y) f(y) \, dy,$$

where $A(x, y) = \sum_{m=0}^{+\infty} \left( e^{-\beta^2_m \tau} / N(\beta_m) \right) X(\beta_m, x) X(\beta_m, y)$. One can therefore, by selecting a suitable quadrature formula (e.g., Simpson’s rule, trapezoidal rule), reduce the integral equation (6) to a system of linear equations

$$T(x_i, \tau) = \sum_j a_{ij} f(x_i).$$

So we can directly invert this linear approximation, that is, we invert the projection, onto a finite-dimensional space, of the operator associated to the direct problem.

4.2. Discrete backward inversion — sequential technique

This so-called second explicit method is based on a backward-time centered-space discretization of the heat equation (BTCS), so that an explicit scheme is obtained such that $f$ is a function of $T$.

**Notation.** $T(x_i, t_n) = T^n_i$, where $t_n = n \Delta t$, $x_i = i \Delta x$ and $\Delta t$, $\Delta x$ are, respectively, the temporal and spatial variation.

Consequently, we have

$$\frac{\partial^2 T(x_i, t_n)}{\partial x^2} \approx \frac{T^{n+1}_{i+1} - 2T^n_i + T^{n+1}_{i-1}}{\Delta x^2} \quad \text{and} \quad \frac{\partial T(x_i, t_n)}{\partial t} \approx \frac{T^{n+1}_i - T^n_i}{\Delta t}.$$

This implies that, with a second-order approximation of the boundary conditions, we can define a matrix $B$ such that

$$B = \begin{bmatrix}
1 + 2D & -2D & 0 & \cdots & 0 \\
-D & 1 + 2D & -D & \cdots & \vdots \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & -D & 1 + 2D & -D \\
0 & \cdots & 0 & -2D & 1 + 2D
\end{bmatrix},$$

with $D = \Delta t / \Delta x^2$ and hence

$$\bar{T}^{(n)} = B \bar{T}^{(n+1)} \rightarrow \bar{f} = \bar{T}^{(0)} = B^n \bar{T}^{(0)},$$

where $\tau = n \Delta t$ and $\bar{T} = \bar{T}^{(0)} = [T(x_i, \tau)]$. 
4.3. The implicit method: least squares

The least-squares approximation, in the sense of the minimum norm, can guarantee the existence and uniqueness, but this solution can be unstable in the presence of noise in the experimental data, requiring thus the use of some regularization technique [13]. The regularized solution is obtained by choosing the function $f$ that minimizes the following functional:

$$M^*[\hat{T}, f] = \|Af - \hat{T}\|_2^2 + x \Omega[f],$$

(7)

where $\hat{T} = \hat{T}(x, \tau)$ is the experimental data ($t = \tau > 0$), $A$ is an operator that maps the parameter set $\{f\}$ into the results set $\{T\}$, $\Omega[f]$ denotes the regularization term (Tikhonov), $x$ is the regularization parameter, and $\| \cdot \|_2$ is the 2-norm.

The regularization parameter $x$ is chosen numerically, through an a posteriori parameter choice rule, assuming that the statistics of the measurement errors is known. These numerical experiments are based on the Morozov’s discrepancy principle: $x^*$ is optimum when

$$N_x \sigma^2 \sim \|Af^* - \hat{T}\|_2^2 =: R(f^*) \quad (= M^*[\hat{T}, f^*] - x^* \Omega[f^*]),$$

where $\sigma$ is the standard deviation of the measurement errors [1, 6].

4.3.1. Regularization functions

As already mentioned, it can be assumed that there is a unique solution, in the minimum norm least-squares sense, for a given inverse problem. According to Tikhonov [13], ill-posed problems can yield stable solution if sufficient a priori information about the true solution is available. Such information is added to the least-squares approximation by means of a regularization term, in order to complete the solution of the inverse problem. Therefore, one can say that it is natural to expect that the regularization parameter $x$ is a good compromise between the data fitting and the smoothing requirement. The regularization functions used in this paper are described below: they correspond to the so-called Tikhonov regularization.

4.3.2. Tikhonov regularization

The regularization technique presented by Tikhonov can be expressed by [13]

$$\Omega[f] = \sum_{k=0}^p x_k \|f^{(k)}\|_2^2,$$

(8)

where $f^{(k)}$ denotes the $k$th derivative relative to $x$, since $f = f(x)$, and the regularization parameters $x_k \geq 0$. The regularization effect for zeroth order is to reduce the oscillations on the parameter vector (smooth function $f(x)$). A first-order regularization tends to make $|df/dx| \approx 0$, that is, $f(x)$ is approximately constant.

Clearly, as $x_k \to 0$ the least-squares term in the objective function is over-estimated, what might not give good results in the presence of noise. On the other hand, if $x_k \to \infty$, all consistency with the information about the system is lost.

Considering the zeroth order Tikhonov regularization, note that one can easily show that the functional $M^*[\hat{T}, f]$, as defined in (7), is monotonically increasing with respect to $x$ and that $f_x$ is monotonically decreasing with respect to $x$. Observe that $f_x$ is the solution of $\min M^*[\hat{T}, f]$. 
5. Numerical realization of the direct problem

Since the implicit methods require iterative solutions, where numerical techniques are applied, a numerical solution to the direct problem (1) is necessary.

Here we will consider two different approximations to the problem:
1. A second-order (time and space) finite difference method of discretization of the heat equation: Crank–Nicolson [12].
2. An integral approach, which is based on a linear approximation of the function $f$ in the subdomains of the whole spatial domain $\Omega$, such that a semi-analytical approximation is established. In this case, the approximation will be outlined below by considering Eq. (2):

The integral $\int_{\Omega} f(x)X(\beta_m,x') \, dx'$ in Eq. (2) is approximated as follows: the interval $\Omega$ is splitted into $N_x$ sub-intervals $\Omega_i = (x_i, x_{i+1})$, such that $x_{i+1} = x_i + \mu_i$ at $i = 0, \ldots, N_x$, where $\mu_i$ is a positive quantity. Hence, $\Omega = \bigcup_{i=0}^{N_x-1} \Omega_i$. Since $f(x_i) = f_i$ and $f(x_{i+1}) = f_{i+1}$ are known, a linear approximation of $f(x)$ in each sub-interval $\Omega_i$, such that $f(x) = a_i x + b_i$ at $x \in \Omega_i$, yields the constants $a_i$ and $b_i$:

$$a_i = \frac{f_{i+1} - f_i}{\mu_i} \quad \text{and} \quad b_i = \frac{x_{i+1} f_i - x_i f_{i+1}}{\mu_i}.$$ 

In the interval $\Omega_i$ the integration of the function $f(x)X(\beta_m,x)$ is computed by $Z_i^{(m)} = \int_{\Omega_i} (a_i x + b_i)X(\beta_m,x) \, dx$, which is analytically solved since $a_i, b_i$ and the eigenfunction $X(\beta_m,x)$ are available. Therefore, the integral in $\Omega$ is given by

$$\int_{\Omega} f(x)X(\beta_m,x) \, dx = \sum_{i=0}^{N_x-1} \int_{\Omega_i} f(x)X(\beta_m,x) \, dx = \sum_{i=0}^{N_x-1} Z_i^{(m)}.$$ (9)

Substituting Eq. (9) into Eq. (2) yields

$$T(x, \tau) = \sum_{m=0}^{+\infty} \frac{e^{-\beta_m^2 \tau}}{N(\beta_m)} X(\beta_m,x) \sum_{i=0}^{N_x-1} Z_i^{(m)}.$$ (10)

6. Numerical results

In this article we have described three models in order to solve the backwards heat equation and we now give an example in order to illustrate the accuracy of the methods.

The numerical experimentation of the proposed inversion methods (Section 4) is based on a triangular test function

$$f(x) = \begin{cases} 2x, & x \in [0, 0.5], \\ 2(1 - x), & x \in (0.5, 1]. \end{cases}$$
The experimental data (measured temperatures at a time $\tau > 0$), which intrinsically contains errors, is obtained by adding a random perturbation to the exact solution of the direct problem, such that

$$T_{\text{experimental}} = T_{\text{exact}} + \sigma U,$$

where $\sigma$ is the standard deviation of the errors and $U$ is a random variable taken from a uniform distribution ($U \in [-1, 1]$).

It is important to observe that the spatial grid consists of 100 points ($N_x = 100$) and the so-called inversion method was developed through the trapezoidal rule.

One can easily conclude that the explicit inversion method, developed in Section 4.1, which is based on a quadrature formula (trapezoidal rule), does not give satisfactory results, since Figs. 1–4 show that numerical solutions using this methodology present disastrous oscillations, even without noise in the data (Figs. 1 and 2). Note that the magnitude of the numerical solution, when $\tau = 0.008$ and $\sigma = 0.05$, is $O(10^5)$! A very undesirable result.

However, it is observed (Figs. 5–8) that the agreement of the numerical solution as obtained by the sequential scheme of inversion (Section 4.2) with the exact solution is ‘good’ only when there is no noise ($\sigma = 0$). This approach also presents a numerical solution with strong oscillatory characteristics in the presence of measurement error in the data $\sigma = 0.05$.

The least-squares (LS) method, as outlined previously, is classified as an implicit methodology. So, the optimization problem defined by Eq. (7) is iteratively solved by the quasi-Newton optimization algorithm [4]. This approach has been previously adopted with success in others works [7–9]. The parameter vector was always subjected to simple bounds: $-0.2 \leq f_k \leq 1.2$ ($k = 1, 2, \ldots, N_x$).
Fig. 2. Explicit inversion: $\tau = 0.008$ without noise.

Fig. 3. Explicit inversion: $\tau = 10^{-4}$, $\sigma = 0.05$. 
Now, note that the Figs. 9–16 present the numerical solutions obtained by two methods which solve the direct problem associated to Eq. (7): the Crank–Nicolson and the spectral method. It is observed that, under errorless data, both methods present a numerical solution very close to the triangular function. However, with noisy data, the spectral method presents the best results, which becomes clear if we compare Figs. 11 and 15. The advantage of the spectral approach is that the time dependence is exactly represented and only the spatial domain is approximated.

It is clear that the LS methodology associated to the spectral method presents better results than the so-called sequential and linear explicit inversion. Nevertheless, the LS methodology or residual minimization does not eliminate the instability of the solution under the presence of noise, see Fig. 16. So, a regularization method will be required: we use a zeroth- and first-order Tikhonov regularization with the Morozov’s discrepancy principle as the parameter choice rule.

If we effectively want to apply some kind of regularization, as Tikhonov, which means $\alpha > 0$, then the discrepancy principle implies that a suitable regularized solution can be obtained. Since the spatial resolution is $N_x = 100$, the optimum $\alpha$ is reached for $R(f^*) \approx n_x \sigma^2 = 0.25$. Table 1 shows the least-squares term $R(f^*)$ obtained for different values of $\alpha$, and the optimum value is pointed out ($X$) for each regularization method.

Fig. 13 shows the estimation of the initial condition without any regularization. Clearly, the least-squares solution did not reconstruct appropriately the initial condition. The regularized solutions are plotted in Figs. 17 and 18. As pointed out in Section 3, a small regularization parameter yields oscillatory solutions, while for $\alpha \rightarrow \infty$ the inverse solution tends to a fully uniform profile. By using the values estimated by the discrepancy criterion $\alpha = 0.073$, and 34 for zeroth- and first-order
Fig. 5. Sequential inversion: $\tau = 10^{-4}$ without noise.

Fig. 6. Sequential inversion: $\tau = 0.008$ without noise.
Fig. 7. Sequential inversion: $\tau = 10^{-4}$, $\sigma = 0.05$.

Fig. 8. Sequential inversion: $\tau = 0.008$, $\sigma = 0.05$. 
Fig. 9. LS with Crank–Nicolson: $\tau = 10^{-4}$ without noise.

Fig. 10. LS with Crank–Nicolson: $\tau = 0.008$ without noise.
Fig. 11. LS with Crank–Nicolson: $\tau = 10^{-4}$, $\sigma = 0.05$.

Fig. 12. LS with Crank–Nicolson: $\tau = 0.008$, $\sigma = 0.05$. 

Fig. 13. LS with spectral method: $\tau = 10^{-4}$ without noise.

Fig. 14. LS with spectral method: $\tau = 0.008$ without noise.
Fig. 15. LS with spectral method: $\tau = 10^{-4}$, $\sigma = 0.05$.

Fig. 16. LS with spectral method: $\tau = 0.008$, $\sigma = 0.05$. 
Fig. 17. Inversion by Tikhonov-0 (with spectral method), $\tau = 0.008$, $\sigma = 0.05$ and where (-) exact solution; (---) $x = 10$; (\*\*) $x = 10^{-5}$; (\*\*\*) $x = 0.073$.

Table 1
Least-squares term in Eq. (7) for different values of the regularization parameter

<table>
<thead>
<tr>
<th>Tikhonov zeroth order</th>
<th>Least squares</th>
<th>Tikhonov first order</th>
<th>Least squares</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_{\text{Ti-k-0}}$</td>
<td>$\tau_{\text{Ti-k-1}}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.001</td>
<td>0.0733</td>
<td>$10^{-2}$</td>
<td>0.0732</td>
</tr>
<tr>
<td>0.010</td>
<td>0.0775</td>
<td>1</td>
<td>0.0740</td>
</tr>
<tr>
<td>0.050</td>
<td>0.1618</td>
<td>5</td>
<td>0.0800</td>
</tr>
<tr>
<td>0.060</td>
<td>0.1975</td>
<td>30</td>
<td>0.2196</td>
</tr>
<tr>
<td>0.065</td>
<td>0.2172</td>
<td>31</td>
<td>0.2275</td>
</tr>
<tr>
<td>0.070</td>
<td>0.2382</td>
<td>33</td>
<td>0.2437</td>
</tr>
<tr>
<td>0.072</td>
<td>0.2469</td>
<td>34</td>
<td>0.2520 (X)</td>
</tr>
<tr>
<td>0.073</td>
<td>0.2513 (X)</td>
<td>35</td>
<td>0.2600</td>
</tr>
<tr>
<td>0.075</td>
<td>0.2602</td>
<td>36</td>
<td>0.2687</td>
</tr>
<tr>
<td>0.077</td>
<td>0.2694</td>
<td>37</td>
<td>0.2773</td>
</tr>
<tr>
<td>0.080</td>
<td>0.2824</td>
<td>38</td>
<td>0.2859</td>
</tr>
<tr>
<td>0.090</td>
<td>0.3330</td>
<td>39</td>
<td>0.2946</td>
</tr>
<tr>
<td>0.100</td>
<td>0.3864</td>
<td>40</td>
<td>0.3033</td>
</tr>
</tbody>
</table>

Tikhonov regularization, respectively, good estimations were obtained for the triangle test function. In real-world problems the choice of a regularization parameter for a specific test function provides good results even when applied to other initial conditions [1, 7].
7. Final comments

The preliminary approaches to the solution of this inverse problem evidence its genuine ill-posedness. When we treat an ill-posed (or inverse problem), we cannot apply a methodology which would seem natural if we were dealing with a well-posed (or direct problem). We have to evaluate all available information about the physical system.

The direct problem seems to be best solved if the spectral methodology is employed: the advantage of this formulation consists of its semi-analytical nature.

The implicit strategy and regularization techniques adopted in this paper yield good results in reconstructing the initial condition. The discrepancy criterion was efficient to estimate the Lagrange multiplier in the analyzed cases, and it was successfully used for other initial conditions. The chosen regularization techniques — zeroth- and first-order Tikhonov regularization — are suitable for solving the proposed inverse problem.

Acknowledgements

The authors recognize the role played by FAPESP, São Paulo State Foundation for Research Support, in supporting this piece of work through a Thematic Project grant (process 96/07200-8). W.B.M. acknowledges the financial support given by CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico).
References