Nonlinear $L^2$-stability under large disturbances

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Abstract

We derive time-asymptotic decay rates in $L^2$ for large disturbances to some important classes of solutions of the Cauchy problem for a number of uniformly parabolic equations, provided only that the disturbances belong to appropriate $L^p$ spaces at initial time. Examples considered include the scalar nonlinear advection-diffusion equation

$$u_t + f(u)_x = (b(u)u)_x,$$

and the parabolic system

$$u_t + (u \phi(|u|))_x = (B(u)u)_x,$$

where $u(x, t) \in \mathbb{R}^n$, $\phi$ is a given scalar function and $B(u)$ is a uniformly positive-definite diagonal matrix. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

One of the basic questions concerning systems of conservation laws is the large time behavior of disturbances to certain fundamental classes of solutions, like shocks or traveling waves, expansion waves and equilibrium or constant solutions. Under appropriate assumptions, these waves are nonlinearly stable, and disturbances decay time asymptotically provided that they are sufficiently small (in an appropriate sense) at initial time. For example, considering the parabolic system

$$u_t + f(u)_x = (B(u)u)_x,$$

where $u(\cdot, t)$ deviates at $t = 0$ from a given constant solution, which, without loss of generality, we assume to be the zero state, a detailed description of the asymptotic behavior of $u(\cdot, t)$ as $t \to \infty$
has been given when the Jacobian matrix \( f'(u) \) is completely hyperbolic [3, 4, 13], provided \( u(\cdot,0) \) is small enough to satisfy

\[
\int_{-\infty}^{+\infty} |u(x,0)|(1 + |x|) \, dx + \int_{-\infty}^{+\infty} (|u(x,0)|^2 + |u_x(x,0)|^2) \, dx \leq \delta,
\]

for \( \delta \ll 1 \). Here, \( u(x,t) \) denotes an \( m \)-dimensional vector of unknown quantities \( (u_1(x,t), \ldots, u_m(x,t)) \), \( f(u(x,t)) \) is a vector function giving the flux of the conserved variables at the location \( x \) and time \( t \), and \( B(u) \) is a positive-definite viscosity matrix describing the viscous dissipation present in the system. In particular, we get from their results the estimate

\[
\|u(\cdot,t)\|_{L_2(\mathbb{R})} = O(t^{-1/4}),
\]

where the decay rate \( t^{-1/4} \) is optimal, since solutions can be found which decay exactly at this rate [4]. Using much weaker assumptions on the initial data, we derive this estimate here for a class of systems which includes

\[
\frac{\partial u}{\partial t} + (u \frac{\partial u}{\partial x}) = (B(u)u_x),
\]

where \( \varphi \) is a scalar function and \( B(u) \) is a positive-definite diagonal matrix for all values of \( u \) concerned. This is possible due to the particular structure of the solutions of (2). One example is the rotationally invariant system [7, 18]

\[
\frac{\partial u}{\partial t} + (u|u|^2) = \mu u_{xx},
\]

where \( \mu \) is a positive constant. The inviscid form of Eq. (3) was considered in 1979 by Keyfitz and Kranzer in connection with the elastic string problem in elasticity [14]. This system has also been studied in one-dimensional multiphase flow [12, 17], magnetohydrodynamics [1, 6] and more generally in continuum mechanics as a basic model for the propagation of plane waves in isotropic, multidimensional systems [1, 2]. For these and more general systems like (2), we show in Section 2 that

\[
\|u(\cdot,t)\|_{L_2(\mathbb{R})} \leq C(1 + t)^{-1/4},
\]

whenever \( u(\cdot,0) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \), however large, where \( C \) is a positive constant which depends only on the magnitude of \( \|u(\cdot,0)\|_{L_1(\mathbb{R})} \) and \( \|u(\cdot,0)\|_{L_2(\mathbb{R})} \), the dimension parameter \( m \) and \( \mu > 0 \) such that

\[
\langle \xi, B(u)\xi \rangle \geq \mu |\xi|^2 \quad \forall \xi \in \mathbb{R}^m
\]

for all \( u \) concerned. Thus, the solution \( u = 0 \) of Eq. (2) is asymptotically stable under arbitrarily large disturbances in \( L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \). A similar property is shown to hold for the scalar advection–diffusion equation

\[
\frac{\partial u}{\partial t} + f(u) = (b(u)u_x),
\]

where \( b(u) \) is positive. Given an arbitrary initial state \( u(\cdot,0) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \), it is shown in Section 3 that, for any \( t > 0 \),

\[
\|u(\cdot,t)\|_{L_2(\mathbb{R})} \leq C(1 + t)^{-1/4}
\]

(7)
for some constant $C$ which depends only on the magnitude of $\|u(\cdot,0)\|_{L^1(\mathbb{R})}$ and $\|u(\cdot,0)\|_{L^\infty(\mathbb{R})}$, and $\mu > 0$ such that
\[
b(\omega) \geq \mu \quad \text{for } |\omega| \leq \|u(\cdot,0)\|_{L^\infty(\mathbb{R})}.
\] (8)

Finally, in Section 4, we simplify the analysis in [19] to establish a similar result for the time decay in $L^2$-norm of arbitrarily large disturbances to scalar rarefaction waves in case of a quadratic flux, i.e., Burgers equation [5, 10]. This time we are concerned with the large time behavior of solutions of the initial value problem
\[
u_t + f(\nu)_x = \nu_{xx}, \tag{9a}
\]
\[
u(x,0) = \nu_0(x), \tag{9b}
\]
where
\[
f(\nu) = a\nu^2 + b\nu + c \tag{10}
\]
and the initial profile $\nu_0$ is a bounded measurable function connecting two given constant states $\nu_{\pm}$ at $\pm\infty$, in the sense that
\[
\int_{-\infty}^0 |\nu_0(x) - \nu_-| \, dx < \infty \tag{11a}
\]
and
\[
\int_0^{+\infty} |\nu_0(x) - \nu_+| \, dx < \infty.
\] (11b)

We denote by $\mathcal{B}(\nu_-\nu_+)$ the space of all such functions, i.e., all $\nu_0 \in L^\infty(\mathbb{R})$ for which both integrals in (11a) and (11b) are finite. In (10), we assume for definiteness that we have $a > 0$, so that our interest here is the case when
\[
\nu_- < \nu_+. \tag{12}
\]

Taking an initial state $v(\cdot,0) \in \mathcal{B}(\nu_-\nu_+)$ which is monotonically increasing, the corresponding solution $v(x,t)$ of problem (9) will stay monotonic in $x$ for all $t > 0$, see e.g., [19], and is called a (viscous) rarefaction or expansion wave. Hence, associated with (9)–(12) above, we consider the problem
\[
v_t + f(v)_x = v_{xx}, \tag{13a}
\]
\[
v(x,0) = v_0(x) \tag{13b}
\]
where $v_0 \in \mathcal{B}(\nu_-\nu_+)$ is monotonic. Thus, we may regard the initial state $u(\cdot,0)$ as a perturbation of $v(\cdot,0)$,
\[
u(x,0) = v(x,0) + \eta(x), \tag{14}
\]
where the disturbance $\eta$ satisfies
\[
\eta \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \tag{15}
\]
but is otherwise arbitrary. Under the conditions above, the solution \( u(\cdot, t) \) is shown to converge to the rarefaction wave \( v(\cdot, t) \) as \( t \to \infty \), with
\[
\|u(\cdot, t) - v(\cdot, t)\|_{L^2(\mathbb{R})} \leq C(1 + t)^{-1/4},
\]
where the constant \( C \) depends only on the magnitude of \( \|\eta\|_{L^1(\mathbb{R})} \) and \( \|\eta\|_{L^\infty(\mathbb{R})} \) and \( a > 0 \) is given in (10). Thus, like the previous examples, the rarefaction waves (13) exhibit asymptotic stability in \( L^2 \) with respect to arbitrarily large disturbances over their initial profile, provided that they are bounded and integrable. This is also true for other \( L^p \)-norms, \( p > 1 \), as can be shown using standard energy methods to estimate \( \|u(\cdot, t) - v(\cdot, t)\|_{L^p(\mathbb{R})} \) once the results of Section 4 have been proved. We note that the \( L^\infty \)-stability was first shown in [11], without a decay rate. Indeed, they showed that
\[
\|u(\cdot, t) - r(\cdot, t)\|_{L^\infty(\mathbb{R})} \to 0 \quad \text{as} \quad t \to \infty,
\]
where \( r(x, t) \) is the self-similar, entropy solution of the inviscid problem
\[
\begin{align*}
  r_t + f(r)_x &= 0, \quad (18a) \\
  r(x, 0) &= \begin{cases} 
    v_-, & x < 0, \\
    v_+, & x > 0.
  \end{cases} \quad (18b)
\end{align*}
\]
When \( f \) is quadratic, a very detailed description of the viscous waves \( u(\cdot, t) \) has been given [9], which yields
\[
\|u(\cdot, t) - r(\cdot, t)\|_{L^p(\mathbb{R})} = O(1)t^{-1/2+1/2p}. \quad (19)
\]
This can be generalized to more general flux functions using the methods given here and in [8] to estimate \( \|u(\cdot, t) - v(\cdot, t)\|_{L^p(\mathbb{R})} \) and \( \|v(\cdot, t) - r(\cdot, t)\|_{L^p(\mathbb{R})} \), respectively, yielding in the case of convex flux the estimate
\[
\|u(\cdot, t) - r(\cdot, t)\|_{L^p(\mathbb{R})} = O(1)(\log t)^{1/2+1/2p}t^{-1/2+1/2p}.
\]

In what follows, we will derive the estimates (4), (7) and (16) adapting the discussion in [18–20] to our present needs. In particular, we will extend the argument in [18] to the more general systems (2), and make use of (10) to give a direct derivation of (16) which is much simpler than the general treatment considered in [19]. (One should note, however, that for a general convex flux \( f \) the derivation given here would require the additional condition that the disturbance \( \eta \) be sufficiently small in \( L^1(\mathbb{R}) \).) This approach avoids many technicalities which are dealt with in the references above and brings out more clearly the similarities in the analysis for each case. As to the notation used in the text, boldface characters will always denote vector quantities, while capital letters will be usually reserved for matrices. A symbol like \( C_{sf} \) will denote a constant whose value depends on a set of parameters specified by the list \( \mathcal{A} \); distinct references to the same constant symbol will not necessarily imply the same numerical value, so that we will write \( 2C_{sf} \) again as \( C_{sf} \), and so on. Also, we will often use subscripted variables to indicate differentiation, as in \( u_t = \partial u/\partial t, \ f(u)_x = (\partial/\partial x)f(u(x, t)) \), and so forth.
2. A class of parabolic systems

We will consider in this section the $L^2$ decay of disturbances to the equilibrium solution $u=0$ of the parabolic system

$$u_t + (u \phi(|u|))_x = (B(u)u)_x,$$

where $\phi$ denotes a continuously differentiable scalar function, $x \in \mathbb{R}$, $t > 0$, $u(x,t)$ is the vector of unknowns $(u_1(x,t), \ldots, u_m(x,t))$, and $B(u)$ is an $m \times m$ diagonal matrix which is uniformly positive definite, i.e.,

$$B(u) = \text{diag} \{ b_i(u), i = 1, \ldots, m \},$$

for all $u \in \mathbb{R}^m$, where $\mu$ is a positive constant. The initial state $u(\cdot,0)$ is any Lebesgue measurable pulse with finite mass and energy, i.e., $u(\cdot,0) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, and we let $K > 0$ be sufficiently large so that

$$\|u(\cdot,0)\|_{L^1(\mathbb{R})} + \|u(\cdot,0)\|_{L^2(\mathbb{R})} \leq K.$$ (22)

Under these conditions, we will show in this section that there exists a positive constant $C_K$, depending only on the parameters $\mathcal{K} = \{m, K, \mu\}$, such that

$$\|u(\cdot,t)\|_{L^1(\mathbb{R})} \leq C_K (1 + t)^{-1/4}$$ (23)

for all $t > 0$. We will prove this estimate in the following way. First, we note that the solution operator of (20) is $L^1$-contractive, i.e., we have

$$\|u(\cdot,t)\|_{L^1(\mathbb{R})} \leq \|u(\cdot,0)\|_{L^1(\mathbb{R})}$$ (24)

for any $t > 0$, where

$$\|u(\cdot,t)\|_{L^1(\mathbb{R})} = \|u_1(\cdot,t)\|_{L^1(\mathbb{R})} + \cdots + \|u_m(\cdot,t)\|_{L^1(\mathbb{R})}$$

and similarly for $\|u(\cdot,0)\|_{L^1(\mathbb{R})}$. In fact, more is true: Eq. (20) is $L^1$-contractive in each component $u_i(\cdot,t)$ individually, i.e., $\|u_i(\cdot,t)\|_{L^1(\mathbb{R})} \leq \|u_i(\cdot,0)\|_{L^1(\mathbb{R})}$ for all $t > 0$ and $i = 1, 2, \ldots, m$. This can be proved in a standard way as in [8, 15], but for convenience of the reader we will briefly review the argument. Taking a regularized sign function $L_\delta$ (see, e.g., [15, 19]), we multiply the $i$th component of Eq. (20) by $L_\delta(u_i(x,t))$ and integrate the result over $\mathbb{R} \times [0,T]$ to get, after a few integrations by parts,

$$\int_{-\infty}^{\infty} L_\delta(u_i(x,T)) \, dx + \int_0^T \int_{-\infty}^{\infty} L_\delta''(u_i(x,t)) b_i(u(x,t)) \left( \frac{\partial u_i}{\partial x} \right)^2 \, dx \, dt$$

$$= \int_{-\infty}^{\infty} L_\delta(u_i(x,0)) \, dx + \int_0^T \int_{-\infty}^{\infty} L_\delta''(u_i(x,t)) u_i(x,t) \frac{\partial u_i}{\partial x} \phi(|u(x,t)|) \, dx \, dt.$$
Since \( b_i(u) \) is nonnegative, we then have
\[
\int_{-\infty}^{\infty} L_\delta(u_i(x, T)) \, dx \leq \int_{-\infty}^{\infty} L_\delta(u_i(x, 0)) \, dx \\
+ \int_0^T \int_{-\infty}^{\infty} L''_\delta(u_i(x, t)) u_i(x, t) \frac{\partial u_i}{\partial x} \varphi(|u(x, t)|) \, dx \, dt.
\]
Letting \( \delta \to 0 \), we get \( \| u(\cdot, T) \|_{L^1(\mathbb{R})} \leq \| u(\cdot, 0) \|_{L^1(\mathbb{R})} \), as stated, since
\[
\int_0^T \int_{-\infty}^{\infty} L''_\delta(u_i(x, t)) u_i(x, t) \frac{\partial u_i}{\partial x} \varphi(|u(x, t)|) \, dx \, dt \to 0
\]
by Lebesgue’s dominated convergence theorem, which concludes the derivation of inequality (24) above. Another property which can be easily derived is the following energy estimate:
\[
\| u(\cdot, T) \|_{L^2(\mathbb{R})}^2 + 2\mu \int_0^T \| Du(\cdot, t) \|_{L^2(\mathbb{R})}^2 \, dt \leq \| u(\cdot, 0) \|_{L^2(\mathbb{R})}^2,
\]
where
\[
\| u(\cdot, t) \|_{L^2(\mathbb{R})}^2 = \sum_{i=1}^m \| u_i(\cdot, t) \|_{L^2(\mathbb{R})}^2
\]
and
\[
\| Du(\cdot, t) \|_{L^2(\mathbb{R})}^2 = \sum_{i=1}^m \left\| \frac{\partial u_i}{\partial x}(\cdot, t) \right\|_{L^2(\mathbb{R})}^2
\]
In fact, multiplying the \( i \)th component of (20) by \( u_i(x, t) \), integrating the result over \( \mathbb{R} \times [0, T] \) and summing from \( i = 1 \) to \( m \), we get, after a few similar computations,
\[
\sum_{i=1}^m \int_{-\infty}^{\infty} |u_i(x, T)|^2 \, dx + 2\mu \sum_{i=1}^m \int_0^T \int_{-\infty}^{\infty} b_i(u_i(x, t)) \left( \frac{\partial u_i}{\partial x} \right)^2 \, dx \, dt \\
= \sum_{i=1}^m \int_{-\infty}^{\infty} |u_i(x, 0)|^2 \, dx + \sum_{i=1}^m \int_0^T \int_{-\infty}^{\infty} \varphi(|u(x, t)|) \frac{\partial}{\partial x} |u_i(x, t)|^2 \, dx \, dt,
\]
from which we immediately get (25), since
\[
\sum_{i=1}^m \int_{-\infty}^{\infty} \varphi(|u(x, t)|) \frac{\partial}{\partial x} |u_i(x, t)|^2 \, dx = 0.
\]
In order to get a decay rate for \( \| \cdot(\cdot, T) \|_{L^1(\mathbb{R})} \), we multiply the \( i \)th component of Eq. (20) by \( (1 + t)u_i(x, t) \) and integrate the result over \( \mathbb{R} \times [0, T] \), which gives, summing from \( i = 1 \) to \( m \),
\[
(1 + T) \int_{-\infty}^{\infty} |u(x, T)|^2 \, dx \\
+ 2 \int_0^T (1 + t) \sum_{i=1}^m b_i(u_i(x, t)) \left( \frac{\partial u_i}{\partial x} \right)^2 \, dx \, dt
\]
\[
Z + 1 - j u(x; 0) dx + \int_0^T (1 + t) \int_{-\infty}^{+\infty} \phi(|u(x, t)|) \frac{\partial}{\partial x} |u(x, t)|^2 dx dt,
\]
so that, in view of (21) and (26), we have
\[
(1 + T) u(\cdot, T)_{L^2(\mathbb{R})}^2 + 2 \mu \int_0^T (1 + t) \|D u(\cdot, t)\|_{L^2(\mathbb{R})}^2 dt \\
\leq \|u(\cdot, 0)\|_{L^2(\mathbb{R})}^2 + \int_0^T \|u(\cdot, t)\|_{L^2(\mathbb{R})}^2 dt.
\] (27)

Using the elementary Sobolev inequality
\[
\|u(\cdot, t)\|_{L^2(\mathbb{R})} \leq C \|u(\cdot, t)\|_{L^1(\mathbb{R})}^{2/3} \left\| \frac{\partial u}{\partial x}(\cdot, t) \right\|_{L^2(\mathbb{R})}^{1/3},
\]
we get from (24),
\[
\|u(\cdot, t)\|_{L^2(\mathbb{R})} \leq C_{m,K} \|D u(\cdot, t)\|_{L^2(\mathbb{R})}^{1/3}
\]
for each \( t > 0 \), where \( C_{m,K} \) denotes a constant which depends only on \( m, K \). Hence, we obtain from (27) the estimate
\[
(1 + T) u(\cdot, T)_{L^2(\mathbb{R})}^2 + 2 \mu \int_0^T (1 + t) \|D u(\cdot, t)\|_{L^2(\mathbb{R})}^2 dt \\
\leq \|u(\cdot, 0)\|_{L^2(\mathbb{R})}^2 + C_{m,K} \int_0^T \|D u(\cdot, t)\|_{L^2(\mathbb{R})}^{2/3} dt.
\] (28)

Since, by Hölder’s inequality, we have
\[
\int_0^T \|D u(\cdot, t)\|_{L^2(\mathbb{R})}^{2/3} dt \leq 2(1 + T)^{1/3} \left\{ \int_0^T (1 + t) \|D u(\cdot, t)\|_{L^2(\mathbb{R})}^2 dt \right\}^{1/3},
\]
we see that, setting
\[
Y(T) \equiv (1 + T) u(\cdot, T)_{L^2(\mathbb{R})}^2 + \int_0^T (1 + t) \|D u(\cdot, t)\|_{L^2(\mathbb{R})}^2 dt,
\] (29)
we get from (28) that
\[
Y(T) \leq C_{\mathcal{K}} \left\{ 1 + (1 + T)^{1/3} Y(T)^{1/3} \right\}
\]
for some constant \( C_{\mathcal{K}} \) which depends on \( \mathcal{K} = \{ m, K, \mu \} \) given in (21) and (22). This immediately gives
\[
Y(T) \leq C_{\mathcal{K}} (1 + T)^{1/2}
\] (30)
for some suitable constant $C$ which, again, depends on $K = \{m, K, \mu\}$. Recalling the definition of $Y(T)$ given in (29) above, we then have
\[
(1 + T)\|u(\cdot, T)\|_{L^2(\mathbb{R})}^2 + \int_0^T (1 + t)\|Du(\cdot, t)\|_{L^2(\mathbb{R})}^2 \, dt \leq C_K (1 + T)^{1/2}.
\]
A little more generally, given $\sigma > 1/2$, if we repeat the derivation above using $(1 + t)^\sigma$ instead of $(1 + t)$, we will arrive at the following result.

**Theorem 1.** Let $u(x, t)$ be the solution of Eq. (20) corresponding to an initial profile $u(\cdot, 0)$ in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then, given any $\sigma > 1/2$, there exists a constant $C_{\sigma, K}$ (depending only on $\sigma$ and $K = \{m, K, \mu\}$ given in (21) and (22)) such that
\[
(1 + T)^\sigma \|u(\cdot, T)\|_{L^2(\mathbb{R})}^2 + \int_0^T (1 + t)^\sigma \|Du(\cdot, t)\|_{L^2(\mathbb{R})}^2 \, dt \leq C_{\sigma, K} (1 + T)^{\sigma - 1/2}
\]
for every $T > 0$.

3. The scalar advection–diffusion equation

In this section we will establish by a similar argument the $L^2$ decay of solutions $u(x, t)$ of the equation
\[
u_t + f(u)_x = (b(u)u_x)_x,
\]
where $u(\cdot, 0)$ is an arbitrary bounded, integrable pulse on the real line, i.e.,
\[
u(\cdot, 0) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}).
\]
Similarly to (21) above, we assume in this section that
\[
b(u) \geq \mu
\]
for all $u$ concerned, where $\mu$ is a positive constant. Under these assumptions, we will derive the estimate
\[
\|u(\cdot, t)\|_{L^2(\mathbb{R})} \leq C(1 + t)^{-1/4}
\]
for all $t > 0$, where $C$ is a constant which depends on $\mu$ and the magnitude of $\|u(\cdot, 0)\|_{L^1(\mathbb{R})}$ and $\|u(\cdot, 0)\|_{L^\infty(\mathbb{R})}$, referring the reader to [20] for the corresponding analysis in the $n$-dimensional case. Given $u(\cdot, 0)$ in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, we let $K > 0$ be large enough such that
\[
\|u(\cdot, 0)\|_{L^1(\mathbb{R})} + \|u(\cdot, 0)\|_{L^\infty(\mathbb{R})} \leq K.
\]
It is well known that, for any $t > 0$, $u(x, t)$ satisfies the maximum principle [16]
\[
\|u(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq \|u(\cdot, 0)\|_{L^\infty(\mathbb{R})}
\]
and the $L^1$-contractive property
\[
\|u(\cdot, t)\|_{L^1(\mathbb{R})} \leq \|u(\cdot, 0)\|_{L^1(\mathbb{R})},
\]
which can be established in a way similar to (24) above. In order to show (34), given \( T > 0 \) and \( \sigma > \frac{1}{2} \), we multiply (31) through by \((1 + t)^{\sigma}u\) and integrate over \( \mathbb{R} \times [0, T] \) to get, after a few computations,

\[
(1 + T)^{\sigma} \int_{-\infty}^{+\infty} |u(x, T)|^2 \, dx + 2 \int_{0}^{T} (1 + t)^{\sigma} \int_{-\infty}^{+\infty} b(u(x, t))u_2^2(x, t) \, dx \, dt
\]

\[
= \int_{-\infty}^{+\infty} |u(x, 0)|^2 \, dx + \sigma \int_{0}^{T} (1 + t)^{\sigma-1} \int_{-\infty}^{+\infty} |u(x, t)|^2 \, dx \, dt,
\]

so that, using (33), we obtain

\[
(1 + T)^{\sigma} \|u(\cdot, T)\|_{L^2(\mathbb{R})}^2 + 2 \mu \int_{0}^{T} (1 + t)^{\sigma} \|D_u(\cdot, t)\|_{L^2(\mathbb{R})}^2 \, dt
\]

\[
\leqslant \|u(\cdot, 0)\|_{L^2(\mathbb{R})}^2 + |\sigma| \int_{0}^{T} (1 + t)^{\sigma-1} \|u(\cdot, t)\|_{L^2(\mathbb{R})}^2 \, dt. \tag{38}
\]

From (35) and (37) and the Sobolev inequality

\[
\|u(\cdot, t)\|_{L^2(\mathbb{R})} \leqslant C \|u(\cdot, t)\|_{L^{2/3}(\mathbb{R})}^{1/3} \|Du(\cdot, t)\|_{L^2(\mathbb{R})}^{1/3},
\]

we get

\[
\|u(\cdot, t)\|_{L^2(\mathbb{R})} \leqslant C_K \|Du(\cdot, t)\|_{L^2(\mathbb{R})}^{1/3},
\]

where \( C_K \) is a constant which depends on \( K \). Hence, (38) yields

\[
(1 + T)^{\sigma} \|u(\cdot, T)\|_{L^2(\mathbb{R})}^2 + \int_{0}^{T} (1 + t)^{\sigma} \|Du(\cdot, t)\|_{L^2(\mathbb{R})}^2 \, dt
\]

\[
\leqslant C_{\sigma, \mathcal{K}} \left\{ 1 + \int_{0}^{T} (1 + t)^{\sigma-1} \|Du(\cdot, t)\|_{L^2(\mathbb{R})}^{2/3} \, dt \right\}, \tag{39}
\]

where \( C_{\sigma, \mathcal{K}} \) is a constant which depends only on \( \sigma \) and \( \mathcal{K} = \{K, \mu\} \) given in (33) and (35). As in the previous section, we use Hölder’s inequality to get

\[
\int_{0}^{T} (1 + t)^{\sigma-1} \|Du(\cdot, t)\|_{L^2(\mathbb{R})}^{2/3} \, dt \leqslant C_{\sigma} (1 + T)^{(2\sigma - 1)/3} \left( \int_{0}^{T} (1 + t)^{\sigma} \|Du(\cdot, t)\|_{L^2(\mathbb{R})}^2 \, dt \right)^{1/3},
\]

so that, from (39), we obtain

\[
(1 + T)^{\sigma} \|u(\cdot, T)\|_{L^2(\mathbb{R})}^2 + \int_{0}^{T} (1 + t)^{\sigma} \|Du(\cdot, t)\|_{L^2(\mathbb{R})}^2 \, dt
\]

\[
\leqslant C_{\sigma, \mathcal{K}} \left\{ 1 + (1 + T)^{(2\sigma - 1)/3} \left( \int_{0}^{T} (1 + t)^{\sigma} \|Du(\cdot, t)\|_{L^2(\mathbb{R})}^2 \, dt \right)^{1/3} \right\}.
\]

Proceeding as in (29) and (30), we immediately get (34) and the following result.
Theorem 2. Given $\sigma > \frac{1}{2}$, the solution $u(x,t)$ of (31)–(33) satisfies, for all $T > 0$,

$$(1 + T)^\sigma \|u(\cdot, T)\|^2_{L^2(\mathbb{R})} + \int_0^T (1 + t)^\sigma \|Du(\cdot, t)\|^2_{L^2(\mathbb{R})} dt,$$

$$\leq C_{\pi, \xi} (1 + T)^{\sigma - 1/2}$$

for some constant $C_{\pi, \xi}$ which depends only on $\sigma$ and $\mathcal{H} = \{K, \mu\}$ given in (33) and (35).

4. Rarefaction waves

Let $f$ be the quadratic function given in (10), where we assume $a > 0$, and that $u(x,t)$ is the solution of the initial value problem

$$u_t + f(u)x = u_{xx},$$

$$u(x,0) = v_0(x) + \eta(x),$$

where $v_0$ is a monotonically increasing profile connecting two given constant states $v_- < v_+$ as $x \to \pm \infty$, so that (11) is satisfied, i.e.,

$$\int_{-\infty}^0 |v_0(x) - v_-| \, dx < \infty,$$  \hspace{1cm} (41a)

and

$$\int_0^{+\infty} |v_0(x) - v_+| \, dx < \infty.$$  \hspace{1cm} (41b)

The disturbance $\eta$ is assumed to be bounded and integrable, $\eta \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, but is otherwise arbitrary. As before, we take $K > 0$ sufficiently large so that

$$\|\eta\|_{L^1(\mathbb{R})} + \|\eta\|_{L^\infty(\mathbb{R})} \leq K.$$  \hspace{1cm} (42)

Associated with (40) above, we consider the initial value problem

$$v_t + f(v)x = v_{xx},$$

$$v(x,0) = v_0(x).$$

Because $v(\cdot, 0) \in \mathcal{B}(v_-, v_+)$ is monotonic, it is well known (see, e.g., [19]) that $v(\cdot, t)$ stays monotonic for all $t > 0$, and is called a (viscous) rarefaction or expansion wave. Moreover, $v(\cdot, t)$ satisfies

$$\|v_x(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{ate} \quad \forall t > 0,$$  \hspace{1cm} (44)

where $a = f''(v)$, see (10), so that the rarefaction wave flattens out at a linear rate as $t$ increases [19]. Another important property is given by

$$\|u(\cdot, t) - v(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq \|\eta\|_{L^1(\mathbb{R})} \quad \forall t > 0,$$  \hspace{1cm} (45)
since the solution operator of (40) is $L^1$-contractive [8, 19]. Under the conditions above, it will be shown in this section that the solution $u(\cdot, t)$ converges in $L^2(\mathbb{R})$ to the rarefaction wave $v(\cdot, t)$ as $t \to \infty$, with

$$
\|u(\cdot, t) - v(\cdot, t)\|_{L^2(\mathbb{R})} \leq C_{a,K} (1 + t)^{-1/4}
$$

for all $t > 0$, where $C_{a,K}$ denotes a constant which depends on $a$ and $K$. Once (46) has been proved, it would be easy to derive by standard energy methods the more general estimate

$$
\|u(\cdot, t) - v(\cdot, t)\|_{L^p(\mathbb{R})} \leq C_{a,K} (1 + t)^{-1/2 + 1/2p}
$$

for each $1 \leq p \leq \infty$ and $t > 0$, where $C_{a,K}$ depends on $a$, $K$ but not on $p$. Hence, the rarefaction wave (43) is asymptotically stable in $L^p(\mathbb{R})$, $p > 1$, with respect to arbitrarily large disturbances $\eta$, provided only that $\eta$ belongs to $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. For a generalization of this result to an arbitrary convex flux $f$, we refer the reader to [19]. In the sequel, we will establish (46) following an argument similar to that used in the previous sections. Setting $\theta = u - v$, we get, subtracting (43) from (40), that $\theta$ satisfies

$$
\theta_t + [f]_x = \theta_{xx},
$$

$$
\theta(x, 0) = \eta(x),
$$

where

$$
[f] \equiv f(\theta + \psi) - f(\psi).
$$

Multiplying (47a) by $(1 + t)^\sigma \theta$, where $\sigma > \frac{1}{2}$, and integrating the result on $\mathbb{R} \times [0, T]$, we obtain, after a few integrations by parts

$$
\begin{align*}
(1 + T)^\sigma \|\theta(\cdot, T)\|_{L^2(\mathbb{R})}^2 + 2 \int_0^T (1 + t)^\sigma \|\theta_x(\cdot, t)\|_{L^2(\mathbb{R})}^2 \, dt \\
= \|\theta(\cdot, 0)\|_{L^2(\mathbb{R})}^2 + \sigma \int_0^T (1 + t)^{\sigma - 1} \|\theta_x(\cdot, t)\|_{L^2(\mathbb{R})}^2 \, dt \\
- 2 \int_0^T (1 + t)^\sigma \int_{-\infty}^{+\infty} \theta(x, t)[f]_x \, dx \, dt.
\end{align*}
$$

Since

$$
\int_{-\infty}^{+\infty} \theta(x, t)[f]_x \, dx = \int_{-\infty}^{+\infty} \theta(x, t)(f'(\theta + v)(\theta_x + \psi) - f'(v)\psi_x) \, dx
$$

$$
= \int_{-\infty}^{+\infty} \theta(x, t)[f'] \psi_x \, dx + \int_{-\infty}^{+\infty} \theta \psi_x f'(\theta + v) \, dx,
$$

we get, using (10),

$$
\int_{-\infty}^{+\infty} \theta(x, t)[f]_x \, dx = a \int_{-\infty}^{+\infty} \theta(x, t)^2 \psi_x \, dx,
$$

for some positive constant $a$. Therefore, (46) follows from (47a) by using (48) and the $L^1$-contractivity of the solution operator.
so that we can write (49) as

\[
(1 + T)^\sigma \|\theta(\cdot, T)\|_{L^2(\mathbb{R})}^2 + 2 \int_0^T (1 + t)^{\sigma - 1} \|\theta(\cdot, t)\|_{L^2(\mathbb{R})}^2 \, dt
= \|\theta(\cdot, 0)\|_{L^2(\mathbb{R})}^2 + (a + \sigma) \int_0^T (1 + t)^{\sigma - 1} \|\theta(\cdot, t)\|_{L^2(\mathbb{R})}^2 \, dt.
\]

(50)

As in the previous sections, we use the Sobolev inequality

\[
\|\theta(\cdot, t)\|_{L^2(\mathbb{R})} \leq C \|\theta(\cdot, t)\|_{L^3(\mathbb{R})}^{2/3} \|\theta(\cdot, t)\|_{L^1(\mathbb{R})}^{1/3},
\]

together with (42) and (45) to get

\[
\int_0^T (1 + t)^{\sigma - 1} \|\theta(\cdot, t)\|_{L^2(\mathbb{R})}^2 \, dt \leq C_k \int_0^T (1 + t)^{\sigma - 1} \|\theta(\cdot, t)\|_{L^2(\mathbb{R})}^{2/3} \, dt,
\]

which gives, as before,

\[
\int_0^T (1 + t)^{\sigma - 1} \|\theta(\cdot, t)\|_{L^2(\mathbb{R})}^2 \, dt \leq C_{\sigma, K} (1 + T)^{(2\sigma - 1)/3} \left( \int_0^T (1 + t)^{\sigma} \|\theta(\cdot, t)\|_{L^2(\mathbb{R})}^2 \, dt \right)^{1/3}
\]

for some constant \( C_{\sigma, K} \) which depends on \( \sigma \) and \( \mathcal{K} = \{a, K\} \). Thus, recalling (50), we obtain

\[
(1 + T)^\sigma \|\theta(\cdot, T)\|_{L^2(\mathbb{R})}^2 + \int_0^T (1 + t)^{\sigma} \|\theta(\cdot, t)\|_{L^2(\mathbb{R})}^2 \, dt
\leq C_{\sigma, \mathcal{K}} \left\{ 1 + (1 + T)^{(2\sigma - 1)/3} \left( \int_0^T (1 + t)^{\sigma} \|\theta(\cdot, t)\|_{L^2(\mathbb{R})}^2 \, dt \right)^{1/3} \right\}.
\]

Proceeding as in (29) and (30), we then immediately get (46) and the following result.

**Theorem 3.** Let \( v(x, t) \) be the rarefaction wave (43) corresponding to an initial, expansive profile \( \nu_0 \in B(v_-, v_+) \), and let \( u_0 = v_0 + \eta \) where the disturbance \( \eta \) is a bounded, integrable function. Then the solution \( u(x, t) \) of (40) satisfies

\[
(1 + T)^\sigma \|u(\cdot, T) - v(\cdot, T)\|_{L^2(\mathbb{R})}^2 + \int_0^T (1 + t)^{\sigma} \|u(\cdot, t) - v(\cdot, t)\|_{L^2(\mathbb{R})}^2 \, dt \leq C_{\sigma, \mathcal{K}} (1 + T)^{\sigma - 1/2}
\]

for some constant \( C_{\sigma, \mathcal{K}} \) which depends only on \( \sigma \) and \( \mathcal{K} = \{a, K\} \) given in (10) and (42).

**References**


