On condition numbers of some quasi-linear transformations

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Received 25 June 1998

Abstract

Convergence acceleration processes are known to be well conditioned for alternating sequences and ill conditioned for monotonic ones. The aim of this paper is to adapt the definition of conditioning and to give a link between this notion and the property of convergence acceleration. The cases of the linear and logarithmic convergence are studied in details. © 1999 Elsevier Science B.V. All rights reserved.

Keywords: Condition number; Convergence acceleration; Quasi-linearity

1. Introduction

Several authors have noticed that most convergence acceleration processes are generally well conditioned for alternating sequences and ill conditioned for monotonic ones. By studying particularly the stability of the \(\varepsilon\)-algorithm, Wynn [18] showed that this method is numerically stable when applied to sequences whose terms oscillate around their limits, but it is relatively numerically unstable for sequences that approach their limits monotonically. A similar result was given by Bell and Phillips [3] as far as the Aitken’s \(A^2\)-process is concerned. Following the theory of Rice [15], Cordellier [7] proposed a transformation denoted by \(C_k^{(m)}\) and studied its condition number with \(k = 1\). His work leads to a preconditioning process for linear monotonic convergence.

This study starts with a suitable and classical definition of condition numbers concerning the class of quasi-linear sequence transformations. This definition allows us to obtain interesting results on the class of linear sequences. More precisely, we verify that monotonic linear sequences cannot be accelerated by well conditioned processes, while this phenomenon is possible with alternating ones. Furthermore, it will be observed that any process accelerating a logarithmic sequence may often be disastrously ill conditioned. To avoid the ill conditioning, it is natural to try to transform monotonic sequences into alternating ones, as a preconditioning technique.

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In the final part of this work some attempts of preconditioning techniques given by various authors are reviewed.

2. Conditioning and acceleration property

2.1. Ponctual and asymptotic condition numbers of a stationary sequence transformation

Let \( x = (x_n) \) be a converging sequence of limit \( x^* \) and let \( g \) be an application defined on \( \mathbb{R}^{p+1} \), where \( p \geq 2 \). In this paper, we shall consider \( g \) as a function of the \( p + 1 \) variables \( u_0, u_1, \ldots, u_p \).

The sequence transformation \( x \to y \) is defined by

\[
y_i = g(x_i, x_{i+1}, \ldots, x_{i+p}), \quad i = 0, 1, 2, \ldots
\]

We would like to mention that in this work we are interested only by stationary processes (defined in the sense of Ortega and Rheinboldt [14]).

Let us assume that \( g \) is of class \( C^1 \) on an open subset \( A \) of \( \mathbb{R}^{p+1} \).

In this section, we are concerned with the way by which \( y_i \) is changed when we perturb the numbers \( x_i, x_{i+1}, \ldots, x_{i+p} \). We denote by \( \delta x_q \) a perturbation on the term \( x_q \), \( q \in \mathbb{N} \). The resulting perturbation on the term \( y_i \) is given by

\[
\delta y_i = \delta x_i \frac{\partial g}{\partial u_0} (\xi_i) + \delta x_{i+1} \frac{\partial g}{\partial u_1} (\xi_i) + \cdots + \delta x_{i+p} \frac{\partial g}{\partial u_p} (\xi_i),
\]

where \( \xi_i = (x_i + \theta_0 \delta x_i, x_{i+1} + \theta_1 \delta x_{i+1}, \ldots, x_{i+p} + \theta_p \delta x_{i+p}) \in \mathbb{R}^{p+1}, \ 0 < \theta_j < 1, \ j = 0, 1, \ldots, p \). From (2) it follows that

\[
|\delta y_i| \leq \max(|\delta x_i|, |\delta x_{i+1}|, \ldots, |\delta x_{i+p}|) \sum_{j=0}^{p} \left| \frac{\partial g}{\partial u_j} (\xi_i) \right|.
\]

Setting \( X_i = (x_i, x_{i+1}, \ldots, x_{i+p}) \in \mathbb{R}^{p+1}, \ i = 0, 1, 2, \ldots \), we are led to the

**Definition 2.1.** The ponctual condition number at the \( i \)th step of the transformation \( g \) for a sequence \( x \) is given by

\[
\mathcal{C}_i(g, x) = \sum_{j=0}^{p} \left| \frac{\partial g}{\partial u_j} (X_i) \right|, \quad i = 0, 1, 2, \ldots
\]

This number is, in fact, the amplification factor of the errors on the \( x_i \)'s. In case of convergence of this sequence of condition numbers, we can define the **asymptotic condition number** of the transformation \( g \) for the sequence \( x \) by

\[
\mathcal{C}_\infty(g, x) = \lim_{i \to \infty} \mathcal{C}_i(g, x).
\]
Here we will see that this asymptotic condition number is well defined when \( g \) is a transformation accelerating the linear convergence. Unfortunately, it will be observed in case of logarithmic convergence that such a quantity does not exist. Thus, the quasi-linearity will be used. It is a formalism defined by Germain–Bonne [10] to describe some general properties of almost all extrapolation processes.

Remark 2.2. Let us point out the fact that in the definition of \( \zeta_i \) the partial derivatives are evaluated in \( X_i \) instead of the \( \xi_i \) of (2).

2.2. Quasi-linear acceleration processes and condition numbers

We particularly take interest in the class of extrapolation processes which verify

(a) a property of translativity on \( A \), i.e.,
\[
\forall (u_0, \ldots, u_p) \in A, \quad \forall b \in \mathbb{R}, \quad g(u_0 + b, \ldots, u_p + b) = g(u_0, \ldots, u_p) + b, \tag{5}
\]

(b) a property of homogeneity on \( A \), that is
\[
\forall (u_0, \ldots, u_p) \in A, \quad \forall a \in \mathbb{R}, \quad g(a.u_0, \ldots, a.u_p) = a.g(u_0, \ldots, u_p). \tag{6}
\]

Translative and homogeneous transformations are called quasi-linear. Such sequence transformations have been introduced by Germain–Bonne [10] and developed by Brezinski [6]. Most extrapolation processes are quasi-linear transformations, such as Aitken’s \( A^2 \)-process, Wynn’s \( \varepsilon \)-algorithm [17], more generally the \( E \)-algorithm (under certain assumptions) [5], Pennacchi transforms [10], \( \Theta \)-algorithm [4], and many other processes.

We would like to point out the fact that since only stationary transformations are considered in this work, the diagonal processes are excluded because the number of terms of the corresponding transformations is increasing.

In order to apply the notion of quasi-linearity, we need the following result proved by Lembarki [11].

Lemma 2.3. Let \( g \) be an application of class \( C^1 \) in an open domain \( A \) of \( \mathbb{R}^{p+1} \).

(i) \( g \) is translative if and only if
\[
\sum_{j=0}^{p} \frac{\partial g}{\partial u_j}(X) = 1,
\]

(ii) \( g \) is homogeneous if and only if
\[
\sum_{j=0}^{p} x_j \frac{\partial g}{\partial u_j}(X) = g(X),
\]

where \( X = (x_0, x_1, \ldots, x_p) \) belongs to \( A \).
2.2.1. The linear case

Let $\text{LIN}$ be the set of linear sequences, that is the set of convergent sequences $x$ such that

$$\exists \rho \neq 0, \quad \rho \in [-1, 1[ \text{ such that } \lim_{n \to \infty} \frac{e_{n+1}}{e_n} = \rho,$$

where $e_n = x_n - x^*$, $n = 0, 1, 2, \ldots$.

If $\rho \in [-1, 0[$, $x$ is an alternating sequence. Let us designate by $\text{LIN}^-$ the set of these sequences, and by $\text{LIN}^+$ the set of monotonic linear sequences (i.e. such that $\rho \in [0, 1[$).

We shall now discuss why a sequence transformation accelerating $\text{LIN}$ is generally better conditioned on $\text{LIN}^-$ than on $\text{LIN}^+$.

Let $x \in \text{LIN}$ and $y$ the sequence defined by

$$y_n = g(x_n, x_{n+1}, \ldots, x_{n+p}), \quad n = 0, 1, 2, \ldots . \tag{7}$$

**Property 2.4.** The sequence transformation $g$ is regular, i.e., $y$ has the same limit as $x$

$$\lim_{n \to \infty} g(x_n, x_{n+1}, \ldots, x_{n+p}) = x^*.$$

**Proof.** By successive application of translativity and homogeneity, we obtain easily

$$y_n - x^* = (x_n - x^*) g\left(1, \frac{x_{n+1} - x^*}{x_n - x^*}, \ldots, \frac{x_{n+p} - x^*}{x_n - x^*}\right).$$

Since the sequence $x$ converges linearly, then the values of $g$ are bounded. Thus it follows that $\lim_{n \to \infty} y_n = x^*$. \(\square\)

In the case of linear convergence, we will show now that the asymptotic condition number of a quasi-linear transformation is well defined.

Since $x \in \text{LIN}$, then $\exists \rho \neq 0, \ \rho \in [-1, 1[ \text{ such that } \lim_{n \to \infty} (e_{n+1}/e_n) = \rho$.

Setting $\rho_i = e_{i+1}/e_i$, and using the quasi-linearity of $g$, it comes

$$g(X_i) = x^* + \rho_i g\left(1, \rho_i, \ldots, \prod_{k=0}^{p-1} \rho_{i+k}\right),$$

where $X_i = (x_i, x_{i+1}, \ldots, x_{i+p}) \in \mathbb{R}^{p+1}$. Denoting by $(E_j)$ the canonical base of $\mathbb{R}^{p+1}$, the partial derivatives of $g$ verify the following equality

$$\frac{\partial g}{\partial u_j}(X_i) = \lim_{h \to 0} \frac{g(X_i + hE_j) - g(X_i)}{h} = \ldots = \frac{\partial g}{\partial u_j}\left(1, \rho_i, \ldots, \prod_{k=0}^{p-1} \rho_{i+k}\right), \quad j = 0, 1, \ldots, p.$$

This equality holds for all indices $i \in \mathbb{N}$. Since the partial derivatives of $g$ are continuous and $\rho = \lim_{i \to \infty} \rho_i$, we obtain the
Property 2.5. The asymptotic condition number of a quasi-linear sequence transformation \( g \) for a linear sequence \( x \) with the asymptotic linear ratio \( \rho \) is

\[
\kappa'_\infty(g, x) = \sum_{j=0}^{\rho} \left| \frac{\partial g}{\partial u_j}(1, \rho, \ldots, \rho^\rho) \right|.
\]

This condition number is well defined since the following subset

\[
\{(1, \lambda, \ldots, \lambda^\rho) \in \mathbb{R}^{\rho+1} : \lambda \neq 0, \lambda \in [-1, 1]\},
\]

is obviously included in the open domain \( A \) where \( g \) is defined.

2.2.2. The logarithmic case

Let \( \text{LOG} \) denotes the set of logarithmic sequences, that is of convergent sequences \( x \) such that

\[
\lim_{n \to \infty} \frac{e_{n+1}}{e_n} = 1,
\]

where \( e_n = x_n - x^* \), \( n = 0, 1, 2, \ldots \)

In this case (which may be considered as a limit case of the linear convergence with \( \lambda = 1 \)) as we shall see below, it will not be possible to define the asymptotic condition number. In fact, it is shown [11] that only linear transformations \( g \) are of class \( C^1 \) in a neighbourhood of points of the form \((x, x, \ldots, x)\). As we shall see in Section 4, the only possibility on \( \text{LOG} \) is to define punctual condition numbers.

3. Condition numbers of transformations accelerating LIN

The main result presented in this section shows that the asymptotic condition number of a sequence transformation \( g \) is strongly related to the sign of its partial derivatives. Furthermore, it exists a connection between these quantities and the ability of \( g \) to accelerate the convergence.

Let us recall that a sequence transformation \( g : x \to y \) is said to accelerate the convergence of \( x \) iff \( \lim_{n \to \infty} (y_n - x^*)/(x_n - x^*) = 0 \). In this case, \( y \) is said to converge faster than \( x \).

Lemma 3.1. Let \( x \) be a linear sequence of limit \( x^* \) and with asymptotic ratio \( \rho \).

The sequence transformation \( g \) accelerates the sequence \( x \) iff one of the following statements holds

1. \( g(1, \rho, \ldots, \rho^\rho) = 0 \).
2. \( \sum_{j=0}^{\rho} \rho^j \cdot \frac{\partial g}{\partial u_j}(1, \rho, \ldots, \rho^\rho) \).

Proof. (i) Let \( y_n = g(x_n, \ldots, x_{n+p}) \), \( n = 0, 1, 2, \ldots \) be the transformed sequence. As in the proof of Property 2.4 we have

\[
\frac{y_n - x^*}{x_n - x^*} = g \left( 1, \frac{x_{n+1} - x^*}{x_n - x^*}, \ldots, \frac{x_{n+p} - x^*}{x_n - x^*} \right).
\]
Then by taking $n \to \infty$, we deduce that the sequence $y$ accelerates $x$ if and only if

$$g(1, \rho, \ldots, \rho^p) = 0.$$  

(ii) Applying Lemma 2.3, we immediately obtain

$$g(1, \rho, \ldots, \rho^p) = \sum_{j=0}^{p} \rho^j \frac{\partial g}{\partial u_j}(1, \rho, \ldots, \rho^p),$$

thus (ii) obviously holds. $\Box$

We remark that (i) is a characterization of the kernel of any quasi-linear transformation accelerating $\text{LIN}$.  

**Lemma 3.2.** Let $g$ be a quasi-linear transformation defined on $\mathbb{R}^{p+1}$, $p \geq 2$. A necessary and sufficient condition that $\mathcal{C}_1(g; x) = 1$, $\forall x \in \text{LIN}$ is that

$$\frac{\partial g}{\partial u_j}(1, \rho, \ldots, \rho^p) \geq 0, \quad \forall j = 0, \ldots, p, \quad \forall \rho \neq 0, \quad \rho \in [-1, 1],$$

with all these derivatives not simultaneously equal to zero.

**Proof.** Let $x$ a sequence of $\text{LIN}$ of asymptotic ratio $\rho$, and let us suppose that, for this sequence, $\exists i \in \{0, \ldots, p\}$, $\partial g/\partial u_i < 0$ and that $\partial g/\partial u_j \geq 0$, $\forall j \in \{0, \ldots, p\}$, $j \neq i$. We have

$$\mathcal{C}_1(g, x) = \left( \sum_{0 \leq j \leq p, \; j \neq i} \frac{\partial g}{\partial u_j} \right) - \frac{\partial g}{\partial u_i} = \sum_{j=0}^{p} \frac{\partial g}{\partial u_j} - 2 \frac{\partial g}{\partial u_i}.$$

Since $\partial g/\partial u_i < 0$, it follows by (i) of Lemma 2.3, that $\mathcal{C}_1(g, x) > 1$.

Conversely the result is obviously a consequence of the definition of the condition number and Lemma 2.3. $\Box$

3.1. The monotonic case

We will see now why an accelerating process on $\text{LIN}^+$ may propagate small errors on the terms of initial sequence.

**Theorem 3.3.** Let $g$ be a quasi-linear transformation defined on $\mathbb{R}^{p+1}$, $p \geq 2$. If there exists $x \in \text{LIN}^+$ such that

$$\mathcal{C}_\infty(g, x) = 1,$$

then this sequence cannot be accelerated by $g$. 


Proof. We suppose that \( g \) accelerates a sequence \( x \in \text{LIN} \) and \( \mathcal{C}_\infty(g,x) = 1 \). Setting \( e_n = x_n - x^* \), there exists \( \rho \in ]0,1[ \) such that \( \lim_{n \to \infty} (e_{n+1}/e_n) = \rho \). Applying Lemma 3.2 since \( \mathcal{C}_\infty(g,x) = 1 \) we deduce easily that

\[
\rho^j \frac{\partial g}{\partial u_j}(1, \rho, \ldots, \rho^p) \geq 0, \quad \forall j \in \{0, \ldots, p\}.
\]

Using (i) of Lemma 2.3, we have

\[
\sum_{j=0}^p \rho^j \frac{\partial g}{\partial u_j}(1, \rho, \ldots, \rho^p) \geq \rho^p > 0.
\]

Finally by Lemma 3.1, \( g \) cannot define an accelerating process for the sequence \( x \). \( \Box \)

In other words, one cannot expect to accelerate linear monotone sequences without increasing small errors on the initial sequence.

3.2. The alternating case

We suppose now that \( x \in \text{LIN}^- \). Does it exist a transformation \( g \) accelerating \( x \) such that \( \mathcal{C}_\infty(g,x) > 1 \)? The answer to this question requires to establish a general expression of quasi-linear transformations in order to capture the sign of each partial derivative.

3.2.1. General form of quasi-linear transformations accelerating \( \text{LIN} \)

Let \( g \) be a quasi-linear transformation defined on \( \mathbb{R}^{p+1}, p \geq 2 \). We give a global form of \( g \) so that

\[
g(1, \lambda, \ldots, \lambda^p) = 0, \quad \forall \lambda \neq 0, \lambda \in [-1, 1]. \tag{9}
\]

Applying successively the translativity and homogeneity to \( g \), we obtain

\[
g(x_0, x_1, \ldots, x_p) = x_0 + (x_1 - x_0)G(v_2, \ldots, v_p), \tag{10}
\]

where \( G \) is a function of the

\[
v_i = \frac{x_i - x_0}{x_1 - x_0}, \quad i = 2, \ldots, p.
\]

Since \( x \in \text{LIN} \), the asymptotic ratio \( \rho \) verify

\[
G(1+\rho, 1+\rho+\rho^2, \ldots, 1+\rho + \cdots + \rho^{p-1}) = \frac{1}{1-\rho}. \tag{11}
\]

Let us now give a characterization of \( G \) such that (11) is verified for

\[
v_i = \sum_{j=0}^{i-1} \rho^j, \quad i = 2, \ldots, p.
\]
Our problem is now the following: we want to construct \( G \) such that
\[
G(v_2, \ldots, v_p) = \frac{1}{2 - v_2}
\] when the next \( v_i \)'s are given by the iterations
\[
v_i = 1 + \rho v_{i-1}, \quad i = 3, \ldots, p.
\]

For any choice of \( p - 2 \) regular applications \( F_i \) defined on \( \mathbb{R}^{p-1} \), \( i = 2, \ldots, p-1 \), the function \( G \) is
\[
G(v_2, \ldots, v_p) = \frac{1}{2 - v_2} + \sum_{i=2}^{p-1} \frac{v_{i+1} - 1 - (v_2 - 1)v_i}{2} F_i(v_2, \ldots, v_p).
\] (13)

Carrying back this expression of \( G \) in (10), we note that \( g \) is a sum containing a well-known function
\[
g(x_0, x_1, \ldots, x_p) = f_1(x_0, x_1, x_2) + \sum_{i=2}^{p-1} \left( \frac{x_{i+1} - x_i}{x_1 - x_0} \right) f_i(x_0, x_1, \ldots, x_p),
\] (14)

where
\[
f_1(x_0, x_1, x_2) = x_0 - \frac{x_1 - x_0}{[(x_2 - x_1)/(x_1 - x_0)] - 1},
\] (15)
represents Aitken’s \( A^2 \)-process, and
\[
f_i(x_0, x_1, \ldots, x_p) = F_i(v_2, \ldots, v_p), \quad i = 2, \ldots, p-1 \quad \text{if} \quad p \geq 3.
\]

**Remark 3.4.** In the particular case \( p = 2 \), it is obvious that Aitken’s \( A^2 \)-process is the only quasi-linear one which accelerates \( \text{LIN} \).

Using for simplicity the abbreviation \( F_i(\rho) \) instead of \( F_i(1+\rho, 1+\rho^2, \ldots, 1+\rho + \cdots + \rho^{p-1}) \), we easily obtain from (14) the partial derivatives of \( g \).

Their evaluation at the point \((1, \rho, \ldots, \rho^p)\) gives directly
\[
\frac{\partial g}{\partial u_0} = \frac{\rho^2}{(\rho-1)^2} - \sum_{i=2}^{p-1} \rho^2 \left( \frac{\rho^{i-1} - 1}{\rho - 1} \right) F_i(\rho),
\]
\[
\frac{\partial g}{\partial u_1} = -\frac{2\rho}{(\rho-1)^2} + \sum_{i=2}^{p-1} \left( \rho^i + 2\rho \left( \frac{\rho^{i-1} - 1}{\rho - 1} \right) \right) F_i(\rho),
\]
\[
\frac{\partial g}{\partial u_2} = \frac{1}{(\rho-1)^2} - (1+2\rho) F_2(\rho) - \sum_{i=3}^{p-1} \left( \frac{\rho^{i-1} - 1}{\rho - 1} \right) F_i(\rho),
\]
\[
\frac{\partial g}{\partial u_i} = F_{i-1}(\rho) - \rho F_i(\rho), \quad i = 3, \ldots, p-1,
\]
\[
\frac{\partial g}{\partial u_p} = F_{p-1}(\rho).
\] (16)
We can see clearly that the sign of each partial derivative depends on the values $F_i(\rho)$. Let us take, e.g., $F_{p-1}(\rho) < 0$, when $x \in \text{LIN}^-$, this choice is sufficient to assert the existence of transformations $g$ such that $\varphi_\infty(g, x) > 1$. Thus we obtain the following negative result:

**Property 3.5.** Let $x \in \text{LIN}^-$. There exists quasi-linear transformations $g$ accelerating LIN such that

$$\varphi_\infty(g, x) > 1.$$ 

### 3.2.2. A class of stable extrapolation processes on \text{LIN}^-

We exhibit now a class of accelerative transformations $g$ which are well conditioned on \text{LIN}^- in the sense that

$$\varphi_\infty(g, x) = 1, \quad \forall x \in \text{LIN}^-.$$ 

Setting

$$F_0(\rho) = \frac{\rho}{(1-\rho)^2} \quad \text{and} \quad F_1(\rho) = -\frac{1}{(1-\rho)^2}, \quad \text{for} \quad \rho \neq 0,$$

the system (16) is rewritten as follows:

\begin{align}
\frac{\partial g}{\partial u_0} &= -p \sum_{i=0}^{p-1} \rho^i \left(1 - \frac{\rho^{i+1}}{1-\rho}\right) F_i(\rho), \\
\frac{\partial g}{\partial u_1} &= p \sum_{i=0}^{p-1} 2\rho \left(1 - \frac{\rho^{i+1}}{1-\rho}\right) F_i(\rho) + \sum_{i=0}^{p-1} \rho^i F_i(\rho), \\
\frac{\partial g}{\partial u_2} &= -p \sum_{i=0}^{p-1} \left(1 - \frac{\rho^{i+1}}{1-\rho}\right) F_i(\rho) - \sum_{i=0}^{p-1} \rho^i F_i(\rho) - \rho F_2(\rho), \\
\frac{\partial g}{\partial u_i} &= F_{i-1}(\rho) - \rho F_i(\rho), \quad i = 3, \ldots, p-1, \\
\frac{\partial g}{\partial u_p} &= F_{p-1}(\rho).
\end{align}

We are now particularly interested in looking for the quantities $F_i(\rho)$ such that all the partial derivatives of $g$ verify

$$\frac{\partial g}{\partial u_j}(1, \rho, \ldots, \rho^p) = C_\rho \frac{(-\rho)^{p-j}}{\rho^{(1-\rho)p}}, \quad j = 0, 1, \ldots, p.$$ 

**Remark 3.6.** Using the general properties of the binomial coefficients, it is easy to verify the conditions (i) of Lemma 2.3 and (ii) of Lemma 3.1.

To calculate the required quantities, we start with

$$F_{p-1}(\rho) = \frac{1}{(1-\rho)^p},$$
and obtain successively $F_p, F_{p-3}, \ldots$ by the iterations

$$F_i = b_i + \rho F_{i+1}, \quad i = p-2, \ldots, 2,$$

where

$$b_i = C_p \frac{(-\rho)^{p-i-1}}{(1-\rho)^p}, \quad i = p-2, \ldots, 2.$$

The last values $F_0$ and $F_1$ are the constants given by (17).

**Property 3.7.** If

$$F_i = C_p \frac{(-\rho)^{p-i-1}}{(1-\rho)^p}, \quad i = p-1, \ldots, 2,$$

with

$$F_0 = \frac{\rho(1-\rho)^{p-2}}{(1-\rho)^p} \quad \text{and} \quad F_1 = -\frac{(1-\rho)^{p-2}}{(1-\rho)^p},$$

then the corresponding transformations $g$ are such that

$$\forall \rho \neq 0, \rho \in [-1, 1], \frac{\partial g}{\partial u_j}(1, \rho, \ldots, \rho^p) = C_p \frac{(-\rho)^{p-j}}{(1-\rho)^p}, \quad j = 0, \ldots, p.$$

**Definition 3.8.** Let us designate by $S_p, \ p \geq 2$, the set of the transformations $g$ such that the associated function $F_i$ are given by (20) and (21).

As a consequence of (19), when using the general properties of the binomial coefficients, the condition number will be easily expressed as a $p$th power. Furthermore, according to Property 3.7 when $g \in S_p$, we note that all the partial derivatives given by (19) are positive when the corresponding asymptotic ratio $\rho$ is negative. Thus we obtain the

**Theorem 3.9.** Let $g$ be a quasi-linear transformation accelerating LIN. If $g \in S_p$, then for a linear sequence $x$ of asymptotic ratio $\rho$, we have

$$C_\infty(g, x) = \left(\frac{1+|\rho|}{1-\rho}\right)^p,$$

and consequently

$$C_\infty(g, LIN^-) = 1, \quad C_\infty(g, LIN^+) > 1.$$

**Remark 3.10.** This last inequality is, in fact, a confirmation of Theorem 3.3 when $g \in S_p$. 
3.2.3. Examples

(a) Aitken’s $A^2$-process. This well known transformation [1, 2] which appears in (15), is now rewritten as the function

$$g(u_0, u_1, u_2) = \frac{u_0 u_2 - u_1^2}{u_2 - 2u_1 + u_0},$$

under the assumption that the denominator is nonzero when applied to sequences. The function $g$ is obviously quasi-linear.

This process is optimal for accelerating linear convergence (see [9]). This method has been generalized by Shanks [16] and corresponds, for $k = 1$, to Shanks transformation $e_k(S_n)$. In the next part of this work, further developments will be given about the condition numbers of this transformation and the related $\omega$-algorithm.

Setting $A^{i+1} u_i = A^i u_{i+1} - A^i u_i$, $i, j = 0, 1, 2, \ldots$, we have

$$\frac{\partial g}{\partial u_0} = \left( \frac{\Delta u_1}{A^2 u_0} \right)^2, \quad \frac{\partial g}{\partial u_1} = -2 \frac{\Delta u_0 \Delta u_1}{(A^2 u_0)^2}, \quad \frac{\partial g}{\partial u_2} = \left( \frac{\Delta u_0}{A^2 u_0} \right)^2; $$

(23)

Remark 3.11. Lemma 3.1 obviously allows us to find again the property of accelerability of $LIN$ by Aitken’s $A^2$-process.

Setting $r_i = \Delta x_{i+1}/\Delta x_i$, then the pointwise condition number of the Aitken’s transformation for a sequence $x$ is

$$C_i(A^2, x) = \frac{r_i^2 + 2|r_i| + 1}{(r_i - 1)^2}, \quad i = 0, 1, \ldots;$$

This sequence transformation is well defined since the sequence $x$ is assumed to converge linearly. Thus it is an easy matter to deduce the following

Property 3.12.

$$\vartheta_\infty(A^2, x) = \left( \frac{1 + |\rho|}{1 - \rho} \right)^2,$$

where $\rho = \lim_{i \to \infty} r_i$ is the asymptotic linear ratio of the sequence $x$. Thus we deduce

$$\vartheta_\infty(A^2, LIN^-) = 1, \quad \vartheta_\infty(A^2, LIN^+) > 1.$$

Remark 3.13. It is clear that $A^2 \in S_2$. Furthermore, by taking $x \in LIN^+$ we have $\rho \in \left[0, 1\right]$. So the condition number is the value of the function $\phi(\rho) = ((1+\rho)/(1-\rho))^2$. It is an unbounded strictly monotone function on $[0, 1]$. Then Aitken’s process becomes strongly unstable in proportion as the asymptotic ratio $\rho$ of the sequence to accelerate is close to 1.
(b) The $\Theta_2$-transform. This quasi-linear transform corresponds to the second step of the $\Theta$-procedure found by Brezinski [4]. It is given by

$$g(u_0, u_1, u_2, u_3) = u_0 - \frac{u_1 - u_0}{h(u_0, u_1, u_2, u_3) - 1},$$

where

$$h(u_0, u_1, u_2, u_3) = \left(\frac{u_2 - u_1}{u_1 - u_0}\right)^2 \frac{u_2 - 2u_1 + u_0}{u_3 - 2u_2 + u_1}.$$

**Remark 3.14.** The $\Theta_2$-transform accelerates LIN for the same reasons as Aitken’s $\Lambda^2$-process in Remark 3.11. The proof is obvious.

Setting

$$k(u_0, u_1, u_2, u_3) = \left(\frac{u_2 - u_1}{u_1 - u_0}\right)^2 \frac{1}{u_3 - 2u_2 + u_1},$$

and by abbreviating $h(u_0, u_1, u_2, u_3) = h$ and $k(u_0, u_1, u_2, u_3) = k$, the intermediate partial derivatives of $h$ are

$$\frac{\partial h}{\partial u_0} = k \left(\frac{2u_2 - 3u_1 + u_0}{u_1 - u_0}\right),$$

$$\frac{\partial h}{\partial u_1} = -k \left(\frac{2(u_2 - 2u_1 + u_0)(u_2 - u_0)}{(u_1 - u_0)(u_2 - u_1)} + \frac{2u_3 - 3u_2 + u_0}{u_3 - 2u_2 + u_1}\right),$$

$$\frac{\partial h}{\partial u_2} = k \left(\frac{2(u_2 - 2u_1 + u_0)}{u_2 - u_1} + \frac{u_3 - 3u_1 + 2u_0}{u_3 - 2u_2 + u_1}\right),$$

$$\frac{\partial h}{\partial u_3} = -k \left(\frac{u_2 - 2u_1 + u_0}{u_3 - 2u_2 + u_1}\right).$$

Thus the partial derivatives of the function $g$ associated to the $\Theta_2$-transform are

$$\frac{\partial g}{\partial u_0} = 1 + (h - 1)^{-1} + (u_1 - u_0)(h - 1)^{-2} \frac{\partial h}{\partial u_0},$$

$$\frac{\partial g}{\partial u_1} = -(h - 1)^{-1} + (u_1 - u_0)(h - 1)^{-2} \frac{\partial h}{\partial u_1},$$

$$\frac{\partial g}{\partial u_2} = (u_1 - u_0)(h - 1)^{-2} \frac{\partial h}{\partial u_2},$$

$$\frac{\partial g}{\partial u_3} = (u_1 - u_0)(h - 1)^{-2} \frac{\partial h}{\partial u_3}.$$

The computation of the partial derivatives at the point $(1, \rho, \rho^2, \rho^3)$ gives

$$h = \rho, k = \frac{\rho}{(1 - \rho)^2}, \frac{\partial h}{\partial u_0} = \rho(2\rho - 1) \frac{\partial h}{\partial u_0} = \frac{1 - 2\rho - 2\rho^3}{(1 - \rho)^2}, \frac{\partial h}{\partial u_1} = \frac{3\rho}{(1 - \rho)^2} \frac{\partial h}{\partial u_2} = \frac{1}{(1 - \rho)^2}.$$
and consequently
\[
\begin{align*}
\frac{\partial g}{\partial u_0} &= -\rho^3, & \frac{\partial g}{\partial u_1} &= 3\rho^2, \\
\frac{\partial g}{\partial u_2} &= -3\rho, & \frac{\partial g}{\partial u_3} &= 1
\end{align*}
\]
(24)

In fact, by formulae (24), the \(\Theta_2\)-transform belongs to the set \(S_3\). Thus we can state the following

**Property 3.15.**

\[
\varrho_\infty(\Theta_2, x) = \frac{|\rho^3| + 3\rho^2 + 3|\rho| + 1}{(1 - \rho)^3},
\]

where \(\rho = \lim_{i\to\infty} \rho_i\) is the asymptotic linear ratio of the sequence \(x\). Thus we have

\[
\varrho_\infty(\Theta_2, \text{LIN}^+) > 1, \quad \varrho_\infty(\Theta_2, \text{LIN}^-) = 1.
\]

**Remark 3.16.** By comparing Property 3.12 with Property 3.15, it is easy to notice that, for a given sequence of \(\text{LIN}^+\), the best condition number is that of the \(\Delta^2\)-process. Moreover, its condition number is smaller than the condition number of any transformation \(g \in S_p\) with \(p \geq 3\).

### 4. Condition numbers of transformations accelerating \(\text{LOG}\)

Most of processes accelerating slowly convergent sequences are known for their numerical instability. In this section, we will explain this phenomenon by studying the behaviour of the sequence of the punctual condition numbers of quasi-linear transformations.

Let \(g \in \mathbb{R}^{p+1}\) be a transformation accelerating the sequence \(x \in \text{LOG}\). From Lemma 2.3, for fixed \(i\),

\[
\sum_{k=0}^p \frac{\partial g}{\partial u_k}(X_i) = 1,
\]
(25)

and the other condition of the same lemma implies that

\[
\sum_{k=0}^p e_{i+k} \frac{\partial g}{\partial u_k}(X_i) = g(e_i, e_{i+1}, \ldots, e_{i+p}).
\]
(26)

Setting

\[
\mu_j = 1 - \frac{e_{j+1}}{e_j}, \quad j = 0, 1, 2, \ldots
\]

\[
\mu_j^{(l)} = 1 - \prod_{j=i}^{i+l-1} (1 - \mu_j), \quad l = 1, \ldots, p,
\]
(26) can be rewritten
\[ \frac{\partial g}{\partial u_0}(X_i) + \sum_{k=1}^{p} (1 - \mu_i^{(k)}) \frac{\partial g}{\partial u_k}(X_i) = \varepsilon_i, \]  
(27)
where
\[ \varepsilon_i = \frac{1}{e_i} g(e_i,e_{i+1},\ldots,e_{i+p}). \]

Since \( g \) is quasi-linear then, by definition of \( e_i \),
\[ \varepsilon_i = \frac{g(X_i) - x^*}{x_i - x^*}. \]

Thus \( \varepsilon_i \) tends to 0 since \( g \) accelerates \( x \). Furthermore, \( x \in \text{LOG} \) implies that \( \mu_i^l \) tends to 0 when \( i \) tends to infinity, \( l = 1,\ldots,p \).

Let \( \varepsilon > 0 \) be an arbitrarily small quantity. An integer \( n_0 \) can be found such that, for any \( i \geq n_0 \), we have
\[ |\mu_i^{(l)}| \leq \varepsilon \quad \text{and} \quad * |\varepsilon_i| \leq \varepsilon, \quad l = 1,\ldots,p. \]  
(28)

**Lemma 4.1.** For \( i \geq n_0 \), the condition
\[ \frac{\partial g}{\partial u_k}(X_i) \geq 0, \quad k = 0,1,\ldots,p \]  
(29)
is not compatible with conditions (25) and (27).

**Proof.** Let us suppose that the condition (29) holds. Thus, necessarily we have
\[ 0 \leq \frac{\partial g}{\partial u_k}(X_i) \leq 1, \quad k = 0,1,\ldots,p. \]

In the other hand, we have from (25) and (27)
\[ 1 - \varepsilon_i = \sum_{k=1}^{p} \mu_i^{(k)} \frac{\partial g}{\partial u_k}(X_i). \]  
(30)

Thus, since \( i \geq n_0 \), and using successively (28) and (25),
\[ 1 = \varepsilon_i + \sum_{k=1}^{p} |\mu_i^{(k)}| \frac{\partial g}{\partial u_k}(X_i) \leq |\varepsilon_i| + \sum_{k=1}^{p} |\mu_i^{(k)}| \frac{\partial g}{\partial u_k}(X_i) \leq \varepsilon \left( 1 + \sum_{k=1}^{p} \frac{\partial g}{\partial u_k}(X_i) \right) \leq 2\varepsilon. \]

Taking \( 0 < \varepsilon < 1/2 \), the previous majoration is impossible. Therefore there exists at least one integer \( 0 \leq l \leq p \) such that \( \frac{\partial g}{\partial u_l}(X_i) < 0. \)

As a consequence of Lemma 3.2, we obtain
\[ \mathcal{C}_i(g,x) > 1, \quad i = n_0, n_0 + 1, \ldots. \]
Thus it may not exist a well conditioned transformation able to accelerate a logarithmic sequence. Furthermore, a strengthened version of this result is

**Theorem 4.2.** Let \( g \) be a quasi-linear transformation accelerating a sequence \( x \in \text{LOG} \), and \( \varepsilon_i \) the acceleration ratio at the \( i \)th step. We have for \( i \) sufficiently large,

\[
C_i(g,x) \geq \left( \max_{1 \leq k \leq p} (|\varepsilon_i|, |\mu_i^{(k)}|)^{-1} \right) - 1.
\]

**Proof.** From (30) and for \( i \) large enough, we have

\[
0 \leq 1 - \max_{1 \leq k \leq p} (|\varepsilon_i|, |\mu_i^{(k)}|) \leq 1 - \varepsilon_i \leq \sum_{k=1}^{p} |\mu_i^{(k)}| \left| \frac{\partial g}{\partial u_k} (X_i) \right| \leq \max_{1 \leq k \leq p} (|\varepsilon_i|, |\mu_i^{(k)}|) \sum_{k=1}^{p} \left| \frac{\partial g}{\partial u_k} (X_i) \right|.
\]

It follows that

\[
C_i(g,x) \geq \sum_{k=1}^{p} \left| \frac{\partial g}{\partial u_k} (X_i) \right| \geq \frac{1}{\max_{1 \leq k \leq p} (|\varepsilon_i|, |\mu_i^{(k)}|)} - 1.
\]

Not surprisingly, the sequence of condition numbers is unbounded. So it is impossible to define an asymptotic condition number on \( \text{LOG} \). This proves the well known fact that numerical results on logarithmic acceleration are often affected by growing rounding errors.

**Remark 4.3.** We wish to be clearer about the qualifiers ‘well’ and ‘ill conditioned’. We mean by ‘well conditioned’ that \( C(g,x)=1 \). So, when \( C(g,x) > 1 \) we say that this process is ‘ill conditioned’. Nevertheless, the reader has to know that those qualifiers have to be considered as a language convention and it is up to him to modify that definition according to the accuracy of the problem studied. So, a value \( N \) relatively large might be defined as a lower bound from which the transformation \( g \) would be ill conditioned, i.e., \( N < C(g,x) \). As well as, we can also define an upper bound \( M \) so that \( g \) would be well conditioned if and only if \( 1 \leq C(g,x) \leq M \).

### 5. The preconditioning problem

The preconditioning problem is a subject which has not been extensively studied. Nevertheless, some attempts have been made in order to improve the conditioning of ill-conditioned acceleration processes. In this section, without analyzing deeply the question, we will give a look at some methods.

Since alternating sequences are generally well conditioned, it is a good idea to transform monotonic sequences into alternating ones. In this way, an assortment of methods have been proposed by various authors.

(a) A method by Opfer [13].

From a given monotonic sequence, it is possible to construct a new sequence of opposed monotonicity, depending on a parameter \( k \). The alternating sequence is obtained by intercalation, term by term, of these two sequences.
Unfortunately, we note that the application of the Aitken’s process to this new sequence with the parameter \( k = \sqrt{p} \) leads to the same results than applying it directly to the initial sequence.

(b) A method by Longman [12].

The principle of this method is based, for a given strictly monotonic series \( x \), on the development of the formal series

\[
y(t) = \frac{x(t)}{1 + t} = (x_0 - t + \Delta x_0 t^2 + \cdots + \Delta x_{i-2} t^i + \cdots)(1 - t + t^2 - t^3 + \cdots). \tag{31}
\]

Taking \( t = 1 \) in (31), it is proved that the sequence of the partial sums of \( y(1) \) oscillates around its limit \( \frac{1}{2}x^* \).

Longman notices that this new series is not more amenable to convergence acceleration than the monotonic series from which it was derived. In conclusion the difficulty to obtain an efficient accelerator is intrinsic to the considered problem.

(c) A method by Daniel [8].

The so-called condensation transformation maps series of positives terms into alternating series. The author shows that for a large class of extremely slowly convergent series (whose convergence is logarithmic for many of them), the transformed series are more easily summed than the original ones.

However, Daniel advises the reader to be careful in a practical use of this method since many difficulties arise from numerical analysis and numerical treatment. It is remarked that the most crucial problem with the use of condensation is essentially the automatic selection of an appropriate acceleration technique.

Furthermore, the Daniel’s method requires the analytic knowledge of the general term of the series considered.

(d) The Cordellier \( C_k^{(m)} \)'s transformations [7].

Another approach of the preconditioning problem is to use minimization methods. They are based on the least squares approximation and they generalize the classical Shanks transformations \( e_k(S_n) \) since we have \( C_k^{(2k)} = e_k \). The simplest of these processes, which is \( C_1^{(m)} \), generalizes Aitken’s \( \Delta^2 \)-process. The effect of this process on the set of geometrically converging sequences is twice. First, it accelerates the convergence of this set and, secondly, it reduces the values of the condition numbers since it is proved that \( \lim_{m \to \infty} \varphi_\infty(C_1^{(m)} , x) = 1 \), when \( x \) belongs to that set. Of course, more terms of the initial sequence will be needed when choosing a convenient value of \( m \).

To conclude, it seems that both properties of being a good accelerator and a well conditioned method are in opposition when applied to monotonic sequences.

Acknowledgements

I would like to thank Prof. Bernard Germain-Bonne for his helpful discussions about this work.

References