Diffraction of water waves due to a presence of two headlands

A.H. Essawy\textsuperscript{a,*}, H. El-Arabawy\textsuperscript{b}, Sh.M. El-Sawaby\textsuperscript{b}

\textsuperscript{a}Mathematics Department, Faculty of Science, University of El-Minia, El-Minia, Egypt
\textsuperscript{b}Ain-Shams University, Egypt

Received 20 March 1998; received in revised form 10 July 1998

Abstract

This paper is concerned with a boundary value problem for the Helmholtz equation on a horizontal infinite strip with obstacles. The derivation of Helmholtz equation from shallow water equations is given and the boundary value problem with an arbitrary shape of headland is stated. The boundary conditions are of the general Neumann type, and thus we use the finite difference method in numerical solution. Helmholtz equation is replaced by the five-points formula and for the points close to the boundary, Taylor's expansions are made useful with non-uniform spacing. For solving the resulting system of linear equations, the "Mathematica" package is used. The graphs show the velocity potential contours in the cases, of semielliptic, semicircular and narrow headland. Also, we discuss the problem in the presence of two headlands. © 1999 Elsevier Science B.V. All rights reserved.

Keywords: Water waves; Headlands; Mathematica package

1. Introduction

The use of integral equations to solve exterior problem in linear acoustics, i.e., to solve the partial differential equation $(\nabla^2 + \lambda^2)\Phi = 0$ outside a surface $S$ given that $\Phi$ satisfies boundary conditions on $S$ is very common. A good description is provided by Martin [11] for solving, by integral equations, the two-dimensional Helmholtz equation that arise in water wave problem where there is a constant depth variation. Linton and Evans [10] discussed a class of problems concerning obstacles in waveguides. Hwang and Tuck [6] and Lee [9] examined the problems of wave oscillations in arbitrarily shaped harbours using such techniques. Buchwald [1], in his paper shows that for low frequencies, Kelvin waves are propagated round the corner without change of amplitude, but that for high frequencies cylindrical waves of the "Poincare" type are generated at the corner, so that the amplitude of the Kelvin waves propagated round the corner are reduced. Essawy [3, 4] showed the diffraction of Kelvin waves by a regular boundary where he took this boundary as a narrow island.
or a narrow headland. The method of solution is essentially based on the singular integral equations technique. Iskandarani and Philip [7] studied the mass transport induced by three-dimensional water waves for a steady flow, which plays a significant role in the migration of particles and sediments.

Also Dalrymple and Martin [2] discussed an obliquely incident linear wave trains encountering an inlet on a straight reflective shoreline which are examined to determine the response of the inlet to the wave forcing. The problem is divided into a symmetric problem and an anti-symmetric problem, with respect to the channel centerline. Fourier transforms are used to solve the Helmholtz equation in the ocean and an eigen-function expansion is used in the channel (which has constant depth and a rectangular cross-section).

Many water-wave/body interaction problems in which the body is a vertical cylinder with constant cross-section can be simplified by factorizing the depth dependence. If the boundary condition is homogeneous we can write the velocity potential

\[ \Phi(x, y, z, t) = \text{Re}(\Phi(x, y) \cosh(\lambda(z + h_0))e^{-i\omega t}), \]

where the \((x, y)\)-plane corresponds to the undisturbed free surface and \(z\) is measured vertically upwards with \(z = -h_0\), the bottom of the channel. Here \(\lambda\) is the unique real positive solution of the dispersion relation \(\omega^2 = g\lambda \tanh(\lambda h_0)\). In such cases the two-dimensional potential \(\Phi(x, y)\) satisfies the Helmholtz equation \((\nabla^2 + \lambda^2)\Phi = 0\) in this work, we discuss the diffraction of Kelvin waves due to the presence of a regular headland and solve this problem numerically which will be explained later, and then we design a program by “Mathematica” package to solve a system of linear equations performing in our region. Also we draw the values of the potential velocity in three dimensions for several cases as shown in the work.

2. The equation of motion and boundary conditions

A monochromatic wave is incident from the left on a two-dimensional obstacle which is fixed in shallow water of uniform depth \(h_0\). Rectangular Cartesian coordinates \(x\) and \(y\) are defined in Fig. 1, so that the origin is at the level of the undisturbed free surface, the \(x\)-axis is horizontal and the \(y\)-axis vertically upwards. The fluid is assumed to be inviscid and incompressible and so the motion is irrotational. Thus, the flow is described by the equations of motion and the continuity equation

\[ u_t = -g\zeta_x, \]

\[ v_t = -g\zeta_y, \]

\[ \zeta_t + h_0(u_x + v_y) = 0, \]

where \(u(x, y)\) and \(v(x, y)\) are the horizontal velocity components in the \(x\) and \(y\) directions, respectively. \(\zeta\) is the elevation above the undisturbed position. The suffices indicate partial derivatives in the usual way. Assuming a time factor \(\exp(i\omega t)\), where \(\omega\) has small negative imaginary part, so that

\[ \omega = \sigma - i\epsilon, \]

Where \(\epsilon \ll \sigma\), this is a common and convenient way in which the relation condition can be applied [1]. The steady state can be obtained afterwards by taking \(\epsilon\) to be zero.
The linearized equations of motion and continuity equation of shallow waves in a sheet of water of uniform depth $h_0$, in rectangular Cartesian coordinates are given by

\[ i\sigma u = -g\tilde{z}, \quad (2.3a) \]
\[ i\sigma v = -g\tilde{z}_y, \quad (2.3b) \]
\[ h_0(u_x + v_y) + i\sigma \tilde{z} = 0. \quad (2.4) \]

By using (2.3a) and (2.3b) in Eq. (2.4) we get

\[ (\nabla^2 + \lambda^2)\tilde{z} = 0, \quad (2.5) \]

where $c_0 = \sqrt{gh_0}$, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, and $\lambda^2 = \frac{\sigma^2}{gh_0}$.

If we use the kinematical condition on the free surface:

\[ \zeta(x, y, t) = -\frac{1}{g} \frac{\partial \Phi(x, y, t)}{\partial t} \quad (2.6) \]

and since

\[ \Phi(x, y, t) = \phi(x, y) \exp(i\sigma t). \quad (2.7) \]

Then the potential velocity satisfies the Helmholtz equation:

\[ (\nabla^2 + \lambda^2)\phi(x, y) = 0 \quad (2.8) \]

which is satisfied in the region $S$ and $\phi$ independent of time.

Now, we shall consider the general Neumann problem symmetric about $x = 0$. Thus, we want to find $\phi(x, y)$ defined in $S = \{0 < y < d, -\infty < x < \infty\}$, excluding the body and $\partial S$ is an irregular obstacle, see Fig. 1, such that

\[ (\nabla^2 + \lambda^2)\phi(x, y) = 0 \quad \text{in } S, \quad (2.9) \]
\[ \frac{\partial \phi(x, y)}{\partial y} = 0 \quad \text{on } y = 0, |x| > a, \quad (2.10) \]
\[ \frac{\partial \phi(x, y)}{\partial y} = 0 \quad \text{on } y = d, \forall x, \quad (2.11) \]
\begin{equation}
\frac{\partial \phi(x,y)}{\partial n} = 0 \quad \text{on } \partial S. \tag{2.12}
\end{equation}

Here, \( \frac{\partial}{\partial n} \) represents normal differentiation in the direction from \( S \) towards \( \partial S \). For solving Eq. (2.9) with boundary conditions (2.10)–(2.12) we use the method of separation of variables, and we get

\begin{equation}
\phi(x, y) \approx \sum_{n=0}^{J} A_n^\pm \cos \left( \frac{n\pi y}{d} \right) \exp(\pm i\lambda x \sqrt{1 - (n\pi/\lambda d)^2}) \tag{2.13}
\end{equation}

as \( x \to \pm \infty \) [10]. It is difficult to solve the problem with an irregular obstacle and therefore in the following steps we discuss the problem in the presence of a regular obstacle in waveguide. This obstacles may be a headland, a narrow headland, or two headlands.

3. Diffraction of Kelvin water waves due to a regular headland

We shall consider the general Neumann boundary value problem in the presence of a semielliptic headland with different dimensions (one of its axes along the \( x \)-axis and the other along the \( y \)-axis). Thus, we want to find \( \phi(x, y) \) defined in \( S = \{0 < y < d, \ -\infty < x < \infty \}, \) excluding the semielliptic headland, see Fig. 2 i.e., \( \partial S \) is \( x^2/\alpha^2 + y^2/\beta^2 = 1 \).

Using the same conditions (2.10)–(2.12), where \( n \) is the normal vector determined as

\begin{equation}
G(x, y) = \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} - 1 = 0, \tag{3.1}
\end{equation}

\begin{equation}
n = \frac{\nabla G(x, y)}{|\nabla G(x, y)|}, \tag{3.2}
\end{equation}

\begin{equation}
\frac{\partial \phi}{\partial n}(x, y) = n \cdot \nabla \phi(x, y) = 0, \tag{3.3}
\end{equation}

\begin{equation}
\beta^2 x \frac{\partial \phi(x, y)}{\partial x} + x^2 y \frac{\partial \phi(x, y)}{\partial y} = 0, \tag{3.4}
\end{equation}

Eq. (3.4) is satisfied on the curved boundary and has the general solution

\begin{equation}
F(\phi, x^2/\alpha^2), \tag{3.5}
\end{equation}

where \( F \) is an arbitrary function [13]. Also, we can write the complete solution of Eq. (3.4) (the approximate values of \( \phi(x, y) \) on the surface) in the form

\begin{equation}
\phi(x, y) = B(x^2/\alpha^2)^C, \tag{3.5}
\end{equation}

where \( B \) and \( C \) are constants. We can calculate \( B \) and \( C \), if we choose two points on boundary (3.1) and evaluate the approximate value of \( \phi \) from Eq. (2.13), then substitute in (3.5), we obtain two equations in the unknowns \( B \) and \( C \). By solving them, we get their values, thus we can evaluate the other approximate values of \( \phi \) on the boundary. This method is adopted by the authors in previous paper [5].
We can derive different shapes from the case of semielliptic headland, one of them by putting $x < \beta$, $x > \beta$ and $x = \beta$, thus in the latter case we obtained a semicircular headland, and condition (3.4) takes the form

$$x \frac{\partial \phi(x, y)}{\partial x} + y \frac{\partial \phi(x, y)}{\partial y} = 0$$

(3.6)

and solution (3.5) becomes

$$\phi(x, y) = B \left( \frac{x}{y} \right)^A,$$

(3.7)

where $B$ and $A$ are arbitrary constants which can be determined as before. Also if we put $x \ll \beta$, we obtain narrow headland as shown in Fig. 3.
4. Numerical treatment

In this section, we discuss a particular form of a boundary value problem that arises to find the potential velocity \( \varphi(x, y) \) in a region \( S \) containing a headland or number of headlands. For our study of finite-difference techniques of elliptic equations, we restrict ourselves to the boundary value problem:

\[
\varphi_{xx} + \varphi_{yy} + \lambda^2 \varphi = 0 \quad -\infty < x < +\infty, \quad 0 < y \leq d.
\] (4.1)

Subject to conditions (2.10)–(2.12) and \( \varphi(x, y) \) is specified from Eq. (3.5) on the curved boundary by \( \varphi_k(x, y), \ k = 1, 2 \), see Fig. 4.

Furthermore, since the derivative of \( \varphi(x, y) \) is prescribed on the boundaries \( y = 0 \) and \( y = d \), we must use a general Neumann boundary value problem and take \( \varphi(x, y) = 0 \) at the point of intersection between the curved boundary and the straight coastal (at the corner) [1]. In fact, the grid points out the ellipse are subdivided into two main categories and because of symmetry about \( y \)-axis, we discuss the left quadrant and similar technique to the right. When the boundary of the ellipse and the grid points are not coincident, we must proceed differently at points, near this boundary.

Thus for these points, we use Taylor series expansions [8, 12], and Eq. (4.1) take the general form

\[
\varphi_{i,j} = r\{e[h_1\varphi_{i-1,j} + h\varphi_1(x, y)] + s[h\varphi_2(x, y) + h_2\varphi_{i,j+1}]\},
\] (4.2)

where \( \varphi_1(x, y), \varphi_2(x, y) \) are the values of \( \varphi \) on the curved boundary, and

\[
e = 1/(h + h_1)hh_2, \quad s = 1/(h + h_2)hh_2,
\]

\[
p = h + h_1, \quad q = h + h_2,
\]

\[
0 \leq h_1 \leq h, \quad 0 \leq h_2 \leq h,
\]

\[
r = 1/(pe + sq - \lambda^2/2),
\]

and

\[
x_i = a_0 + (i - 1)h, \quad 1 \leq i \leq N; \quad y_j = (j - 1)h, \quad 1 \leq j \leq M;
\]

\[
a_0 = -10, \quad h = 0.1 \text{unit}.
\]

(1) For the interior points of the domain where \( \Delta x = \Delta y = h = 0.1 \), we use the equation:

\[
\varphi_{i,j} = \frac{1}{4 - h^2 \lambda^2}[(\varphi_{i-1,j} + \varphi_{i+1,j} + \varphi_{i,j-1} + \varphi_{i,j+1})].
\] (4.3)

(2) For the lower boundary points, we have the condition \( \partial \varphi/\partial y = 0 \), i.e. \( \varphi_{i,-1} = \varphi_{i,1} \)

\[
\varphi_{i,0} = \frac{1}{4 - h^2 \lambda^2}[\varphi_{i-1,0} + \varphi_{i+1,0} + 2\varphi_{i,1}].
\] (4.4)
(3) For the upper boundary points, we also use the boundary condition \( \frac{\partial \varphi}{\partial y} = 0 \), which performs

\[
\varphi_{i,M-1} = \varphi_{i,M+1},
\]

\[
\varphi_{i,M} = \frac{1}{4 - h^2 \lambda^2} [\varphi_{i-1,M} + \varphi_{i+1,M} + 2\varphi_{i,M-1}].
\]

(4.5)

In computer work, the region must be bounded and therefore we take one solution of approximate solution (2.12) on the left and right boundaries. At the lower and the upper boundaries, we use Eqs. (4.4) and (4.5), while on the headland, we use (3.5), where \( \lambda = 0.001 \), \( d = 1.2 \) unit.

5. Conclusion

The results presented in this paper show that the problem of diffraction of Kelvin waves is due to the presence of a semielliptic, semicircular, and narrow headland, by using the boundary conditions at the line of contact between the fluid and the headland. The form of this condition employed here contains some of the features of the known behavior of fluid near the headland, it includes the important special cases at these points illustrated in the numerical solution.

Firstly, for the problem of diffraction of water waves by a semielliptic headland, when we fixed \( \alpha < \beta \) and took \( \beta < d/2 \), \( \beta = d/2 \), and \( \beta > d/2 \), we noticed that the peak near the headland increased when \( \beta \) increased, see Figs. 5(a) and (b).

Also if we fix \( \alpha > \beta \), \( \beta < d/2 \), \( \beta = d/2 \), \( \beta > d/2 \), we will get an increase in the peak as the increase in \( \beta \), see Figs. 6(a) and (b).

Secondly, for the semicircular headland, a different behavior occurs. The peak near the headland increases when the radius of it decreases, see Figs. 7(a) and (b).

But if we compare the results of the previous three cases, we notice that the values in the cases \( \alpha > \beta \), and \( \alpha = \beta \), are small compared with these in the case \( \alpha < \beta \), which lead us to expect that the values of \( \varphi \) near a narrow headland (\( \alpha \ll \beta \)) are very large, which is made clear as shown in Figs. 8(a) and (b).
Fig. 5. (a) 3D plot of semielliptic headland \((x = 0.3 \text{ unit}, \beta = 0.4 \text{ unit}, \beta < d/2)\). (b) 3D plot of semielliptic headland \((x = 0.3 \text{ unit}, \beta = 0.7 \text{ unit}, \beta > d/2)\).

Fig. 6. (a) 3D plot of semielliptic headland \((x = 0.9 \text{ unit}, \beta = 0.3 \text{ unit}, \beta < d/2)\). (b) 3D plot of semielliptic headland \((x = 0.9 \text{ unit}, \beta = 0.7 \text{ unit}, \beta > d/2)\).

6. Generalization

In the previous section we discussed the diffraction of Kelvin waves due to the presence of one headland by a numerical method. In this section we take into account two headlands, which are symmetric about \(y\)-axis. In fact, the number of headlands can be generalized with different dimensions. Thus, we want to find \(\varphi(x, y)\) defined in \(S = \{0 < y < d, -\infty < x < \infty, \text{ excluding two semielliptic headlands}\}\), see Fig. 9, where

\[
\partial S_1 \text{ is } \frac{(x-x_c)^2}{x_1^2} + \frac{y^2}{\beta_1^2} = 1, \tag{6.1}
\]
Fig. 7. (a) 3D plot of semicircular headland ($x = 0.3 \text{ unit}, \beta = 0.3 \text{ unit}, \beta < d/2$). (b) 3D plot of semicircular headland ($x = 0.9 \text{ unit}, \beta = 0.9 \text{ unit}, \beta > d/2$).

Fig. 8. (a) 3D plot of narrow headland ($x = 0.1 \text{ unit}, \beta = 0.4 \text{ unit}, \beta < d/2$). (b) 3D plot of narrow headland ($x = 0.1 \text{ unit}, \beta = 0.7 \text{ unit}, \beta > d/2$).

Fig. 9. Region sketch of semielliptic two-head lands.
Fig. 10. (a) 3D plot of two headland \((x_1 = 0.3, \beta_1 = 0.5, x_2 = 0.3, \beta_2 = 0.5)\) unit. (b) 3D plot of two headland \((x_1 = 0.4, \beta_1 = 0.4, x_2 = 0.4, \beta_2 = 0.4)\) unit.

\[ \partial S_2 \text{ is } \frac{(x - x_{c_2})^2}{\beta_2^2} + \frac{y^2}{\beta_2^2} = 1, \]  

where \((x_{c_1}, 0)\) and \((x_{c_2}, 0)\) are the centers of the two headlands.

By the same technique as mentioned in Section 3, the boundary conditions \(\partial \varphi(x, y)/\partial n\) on the two headlands take the form:

\[ \beta_1^2(x - x_{c_1}) \frac{\partial \varphi(x, y)}{\partial y} + x_{c_1}^2 y \frac{\partial \varphi(x, y)}{\partial y} = 0 \text{ on } \partial S_1, \]

\[ \beta_2^2(x - x_{c_2}) \frac{\partial \varphi(x, y)}{\partial y} + x_{c_2}^2 y \frac{\partial \varphi(x, y)}{\partial y} = 0 \text{ on } \partial S_2. \]

In fact, we can take the two headlands as semielliptic shapes with different values of the two major axes \(x_1\) and \(x_2\), also different values of the minor axes \(\beta_1\) and \(\beta_2\). Also, we can obtain circular shapes of the two headlands if we put \(x_1 = \beta_1\) and \(x_2 = \beta_2\). If we put \(x_1 \ll \beta_1\) and \(x_2 \ll \beta_2\) we get narrow headlands.

7. Conclusion

We notice that the water waves diffract near the two headlands (which are symmetric about the \(y\)-axis) and tend to the normal state at the points which are not near the obstacles. When, we take the area of the two headlands (semielliptic or semicircular) as equal, the peak appears near the two headlands, see Figs. 10(a) and (b).

But, when the area of the first headland becomes smaller than the second, the peak appears large before the first and very small at the second. If we inverse the position of the two areas, we get the same result, see Figs. 11(a) and (b).

We beg to mention that the above observation applies to the semielliptic and semicircular headlands. But for the case, \(x_1 \ll \beta_1\) and \(x_2 \ll \beta_2\) (narrow headlands), we notice that the values of the potential velocity become very large compared with those mentioned before. The peak appears around
the two headlands when the length of them are equalled. But, when the first is longer than the second the peak appears before the first and not on the second, see Figs. 12(a) and (b).

Also, we have discussed the distance between the two headlands, which are symmetric about the $y$-axis. When we increase this distance the peak also increases and the first headland decreases the diffraction about the other, see Figs. 13(a) and (b).

When the two headlands are not symmetric about the $y$-axis, we observe that peak depends on the areas of the two headlands and not on the position of the $y$-axis, see Fig. 14.

Acknowledgements

The authors wish to express their indebtedness to Professor S. Sadek, and Dr. G.F. Ismail, for their many valuable discussions concerning the above problems.
Fig. 13. (a) 3D plot of two headland with small distance between them ($x_1 = 0.2$, $\beta_1 = 0.3$, $x_2 = 0.7$, $\beta_2 = 0.3$ unit). (b) 3D plot of two headland with large distance between them ($x_1 = 0.2$, $\beta_1 = 0.3$, $x_2 = 0.7$, $\beta_2 = 0.3$ unit).

Fig. 14. 3D plot of nonsymmetric two headlands about $y$-axis.

References