Laurent–Hermite–Gauss Quadrature

Brian A. Hagler
Department of Mathematics, University of Colorado, Boulder, CO 80309-0395, USA
Received 6 October 1998

Abstract
This paper extends the results presented in Gustafson and Hagler (in press) by explicating the $(2n)$-point Laurent–Hermite–Gauss quadrature formula of parameters $\gamma, \lambda > 0$:

$$
\int_{-\infty}^{\infty} f(x)e^{-[1/(\lambda x - \gamma)]^2} \, dx = \sum_{j=\pm 1}^{n} \sum_{k=1}^{n} f(h_{n,k,j}^{(\gamma,\lambda)}) H_{n,k,j}^{(\gamma,\lambda)} + E_{2n}^{(\gamma,\lambda)}[f(x)],
$$

where the abscissas $h_{n,k,j}^{(\gamma,\lambda)}$ and weights $H_{n,k,j}^{(\gamma,\lambda)}$ are given in terms of the abscissas and weights associated with the classical Hermite–Gauss Quadrature, as prescribed in Gustafson and Hagler (J. Comput. Appl. Math. 105 (1999) to appear). By standard numerical methods, it is shown in the present work that, for fixed $\gamma, \lambda > 0$,

$$
E_{2n}^{(\gamma,\lambda)}[f(x)] = \frac{g^{(4n)}(x) n!}{(4n)!} \frac{2^n \pi^{2n+1}}{2n},
$$

for some $\nu$ in $(-\infty, \infty)$, provided $g(x) := x^{2n} f(x)$ has a continuous $(4n)$-th derivative. The resolution as $\gamma \to 0^+$, with $\lambda = 1$, of the transformed quadratures introduced in Gustafson and Hagler (in press) to the corresponding classical quadratures is presented here for the first time, with the $(2n)$-point Laurent–Hermite–Gauss quadrature providing an example, displayed graphically in a figure. Error comparisons displayed in another figure indicate the advantage in speed of convergence, as the number of nodes tends to infinity, of the Laurent–Hermite–Gauss quadrature over the corresponding classical quadrature for certain integrands. © 1999 Elsevier Science B.V. All rights reserved.

MSC: primary 65D32; secondary 41A30

Keywords: Quadrature; Strong distribution; Laurent polynomial

1. Introduction

Gauss quadrature is an important tool for approximating definite integrals, and scholarly works related to this technique continue to abound in the literature. In particular [1–3,6–10] were consulted.

E-mail address: bah@euclid.colorado.edu (B.A. Hagler).

Supported in part by NSF Grant DMS-9701028.

0377-0427/99 $-$ see front matter © 1999 Elsevier Science B.V. All rights reserved.

PII: S0377-0427(99)00054-0
in the preparation of this article. The developing theories of orthogonal functions have yielded new
such quadrature formulas, and the theory of orthogonal Laurent polynomials is no exception, as we
shall now indicate.

Assume \( w(x) \) is a non-negative weight function on an interval \( (a, b) \), \( -\infty \leq a < b \leq \infty \), giving an
inner-product,
\[
(P, Q) := \int_a^b P(x)Q(x)w(x) \, dx,
\]
for the space of real polynomials. We let \( \{ P_n(x) \}_{n=0}^{\infty} \) denote the orthogonal basis of monic polynomials,
ordered by polynomial degree, orthogonal with respect to the above inner-product.
\[
P_n(x) = \prod_{k=1,2,\ldots,n} (x - x_{n,k}),
\]
for distinct real roots \( x_{n,1}, x_{n,2}, \ldots, x_{n,n} \). For functions \( f(x) \) where the integral exists, Gauss quadrature,
as in [7] for example, then gives unique positive real numbers \( w_{n,1}, w_{n,2}, \ldots, w_{n,n} \) such that
\[
\int_a^b f(x)w(x) \, dx = \sum_{k=1,2,\ldots,n} f(x_{n,k})w_{n,k} + E_n[f(x)],
\]
where \( E_n(x^m) = 0 \), for \( m = 0, 1, \ldots, 2n - 1 \), or, more generally,
\[
E_n[f(x)] = \frac{f^{(2n)}(v)}{(2n)!} (P_n, P_n),
\]
for \( f(x) \) having a continuous \((2n)\)-th derivative and some \( v \in (a, b) \).

Hagler, in [4], and Hagler et al. in [5], presented a two parameter family of transformations
taking systems of orthogonal polynomials to systems of orthogonal Laurent polynomials. Their work
is based on a Laurent polynomial called the doubling transformation of parameters \( \gamma, \lambda > 0 \), which
we denote here by
\[
v^{(\gamma, \lambda)}(x) := \frac{1}{\lambda} \left( x - \frac{\gamma}{x} \right).
\]
Its inverses are
\[
v_j^{(\gamma, \lambda)}(y) := \frac{\lambda}{2} \left( y + j \sqrt{y^2 + \frac{4\gamma}{\lambda^2}} \right),
\]
for \( j = \pm 1 \). In our current context with \( w(x) \), a non-negative weight function, results in [4,5] include
an inner-product,
\[
\langle R, S \rangle := \sum_{j=\pm 1} \int_{v_j^{(-, \lambda)}}^{v_j^{(+, \lambda)}} R(x)S(x)w^{(\gamma, \lambda)}(x) \, dx,
\]
for the space of real Laurent polynomials, where
\[
w^{(\gamma, \lambda)}(x) := w(v^{(\gamma, \lambda)}(x)).
\]
With the terms \( L\)-degree and monic implied by the ordered basis \( \{ 1, x^{-1}, x, x^{-2}, x^2, \ldots \} \) for the
space of Laurent polynomials, we let \( \{ P_{n}^{(\gamma, \lambda)}(x) \}_{n=0}^{\infty} \) denote the orthogonal basis of monic Laurent
polynomials, ordered by L-degree, orthogonal with respect to Eq. (1.7). [4,5] give

\[ P_{2n}^{(\gamma)}(x) = \hat{\lambda}^n P_n^{(\gamma)}(v^{(\gamma)}(x)), \]  

(1.9)

\[ P_{2n+1}^{(\gamma)}(x) = \left( -\frac{\hat{\lambda}}{\gamma} \right)^n \frac{1}{x} P_n^{(\gamma)}(v^{(\gamma)}(x)). \]  

(1.10)

It is also shown in [4,5] that

\[ P_{2n}^{(\gamma)}(x) = x^{-n} \prod_{j=\pm 1} \prod_{k=1,2,...,n} (x - x_{n,k,j}^{(\gamma)}), \]  

(1.11)

where the \( 2n \) distinct real roots are

\[ x_{n,k,j}^{(\gamma)} := v_j^{(\gamma)}(x_{n,k}), \]  

(1.12)

for \( k = 1,2,\ldots,n \) and \( j = \pm 1 \). Another result reported in [4,5] that will be used below is

\[ (P_{2n}^{(\gamma)}, P_{2n+1}^{(\gamma)}) = \hat{\lambda}^{2n+1}(P_n, P_n). \]  

(1.13)

Gustafson and Hagler in [2] show that, when the integrals exist, there are unique positive numbers \( w_{n,k,j}^{(\gamma)} \), \( k = 1,2,\ldots,n \) and \( j = \pm 1 \), such that

\[ \sum_{j=\pm 1} \int_{v_j^{(\gamma)}(a)}^{v_j^{(\gamma)}(b)} f(x) w_{n,k,j}^{(\gamma)}(x) \, dx = \sum_{j=\pm 1} \sum_{k=1,2,...,n} f(x_{n,k,j}^{(\gamma)}) w_{n,k,j}^{(\gamma)} + E_{2n}^{(\gamma)}[f(x)], \]  

(1.14)

where \( E_{2n}^{(\gamma)}(x^m) = 0 \), for \( m = -2n, -2n+1, \ldots, 2n-1 \), or, more generally,

\[ E_{2n}^{(\gamma)}(f) = \sum_{j=\pm 1} \frac{g^{(4n)}(v_j)}{(4n)!} \int_{v_j^{(\gamma)}(a)}^{v_j^{(\gamma)}(b)} [P_{2n}^{(\gamma)}(x)]^2 w_{n,k,j}^{(\gamma)}(x) \, dx, \]  

(1.15)

for \( g(x) := x^{2n} f(x) \) having a continuous \((4n)\)-th derivative and some numbers \( v_{\pm 1} \) in the interval \( (v_{-1}^{(\gamma)}(a), v_1^{(\gamma)}(b)) \). They also show that the weights in the above quadrature formula satisfy

\[ w_{n,k,j}^{(\gamma)} = \frac{w_{n,k}}{v'(x_{n,k,j}^{(\gamma)})}, \]  

(1.16)

where we have used the abbreviation \( v' \) for the derivative \( dx^{(\gamma)}(x)/dx \) of \( v^{(\gamma)}(x) \).

Cursory numerical examples of the transformed quadrature (1.14) are included in [2]. In the present paper, we concentrate on quadrature in the Hermite case, where the strong weight function (1.8) is \( e^{-x^2/2(x^{(\gamma)})^2} \) on \( (-\infty, \infty) \). A Table of numerical values for the abscissas and weights, an expression for the remainders \( E_{2n}^{(\gamma)}[f(x)] \) and specific case studies with comparisons to results yielded by various other quadrature rules are included. However, before narrowing our focus, we first present results which describe how the classical quadrature (1.3) is a limiting case of the transformed quadrature (1.14). An example of this convergence will be provided in Section 4.
2. Resolution to classical quadrature

**Theorem 2.1 (Resolution).** Let \( x_{n,k} \) and \( w_{n,k} \) denote, respectively, an abscissas and a corresponding weight in Gauss quadrature, (1.3). Let \( x^{(j,1)}_{n,k,j} \) and \( w^{(j,1)}_{n,k,j} \) be given by Eqs. (1.12) and (1.16), respectively. Taking \( \lambda = 1 \), the following hold as \( \gamma \to 0^+ \):

1. If \( x_{n,k} = 0 \), then \( x^{(1)}_{n,k,j} = j \sqrt{\gamma} \to 0 = x_{n,k} \) and \( w^{(1)}_{n,k,j} = \frac{1}{2} w_{n,k} \), for \( j = \pm 1 \).
2. If \( x_{n,k} < 0 \), then \( x^{(1)}_{n,k,j} \to x_{n,k} \) and \( w^{(1)}_{n,k,j} \to w_{n,k} \), but \( x^{(1)}_{n,k,1} \to 0 \) and \( w^{(1)}_{n,k,1} \to 0 \).
3. If \( x_{n,k} > 0 \), then \( x^{(1)}_{n,k,j} \to x_{n,k} \) and \( w^{(1)}_{n,k,j} \to w_{n,k} \), but \( x^{(1)}_{n,k,1} \to 0 \) and \( w^{(1)}_{n,k,1} \to 0 \).

**Proof.** \( x^{(1)}_{n,k,j} := \frac{1}{2}(x_{n,k} + j \sqrt{x_{n,k}^2 + 4 \gamma}) \to \frac{1}{2}(x_{n,k} + j|x_{n,k}|) \) as \( \gamma \to 0^+ \). Hence, the statements concerning \( x^{(1)}_{n,k,j} \) are seen to hold. If \( x_{n,k} = 0 \), \( w^{(1)}_{n,k,j} = \frac{1}{2} w_{n,k} \) by consideration of the definitions. If \( x_{n,k} < 0 \) and \( j = -1 \), or if \( x_{n,k} > 0 \) and \( j = 1 \), we have that \( x^{(1)}_{n,k,j} \to x_{n,k} \neq 0 \) as \( \gamma \to 0^+ \), and it follows that \( w^{(1)}_{n,k,j} := w_{n,k}/(1 + \gamma/[x^{(1)}_{n,k,j}]) \to w_{n,k} \) in these cases. If \( x_{n,k} < 0 \) and \( j = 1 \), or if \( x_{n,k} > 0 \) and \( j = -1 \), then \( x^{(1)}_{n,k,j} \to 0 \) as \( \gamma \to 0^+ \), and it follows by an application of L’Hospital’s rule that

\[
\lim_{\gamma \to 0^+} \frac{[x^{(1)}_{n,k,j}]^2}{[x_{n,k,j}^2 + \gamma]} = \lim_{\gamma \to 0^+} \frac{2x^{(1)}_{n,k,j} d/dx^{(1)}_{n,k,j}}{2x^{(1)}_{n,k,j} d/dx^{(1)}_{n,k,j} + 1} = 0. \tag{2.1}
\]

Hence, in these cases, \( w^{(1)}_{n,k,j} := w_{n,k}/(1 + \gamma/[x^{(1)}_{n,k,j}]) \to 0 \). \( \Box \)

**Theorem 2.2.** If \( f(x) \) is continuous at 0 and at \( x_{n,k} \), \( k = 1, \ldots, n \), then

\[
\sum_{j=\pm 1} \sum_{k=1,2,\ldots,n} f(x^{(1)}_{n,k,j})w^{(1)}_{n,k,j} \to \sum_{k=1,2,\ldots,n} f(x_{n,k})w_{n,k}, \quad \text{as} \quad \gamma \to 0^+. \tag{2.2}
\]

**Proof.** Apply the previous theorem. \( \Box \)

3. Laurent–Hermite–Gauss quadrature

Let \( \{H_n(x)\}_{n=0}^{\infty} \) be the sequence of monic Hermite polynomials, orthogonal with respect to

\[
(P, Q)_H := \int_{-\infty}^{\infty} P(x)Q(x)e^{-x^2} \, dx. \tag{3.1}
\]

Using the known formula

\[
(H_n, H_n)_H = \frac{n!}{2^n \pi}, \tag{3.2}
\]

we write the quadrature (1.3) in the compact form

\[
\int_{-\infty}^{\infty} f(x)e^{-x^2} \, dx = \sum_{k=1,2,\ldots,n} f(h_{n,k})H_{n,k} + \frac{f^{(2n)}(y) n!}{(2n)! \frac{1}{2^n \pi}}, \tag{3.3}
\]
for $f(x)$ having a continuous $(2n)$-th derivative, and some $v \in (-\infty, \infty)$. We have thus denoted the zeros of $H_n(x)$ by $h_{n,k}$ and the corresponding quadrature weights by $H_{n,k}$, for $k = 1, 2, \ldots, n$. The formula

$$H_{n,k} = \frac{2^{n-1}n!\sqrt{\pi}}{n^2[H_{n-1}(h_{n,k})]^2}$$

is also well known in the literature.

Since

$$\lim_{x \to 0} x^m e^{-[1/\lambda(x - \gamma/x)]^2} = 0,$$

for $m = 0, \pm 1, \pm 2, \ldots$, we can then write Eq. (1.7), in this case, as

$$\langle R, S \rangle_{\lambda, \gamma} := \int_{-\infty}^{\infty} R(x)S(x)e^{-[1/\lambda(x - \gamma/x)]^2} \, dx,$$

as an inner-product for the space of real Laurent polynomials.

Denote by $\{H_n^{\gamma, \lambda}(x)\}_{n=0}^\infty$ the sequence of monic orthogonal Laurent polynomials with respect to Eq. (3.6). By Eq. (1.12), the zeros of $H_n^{\gamma, \lambda}(x)$ are given by

$$h_{n,k,j}^{\gamma, \lambda} = \frac{\lambda}{2} \left( h_{n,k} + j\sqrt{h_{n,k}^2 + \frac{4\gamma}{\lambda^2}} \right)$$

for $k = 1, 2, \ldots, n$ and $j = \pm 1$. The corresponding weight factors in Eq. (1.14), considering Eq. (1.16), we can write as

$$H_{n,k,j}^{\gamma, \lambda} = \frac{\lambda(h_{n,k,j}^{\gamma, \lambda})^2}{(h_{n,k,j}^{\gamma, \lambda})^2 + \gamma} - H_{n,k},$$

or, by Eq. (3.4),

$$H_{n,k,j}^{\gamma, \lambda} = \frac{\lambda 2^{n-1}n!\sqrt{\pi}(h_{n,k,j}^{\gamma, \lambda})^2}{n^2[(h_{n,k,j}^{\gamma, \lambda})^2 + \gamma][H_{n-1}(h_{n,k})]^2},$$

for $k = 1, 2, \ldots, n$ and $j = \pm 1$.

The following theorem introduces the $(2n)$-point Laurent–Hermite–Gauss quadrature formula of parameters $\gamma, \lambda > 0$.

**Theorem 3.1.** Let $\gamma, \lambda > 0$, and let $n$ be a positive integer. With $h_{n,k,j}^{\gamma, \lambda}$ and $H_{n,k,j}^{\gamma, \lambda}$ given by (3.7) and (3.9), respectively,

$$\int_{-\infty}^{\infty} f(x)e^{-[1/\lambda(x - \gamma/x)]^2} \, dx = \sum_{j=\pm 1} \sum_{k=1}^n f(h_{n,k,j}^{\gamma, \lambda})H_{n,k,j}^{\gamma, \lambda} + E_{2n}^{\gamma, \lambda}[f(x)]$$

with

$$E_{2n}^{\gamma, \lambda}[f(x)] := \frac{g^{(4n)}(v) \pi^{2n+1}}{(4n)!},$$

for some $v$ in $(-\infty, \infty)$, provided $g(x) := x^{2n}f(x)$ has a continuous $(4n)$-th derivative.
Table 1
Abscissas and weights for Laurent–Hermite–Gauss quadrature of parameters \( \gamma, \lambda > 0 \)

\[
\int_{-\infty}^{\infty} f(x) \, e^{-\gamma (x-x_c)^2} \, dx \approx \sum_{j=\pm 1} \sum_{k=1}^{n} f(h_{n,k,j}) H_{n,k,j}
\]

Abscissas : \( h_{n,k,j}^{(\lambda)} = \frac{1}{2} \lambda [h_{n,k} + j \sqrt{h_{n,k}^2 + (4 \gamma \lambda^2)}] \)

Weights : \( H_{n,k,j}^{(\lambda)} = \lambda H_{n,k}(h_{n,k,j}^{(\lambda)})^2 / [(h_{n,k,j}^{(\lambda)})^2 + \gamma] \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \pm h_{n,k} )</th>
<th>( H_{n,k} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.70710 67811 86548</td>
<td>8.86226 92545 28 (−1)</td>
</tr>
<tr>
<td>3</td>
<td>0.00000 00000 00000 1.22474 48713 91589</td>
<td>1.18163 58006 04 (0)</td>
</tr>
<tr>
<td>4</td>
<td>0.52464 76232 75290 1.65068 01238 85785</td>
<td>8.04919 09000 55 (−1)</td>
</tr>
<tr>
<td>5</td>
<td>0.95857 24646 13819 2.02018 28704 56086</td>
<td>3.93619 32315 22 (−1)</td>
</tr>
<tr>
<td>6</td>
<td>1.33584 90760 13697 2.35060 49736 74492</td>
<td>3.93619 32315 22 (−1)</td>
</tr>
</tbody>
</table>

Proof. The result follows by standard methods, using Hermite’s interpolation formula for \( g(x) := x^2 f(x) \), considering (1.13) and (3.2).

Table 1 shows how the formulas (3.7) and (3.8) incorporate numerical values for \( h_{n,k} \) and \( H_{n,k} \) in (3.3) to give \( h_{n,k,j}^{(\lambda)} \) and \( H_{n,k,j}^{(\lambda)} \) in Eq. (3.10). The abscissas \( h_{n,k} \) and weight factors \( H_{n,k} \), for \( n = 2, 3, 4, 5, 6 \), were taken from the values tabulated in [10].

4. Examples and comparisons

For convenience, we will henceforth consider formulas written in the form

\[
\int_{-\infty}^{\infty} f(x) \, dx = \sum_{i=1}^{n} \frac{f(x_i)}{w(x_i)} w_i + e_n(f).
\]

Fig. 1 shows several gross features of the approximation

\[
\int_{-\infty}^{\infty} f(x) \, dx \approx Q_7 = \text{Truncated} \left[ \sum_{j=\pm 1} \sum_{k=1}^{8} \frac{f(h_{8,k,j}^{(-1)})}{[h_{8,k,j}^{(1)}] \cdot \sqrt{2} \lambda} e \left[ \frac{h_{8,k,j}^{(-1)}}{h_{8,k,j}^{(1)}} \right] H_{8,k,j}^{(-1)} \right],
\]

(4.1)
for \( f(x) = x^{-10}e^{-(x-1)^2} \), as \( \gamma \) is varied. Scaling is uniform for the six plots, and \( h_{8,k,j}^{(\gamma,1)} \) and \( H_{8,k,j}^{(\gamma,1)} \) used in the computation of \( Q_\gamma \), for each of the values of \( \gamma \), were derived using Eqs. (3.7) and (3.8) and numerical values for \( h_{8,k} \) and \( H_{8,k} \) given in [10]. The 16-point Laurent–Hermite–Gauss quadrature for this integral, represented by the sum on the right-hand side of Eq. (4.1), is theoretically exact for \( \gamma = 1 \), and the value 8.8604 of \( Q_\gamma \) given in the figure is correct in each of its digits. \( \gamma = 0 \) gives the classical Hermite–Gauss quadrature.

The Resolution Theorem shows the convergence, as \( \gamma \) tends to 0, with \( \lambda = 1 \), of the quadrature associated with the Laurent polynomial (1.5) to the corresponding classical quadrature. The sequence of graphs in Fig. 1 demonstrates this convergence. The plots also give an indication of why we call (1.5) the doubling transformation, especially when comparing the structures of the \( \gamma = 10 \) and \( \gamma = 0 \) graphs.

Since \( Q_\gamma \) varies so greatly, perhaps the first question that Fig. 1 suggests is this: For a given integrand, which values of the parameters will give the best approximation of the integral? Such problems are beyond the goals of this paper.

Fig. 2 shows six large scale graphical comparisons of the magnitude of the error terms \( e_n(f_p) \) and \( e^{(1,1)}_{2n}(f_p) \) in the formulas

\[
\int_{-\infty}^{\infty} f_p(x) \, dx = \sum_{i=1,\ldots,n} \frac{f_p(h_{n,k})}{e^{-(h_{n,k})^2}} H_{n,k} + e_n(f_p),
\]

for \( f_p(x) = x^{-10}e^{-(x-1)^2} \), as \( \gamma \) is varied. Scaling is uniform for the six plots, and \( h_{8,k,j}^{(\gamma,1)} \) and \( H_{8,k,j}^{(\gamma,1)} \) used in the computation of \( Q_\gamma \), for each of the values of \( \gamma \), were derived using Eqs. (3.7) and (3.8) and numerical values for \( h_{8,k} \) and \( H_{8,k} \) given in [10]. The 16-point Laurent–Hermite–Gauss quadrature for this integral, represented by the sum on the right-hand side of Eq. (4.1), is theoretically exact for \( \gamma = 1 \), and the value 8.8604 of \( Q_\gamma \) given in the figure is correct in each of its digits. \( \gamma = 0 \) gives the classical Hermite–Gauss quadrature.

The Resolution Theorem shows the convergence, as \( \gamma \) tends to 0, with \( \lambda = 1 \), of the quadrature associated with the Laurent polynomial (1.5) to the corresponding classical quadrature. The sequence of graphs in Fig. 1 demonstrates this convergence. The plots also give an indication of why we call (1.5) the doubling transformation, especially when comparing the structures of the \( \gamma = 10 \) and \( \gamma = 0 \) graphs.

Since \( Q_\gamma \) varies so greatly, perhaps the first question that Fig. 1 suggests is this: For a given integrand, which values of the parameters will give the best approximation of the integral? Such problems are beyond the goals of this paper.

Fig. 2 shows six large scale graphical comparisons of the magnitude of the error terms \( e_n(f_p) \) and \( e^{(1,1)}_{2n}(f_p) \) in the formulas

\[
\int_{-\infty}^{\infty} f_p(x) \, dx = \sum_{i=1,\ldots,n} \frac{f_p(h_{n,k})}{e^{-(h_{n,k})^2}} H_{n,k} + e_n(f_p),
\]

for \( f_p(x) = x^{-10}e^{-(x-1)^2} \), as \( \gamma \) is varied. Scaling is uniform for the six plots, and \( h_{8,k,j}^{(\gamma,1)} \) and \( H_{8,k,j}^{(\gamma,1)} \) used in the computation of \( Q_\gamma \), for each of the values of \( \gamma \), were derived using Eqs. (3.7) and (3.8) and numerical values for \( h_{8,k} \) and \( H_{8,k} \) given in [10]. The 16-point Laurent–Hermite–Gauss quadrature for this integral, represented by the sum on the right-hand side of Eq. (4.1), is theoretically exact for \( \gamma = 1 \), and the value 8.8604 of \( Q_\gamma \) given in the figure is correct in each of its digits. \( \gamma = 0 \) gives the classical Hermite–Gauss quadrature.

The Resolution Theorem shows the convergence, as \( \gamma \) tends to 0, with \( \lambda = 1 \), of the quadrature associated with the Laurent polynomial (1.5) to the corresponding classical quadrature. The sequence of graphs in Fig. 1 demonstrates this convergence. The plots also give an indication of why we call (1.5) the doubling transformation, especially when comparing the structures of the \( \gamma = 10 \) and \( \gamma = 0 \) graphs.

Since \( Q_\gamma \) varies so greatly, perhaps the first question that Fig. 1 suggests is this: For a given integrand, which values of the parameters will give the best approximation of the integral? Such problems are beyond the goals of this paper.

Fig. 2 shows six large scale graphical comparisons of the magnitude of the error terms \( e_n(f_p) \) and \( e^{(1,1)}_{2n}(f_p) \) in the formulas

\[
\int_{-\infty}^{\infty} f_p(x) \, dx = \sum_{i=1,\ldots,n} \frac{f_p(h_{n,k})}{e^{-(h_{n,k})^2}} H_{n,k} + e_n(f_p),
\]
Fig. 2. Six error comparison graphs with $e_n(f_p)$, $e_{2n}^{(1,1)}(f_p)$ and constants $c_p$ as in Eqs. (4.2)–(4.4).

\[
\int_{-\infty}^{\infty} f_p(x) \, dx = \sum_{j=\pm1} \sum_{k=1,2,...,n} \frac{f_p(h_{n,k,j}^{(1,1)})}{e^{-(h_{n,k,j}^{(1,1)}/h_{n,k,j}^{(1,1)})^2}} H_{n,k,j}^{(1,1)} + e_{2n}^{(1,1)}(f_p),
\]  

(4.3)

for the integrands $f_p(x) = c_p x^{-2p} e^{-x^2-\frac{1}{x^2}}$, corresponding to $p = 3, 4, 5, 6, 7, 8$. The constants $c_p$ are chosen so that

\[
\int_{-\infty}^{\infty} f_p(x) \, dx = 1.
\]

(4.4)

Scaling is uniform for the six plots. Once again, $h_{n,k,j}^{(1,1)}$ and $H_{n,k,j}^{(1,1)}$ were computed using Eqs. (3.7) and (3.8) and numerical values for $h_{n,k}$ and $H_{n,k}$ given in [10]. In each graph, the points $(n, |e_n(f_p)|)$, for $n=2, 3, \ldots, 10$, have been joined by a dashed line, while the points $(n, |e_{2n}^{(1,1)}(f_p)|)$, for $n=2, 3, \ldots, 10$, have been joined by a solid line.

It can be seen from Eq. (3.11), and reflected in the six graphs of Fig. 2, that

\[
e_{2n}^{(1,1)}(f_p) = 0, \quad n \geq p.
\]

(4.5)

In dramatic contrast, it is not at all clear from the numerical evidence represented in Fig. 2 that the error $e_n(f_p)$, associated with the classical Hermite–Gauss quadrature, for any of the six values of $p$, converges to 0 as $n$ tends to infinity.
References