Classes of functions defined by certain differential–integral operators

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Abstract

In this paper we investigate a class of \( p \)-valent analytic functions with fixed argument of coefficient, which is defined in terms of certain differential–integral operators. Coefficient estimates, distortion theorems, extreme points, the radii of convexity and starlikeness in this class are given. © 1999 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Let \( \mathcal{A}(p,k) \), where \( p,k \in \mathbb{N} = \{1,2,\ldots\} \), \( p < k \), denote the class of functions \( f \) of the form

\[
f(z) = z^p + \sum_{n=k}^{\infty} d_n z^n,
\]

which are analytic in \( \mathcal{U} = \mathcal{U}(1) \), where \( \mathcal{U}(r) = \{ z : |z| < r \} \). Let \( \mathcal{A} = \mathcal{A}(1,2) \).

We say that a function \( f \in \mathcal{A}(p,p+1) \) is convex in \( \mathcal{U}(r) \), \( r \in (0,1] \), if

\[
\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 0, \quad z \in \mathcal{U}(r)
\]

and we say that a function \( f \in \mathcal{A}(p,p+1) \) is starlike in \( \mathcal{U}(r) \), \( r \in (0,1] \), if

\[
\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > 0, \quad z \in \mathcal{U}(r).
\]

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By $C_p$ we denote the class of all functions convex in $\mathcal{U}$ and by $S_p^*$ we denote the class of all starlike functions in $\mathcal{U}$.

We say that a function $f \in \mathcal{A}$ is subordinate to a function $F \in \mathcal{A}$, and write $f \prec F$, if and only if there exists a function $\omega \in \mathcal{A}$, $\omega(0) = 0$, $|\omega(z)| < 1$, $z \in \mathcal{U}$, such that $f(z) = F(\omega(z))$, $z \in \mathcal{U}$.

Let $p, k \in \mathbb{N}$, $A, B, \alpha, \beta, \theta \in \mathbb{R}$, $-\beta < p < k$, $\alpha + \beta > -p$, $0 \leq B \leq 1$, $-B \leq A < B$, ($B \neq 1$ or $\cos \theta < 0$) and let $\Gamma$ denote the gamma function.

In [3, 4] Kim and Srivastava et al. studied the following integral operator:

**Definition 1.1.** Let $f \in \mathcal{A}(p, k)$ and let $\log(z - \zeta)$ be real when $z - \zeta > 0$, $z, \zeta \in \mathcal{U}$. The integral operator $Q_p^\alpha f(z)$ is defined for $\alpha > 0$ and the function $f$ by

$$Q_p^\alpha f(z) = \frac{1}{\Gamma(\alpha)} \int_0^z \left(1 - \frac{\zeta}{z}\right)^{\alpha - 1} \zeta^{\beta - 1} f(\zeta) \, d\zeta.$$

Now, we define the differential–integral operator $\Phi_p^\alpha f(z)$ for $\alpha \leq 0$. Then there exists a nonnegative integer $m$ and a real number $\gamma$, $-1 < \gamma \leq 0$, such that $\alpha = \gamma - m$.

**Definition 1.2.** Let $f \in \mathcal{A}(p, k)$ and let $\log(z - \zeta)$ be real when $z - \zeta > 0$, $z, \zeta \in \mathcal{U}$. The integral operator $\Phi_p^\alpha f(z)$ is defined for $\alpha \leq 0$ ($\alpha = \gamma - m$, $m \in \mathbb{N} \cup \{0\}$, $-1 < \gamma \leq 0$) and for the function $f$ by

$$\Phi_p^\alpha f(z) = \frac{1}{\Gamma(1 + \gamma)} \frac{d^{m+1}}{dz^{m+1}} \int_0^z (z - \xi)^{\gamma - 1} f(\xi) \, d\xi.$$

The multiplicities of $(z - \xi)^{\gamma-1}$, $(z - \xi)^{-1}$ in Definitions 1.1 and 1.2 are removed by requiring $\log(z - \zeta) \in \mathbb{R}$ when $z - \zeta > 0$.

By using these definitions we define the linear operator

$$\Omega_p^\alpha : \mathcal{A}(p, k) \rightarrow \mathcal{A}(p, k)$$

by

$$\Omega_p^\alpha f(z) = \frac{\Gamma(p + \alpha + \beta)}{\Gamma(p + \beta)} z^{-\beta} Q_p^\alpha f(z) \quad \text{for} \ z > 0$$

and

$$\Omega_p^\alpha f(z) = \frac{\Gamma(p + \alpha + \beta)}{\Gamma(p + \beta)} z^{1-\beta} \Phi_p^\alpha f(z) \quad \text{for} \ z \leq 0.$$

Putting $\beta = 1$ in Definitions 1.1 and 1.2 we obtain the integral operator defined by Owa [5] and investigated by Srivastava and Owa [7], Dziok [1, 2] and others.

**Definition 1.3.** Let $T(\alpha, \beta)$ denote the class of functions $f \in \mathcal{A}(p, k)$ satisfying the following condition:

$$\frac{\Omega_p^\alpha f(z)}{z^p} < \frac{1 + Az}{1 + Bz} \quad \text{for} \ z > 0.$$  \hfill (1.4)
Definition 1.4. Let \( T_0(\alpha, \beta) \) denote the subclass of the class \( T(\alpha, \beta) \) of functions \( f \) of the form (1.1), such that \( \arg a_n = \theta \) for \( a_n \neq 0, n = k, k + 1, k + 2, \ldots \).

We can write every function \( f \) from the class \( T_0(\alpha, \beta) \) in the form
\[
f(z) = z^{\nu} + e^{i\psi} \sum_{n=k}^{\infty} |a_n| z^n, \quad z \in \mathcal{U}.
\]

(1.5)

In the present paper we obtain coefficient estimates, distortion theorems, extreme points, the radii of convexity and starlikeness for the class \( T_0(\alpha, \beta) \) and coefficient estimates for the class \( T(\alpha, \beta) \).

2. Coefficient estimates

By Definition 1.4 we obtain:

Lemma 2.1. If \( A \leq \check{A} \) and \( \check{B} \leq B \), then the class \( T_0(\alpha, \beta) \) for parameters \( A, B \) is included in the class \( T_0(\alpha, \beta) \) for parameters \( \check{A}, \check{B} \).

Lemma 2.2 (Ratti [6]). Let \( f \) be a function of the form (1.1). If \( f \prec g \) and \( g \) is convex function, then \( |a_n| \leq 1, n = k, k + 1, k + 2, \ldots \).

After some calculations we obtain:

Lemma 2.3. If a function \( f \) of the form (1.1) belongs to the class \( A(p,k) \), then
\[
\sum_{n=k}^{\infty} \frac{\Gamma(n + \alpha + \beta)}{\Gamma(n + \alpha + \beta)} a_n z^n, \quad z \in \mathcal{U}.
\]

Theorem 2.1. If a function \( f \) of the form (1.5) belongs to the class \( T_0(\alpha, \beta) \), then
\[
\sum_{n=k}^{\infty} \frac{\Gamma(n + 1)}{\Gamma(n + 1)} |a_n| \leq \delta(\theta, A, B),
\]

where
\[
\delta(\theta, A, B) = \frac{B - A}{\sqrt{1 - B^2 \sin^2 \theta} - B \cos \theta}
\]

and
\[
\Gamma_n = \frac{\Gamma(n + \alpha + \beta) \Gamma(p + \beta)}{\Gamma(n + \beta) \Gamma(p + \alpha + \beta)}.
\]

Proof. Let \( f \in T_0(\alpha, \beta) \). By Definition 1.4 we obtain
\[
\frac{\Omega_0^\alpha f(z)}{z^{\nu}} = \frac{1 + A\omega(z)}{1 + B\omega(z)}.
\]
where $\omega(z)$ is an analytic function in $\mathcal{U}$, such that $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in \mathcal{U}$. Thus, we have
\[
\left| \frac{\Omega^2_\mu \Omega^2_\nu f(z) - z^p}{B \Omega^2_\mu f(z) - Az^p} \right| = |\omega(z)| < 1.
\]
Using Lemma 2.3 we obtain
\[
\sum_{n=k}^{\infty} \Gamma_n^{-1} |a_n| r^{n-p} < \left| B - A + B e^{i\theta} \sum_{n=k}^{\infty} \Gamma_n^{-1} |a_n| r^{n-p} \right|.
\]
where $\Gamma_n$ is defined by (2.3). Thus, putting $z = r, 0 < r < 1$, we have
\[
|w| < |B - A + B \cdot e^{i\theta}|,
\]
where $w = \sum_{n=k}^{\infty} \Gamma_n^{-1} |a_n| r^{n-p}$.

Since $w \in \mathbb{R}$, by (2.4) we have
\[
(1 - B^2)w^2 - (2B(2A - B)\cos \theta)w - (B - A)^2 < 0.
\]
Solving this inequality with respect to $w$ we obtain
\[
\sum_{n=k}^{\infty} \Gamma_n^{-1} |a_n| r^{n-p} < \delta(\theta, A, B),
\]
where $\delta(\theta, A, B)$, $\Gamma_n$ are defined by (2.2) and (2.3). Thus letting $r \to 1$ we obtain (2.1). \qed

**Theorem 2.2.** If a function $f$ of the form (1.1) belongs to the class $T(\alpha, \beta)$, then
\[
|a_n| \leq (B - A) \Gamma_n, \quad n = k, k + 1, k + 2, \ldots,
\]
where $\Gamma_n$ is defined by (2.3). The result is sharp.

**Proof.** Let a function $f$ of the form (1.1) belong to the class $T(\alpha, \beta)$ and let us put
\[
g(z) = (z^{-p} \Omega^2_\mu f(z) - 1)(A - B)^{-1}, \quad h(z) = \frac{z}{1 + Bz}.
\]
By (1.4) we have $g \prec h$. Since the function $g$ is the function of the form
\[
g(z) = \sum_{n=k}^{\infty} [(A - B) \Gamma_n]^{-1} a_n z^{n-p}
\]
and the function $h$ is convex in $\mathcal{U}$, by Lemma 1.2 we obtain
\[
[(A - B) \Gamma_n]^{-1} |a_n| \leq 1, \quad n = k, k + 1, k + 2, \ldots.
\]
Thus we have (2.5). Equality in (2.6) holds for the functions $g_n$ of the form
\[
g_n(z) = h(z^{n-p}) = z^{n-p} + \sum_{i=-p+1}^{\infty} (-B)^{i-n+p} z^i, \quad n = k, k + 1, k + 2, \ldots.
\]
Thus equality in (2.5) holds for the functions $f_n$ of the form

$$f_n(z) = z^p + (A - B) \Gamma_n z^n + \sum_{i=n+1}^{\infty} (A - B) \Gamma_i (-B)^{i-n} z^i, \quad n = k, k + 1, \ldots.$$  

By Theorem 2.1 we obtain:

**Corollary 2.1.** If a function $f$ of the form (1.5) belongs to the class $T_0(\alpha, \beta)$, then

$$|a_n| \leq \delta(0, A, B) \Gamma_n, \quad n = k, k + 1, k + 2, \ldots,$$

where $\delta(0, A, B), \Gamma_n$ are defined by (2.2) and (2.3).

Putting $\theta = \pi$ in Corollary 2.1 we obtain:

**Corollary 2.2.** If a function $f$ of the form (1.5) belongs to the class $T_\pi(\alpha, \beta)$, then

$$|a_n| \leq \frac{B - A}{1 + B} \Gamma_n, \quad n = k, k + 1, k + 2, \ldots,$$

where $\Gamma_n$ is defined by (2.3). The result is sharp. The extremal functions are functions $f_n$ of the form

$$f_n(z) = z^p - \frac{B - A}{1 + B} \Gamma_n z^n, \quad n = k, k + 1, k + 2, \ldots. \quad (2.7)$$

3. Distortion theorems and extreme points

**Theorem 3.1.** If $f \in T_\theta(\alpha, \beta), |z| = r < 1$, then for $\alpha \leq 0$

$$r^\alpha - \delta(\theta, A, B) \Gamma_k r^k \leq |f(z)| \leq r^\alpha + \delta(\theta, A, B) \Gamma_k r^k \quad (3.1)$$

and for $\alpha \leq -1$

$$pr^{\alpha - 1} - k\delta(\theta, A, B) \Gamma_k r^{k+1} \leq |f'(z)| \leq pr^{\alpha - 1} + k\delta(\theta, A, B) \Gamma_k r^{k+1}, \quad (3.2)$$

where $\delta(\theta, A, B), \Gamma_n$ are defined by (2.2) and (2.3).

**Proof.** Let $f \in T_\theta(\alpha, \beta), |z| = r < 1$. Since the sequences $\{\Gamma_n\}$ for $\alpha \leq 0$ and $\{\Gamma_n/n\}$ for $\alpha \leq -1$ are decreasing and positive, by Theorem 2.1 we obtain

$$\sum_{n=k}^{\infty} |a_n| \leq \delta(\theta, A, B) \Gamma_k \quad \text{for } \alpha \leq 0, \quad (3.3)$$

and

$$\sum_{n=k}^{\infty} n |a_n| \leq k\delta(\theta, A, B) \Gamma_k \quad \text{for } \alpha \leq -1. \quad (3.4)$$
Since
\[ |f(z)| = \left| z^p + e^{i\theta} \sum_{n=k}^{\infty} |a_n|z^n \right| \leq r^p + \sum_{n=k}^{\infty} |a_n|r^n \]
\[ = r^p + r^k \sum_{n=k}^{\infty} |a_n|r^{n-k} \leq r^p + r^k \sum_{n=k}^{\infty} |a_n| \]
and
\[ |f(z)| = \left| z^p + e^{i\theta} \sum_{n=k}^{\infty} |a_n|z^n \right| \geq r^p - \sum_{n=k}^{\infty} |a_n|r^n \]
\[ = r^p - r^k \sum_{n=k}^{\infty} |a_n|r^{n-k} \geq r^p - r^k \sum_{n=k}^{\infty} |a_n|, \]
by (3.3) we obtain (3.1). Using (3.4) the estimations (3.2) we prove analogously. \( \square \)

Putting \( \theta = \pi \) in Theorem 3.1 we obtain:

Corollary 3.1. If \( f \in T_\pi(\alpha, \beta) \), \( |z| = r < 1 \), then for \( \alpha \leq 0 \)
\[ r^p - B - A \Gamma_k r^k \leq |f(z)| \leq r^p + B - A \Gamma_k r^k \]
and for \( \alpha \leq -1 \)
\[ pr^{p-1} - k \frac{B - A}{1 + B} \Gamma_k r^{k-1} \leq |f'(z)| \leq pr^{p-1} + k \frac{B - A}{1 + B} \Gamma_k r^{k-1}, \]
where \( \Gamma_n \) is defined by (2.3). The result is sharp. The extremal function is function \( f_k \) of the form (2.7).

Theorem 3.2. Let \( f_{k-1}(z) = z^p \) and let \( f_n, n = k, k + 1, k + 2, \ldots \), be defined by (2.7). A function \( f \) belongs to the class \( T_\pi(\alpha, \beta) \) if and only if it is of the form
\[ f(z) = \sum_{n=k}^{\infty} \gamma_n f_n(z), \quad z \in \mathcal{U}, \] (3.5)
where
\[ \sum_{n=k-1}^{\infty} \gamma_n = 1 \text{ and } \gamma_n \geq 0, \quad n = k - 1, k, k + 1, \ldots. \]

Proof. (\( \Rightarrow \)) Let a function \( f \) of the form
\[ f(z) = z^p - \sum_{n=k}^{\infty} |a_n|z^n \] (3.6)
belong to the class \( T_\pi(\alpha, \beta) \). Let us put
\[ \gamma_n = \frac{1 + B}{B - A} r_n^{-1} |a_n|, \quad n = k, k + 1, k + 2, \ldots \]
and
\[ \gamma_{k-1} = 1 - \sum_{n=k}^{\infty} \gamma_n. \]

We have \( \gamma_n \geq 0 \), \( n = k, k + 1, k + 2, \ldots \) and by (1.7) we have \( \gamma_{k-1} \geq 0 \). Thus,
\[
\sum_{n=k}^{\infty} \gamma_n f_n(z) = \gamma_{k-1} f_{k-1}(z) + \sum_{n=k}^{\infty} \gamma_n f_n(z)
\]
\[ = \left( 1 - \sum_{n=k}^{\infty} \gamma_n \right) z^p + \sum_{n=k}^{\infty} \frac{1 + B}{B - A} \Gamma_n^{-1} |a_n| \left( z^p - \frac{B - A}{1 + B} \Gamma_n z^n \right) \]
\[ = z^p - \sum_{n=k}^{\infty} \frac{1 + B}{B - A} \Gamma_n^{-1} |a_n| z^p + \sum_{n=k}^{\infty} \frac{1 + B}{B - A} \Gamma_n^{-1} |a_n| z^p
\]
\[ - \sum_{n=k}^{\infty} |a_n| z^n = f(z) \]
and the condition (3.5) follows.

(\( \Leftarrow \)) Let a function \( f \) satisfy (3.5). Thus,
\[
f(z) = \sum_{n=k}^{\infty} \gamma_n f_n(z) = \gamma_{k-1} f_{k-1} + \sum_{n=k}^{\infty} \gamma_n f_n(z)
\]
\[ = \left( 1 - \sum_{n=k}^{\infty} \gamma_n \right) z^p + \sum_{n=k}^{\infty} \left( z^p - \frac{B - A}{1 + B} \Gamma_n z^n \right) \gamma_n \]
\[ = z^p - \sum_{n=k}^{\infty} \frac{B - A}{1 + B} \Gamma_n \gamma_n z^n
\]
and we can write the function \( f \) in the form (3.6), where
\[ |a_n| = \gamma_n \frac{B - A}{1 + B} \Gamma_n. \]

Since
\[
\sum_{n=k}^{\infty} \Gamma_n^{-1} |a_n| = \sum_{n=k}^{\infty} \gamma_n \frac{B - A}{1 + B} \frac{B - A}{1 + B} (1 - \gamma_{1-k}) \leq \frac{B - A}{1 + B},
\]
we have \( f \in T_\alpha(x, \beta) \), which ends the proof. \( \square \)

By Theorem 3.1 we have:

**Corollary 3.2.** The class \( T_\gamma(x, \beta) \) is convex. The extremal points are functions \( f_n \) of the form (2.7) and the function \( f_{k-1}(z) = z^p. \)
4. The radii of convexity and starlikeness

**Theorem 4.1.** If \( f \in T_\theta(\alpha, \beta) \), then a function \( f \) is convex in the disc \( \mathcal{U}(r_c) \), where

\[
 r_c = \inf_{n \geq k} \left( \delta(\theta, A, B) \frac{n^2}{p^2} \Gamma_n \right)^{1/(p-n)} \tag{4.1}
\]

and \( \delta(\theta, A, B), \Gamma_n \) are defined by (2.2) and (2.3).

**Proof.** Let a function \( f \) of the form (1.5) belong to the class \( T_\theta(\alpha, \beta) \). By (1.2) the function \( f \) is convex in \( \mathcal{U}(r) \), \( 0 < r \leq 1 \), if

\[
 1 + \frac{zf''(z)}{f'(z)} - p < p, \quad z \in \mathcal{U}(r). \tag{4.2}
\]

Since

\[
 1 + \frac{zf''(z)}{f'(z)} - p = \left| \frac{e^{i\theta} \sum_{n=k}^{\infty} (n^2 - np) |a_n| z^{n-1}}{p z^{p-1} + e^{i\theta} \sum_{n=k}^{\infty} n |a_n| z^{n-1}} \right|
\]

\[
 \leq \left| \frac{\sum_{n=k}^{\infty} (n^2 - np) |a_n| z^{n-1}}{p - \sum_{n=k}^{\infty} n |a_n| z^{n-1}} \right|,
\]

putting \( |z| = r \), condition (4.2) is true if

\[
 \sum_{n=k}^{\infty} \frac{n^2}{p^2} |a_n| r^{n-p} \leq 1. \tag{4.3}
\]

From Theorem 2.1 we have

\[
 \sum_{n=k}^{\infty} [\delta(\theta, A, B) \Gamma_n]^{-1} |a_n| \leq 1, \tag{4.4}
\]

where \( \delta(\theta, A, B), \Gamma_n \) are defined by (2.2) and (2.3). Thus it is sufficient to show that

\[
 \frac{n^2}{p^2} r^{n-p} \leq [\delta(\theta, A, B) \Gamma_n]^{-1} \quad \text{for } n = k, k+1, k+2, \ldots,
\]

that is

\[
 r \leq (b_n)^{1/(p-n)} \quad \text{for } n = k, k+1, k+2, \ldots, \tag{4.5}
\]

where

\[
 b_n = \delta(\theta, A, B) \frac{n^2}{p^2} \Gamma_n. \tag{4.6}
\]

Condition (4.5) is true for \( r = r_c \), where \( r_c \) is defined by (4.1). Since

\[
 b_n > 0, \quad n = k, k+1, k+2, \ldots
\]

and

\[
 \lim_{n \to \infty} (b_n)^{1/(p-n)} = 1, \tag{4.7}
\]

we have \( 0 < r_c \leq 1 \), which ends the proof. \( \square \)
By Theorem 4.1 we have:

**Corollary 4.1.** Let \( \alpha \leq -2 \) and let \( \delta(\theta,A,B) \), \( \Gamma_n \) be defined by (2.2) and (2.3). If

\[
\frac{k^2}{p^2} \frac{\Gamma_k}{\delta(\theta,A,B)} \leq 1, \tag{4.8}
\]

then

\[
T_\theta(\alpha,\beta) \subseteq C_p. \tag{4.9}
\]

In the other case a function \( f \in T_\theta(\alpha,\beta) \) is convex in \( \mathcal{U}(r_c) \), where

\[
r_c = \left( \frac{\delta(\theta,A,B) k^2}{p^2 \Gamma_k} \right)^{1/(p-k)}. \tag{4.10}
\]

**Proof.** Let \( f \in T_\theta(\alpha,\beta) \). By Theorem 4.1 the function \( f \) is convex in \( \mathcal{U}(r_c) \), where \( r_c \) is defined by (4.1). For \( \alpha \leq -2 \) the sequence \( \{b_n\} \), defined by (4.6), is decreasing. Hence, if \( b_k \leq 1 \), that is condition (4.8) is true, then

\[
(b_n)^{(p-n-1)/(p-k)} \geq 1 \quad \text{for} \quad n = k, k+1, k+2, \ldots.
\]

This, by (4.7) we have \( r_c = 1 \), that is condition (4.9) is true. In the other case we have

\[
(b_n)^{(p-k)} \leq (b_n)^{(p-n)} \quad \text{for} \quad n = k, k+1, k+2, \ldots,
\]

which gives (4.10). \( \square \)

Since for \( \alpha \leq -2 \)

\[
\delta(\pi,A,B) \frac{k^2}{p^2} \Gamma_k < 1,
\]

by Corollary 4.1 we have:

**Corollary 4.2.** If \( \alpha \leq -2 \), then

\[
T_\pi(\alpha,\beta) \subseteq C_p.
\]

We now determine the radius of starlikeness for the class \( T_\theta(\alpha,\beta) \).

**Theorem 4.2.** If \( f \in T_\theta(\alpha,\beta) \), then the function \( f \) is starlike in \( \mathcal{U}(r^*) \), where

\[
r^* = \inf_{n \geq k} \left( \frac{\delta(\theta,A,B) n \Gamma_n}{p} \right)^{1/(p-n)}. \tag{4.11}
\]

and \( \delta(\theta,A,B), \Gamma_n \) are defined by (2.2) and (2.3).

**Proof.** Let a function \( f \) of the form (1.5) belong to the class \( T_\theta(\alpha,\beta) \). By (1.3) the function \( f \) is starlike in \( \mathcal{U}(r) \), \( 0 < r \leq 1 \), if

\[
\left| \frac{zf''(z)}{f'(z)} - p \right| < p, \quad z \in \mathcal{U}(r). \tag{4.12}
\]
Since
\[
\left| \frac{zf'(z)}{f(z)} - p \right| = \left| \frac{e^{i\theta} \sum_{n=k}^{\infty} (n-p) |a_n| z^n}{z^p + e^{i\theta} \sum_{n=k}^{\infty} |a_n| z^{n-1}} \right| 
\leq \frac{\sum_{n=k}^{\infty} (n-p) |a_n| |z|^{n-p}}{1 - \sum_{n=k}^{\infty} |a_n| |z|^{n-p}},
\]
putting \( |z| = r \), the condition (4.12) is true if
\[
\sum_{n=k}^{\infty} \frac{n}{p} |a_n| r^{n-p} \leq 1. \tag{4.13}
\]
By (4.4) it is sufficient to prove
\[
\frac{n}{p} r^{n-p} \leq \left[ \delta(\theta,A,B) \Gamma_n \right]^{-1} \quad \text{for } n = k, k+1, k+2, \ldots,
\]
that is
\[
r \leq (h_n)^{1/(p-n)},
\]
where
\[h_n = \delta(\theta,A,B) \frac{n}{p} \Gamma_n.\]
The last inequality is true for \( r = r^* \), where \( r^* \) is defined by (4.11). Since
\[h_n > 0 \text{ for } n = k, k+1, k+2, \ldots\]
and
\[
\lim_{n \to \infty} (h_n)^{1/(p-n)} = 1,
\]
we have \( 0 < r^* \leq 1 \), which completes the proof. \( \square \)

From Theorem 4.2 we obtain the following corollaries analogously as Corollaries 4.1, 4.2 from Theorem 4.1.

**Corollary 4.3.** Let \( \alpha \leq -1 \). If
\[
\delta(\theta,A,B) \frac{k}{p} \Gamma_k \leq 1,
\]
where \( \delta(\theta,A,B) \), \( \Gamma_n \) are defined by (2.2) and (2.3), then
\[T_\delta(\alpha,\beta) \subset S^*_p.\]
In the other case a function \( f \in T_\delta(\alpha,\beta) \) is starlike in \( \mathcal{U}(r^*) \), where
\[
r^* = \left( \delta(\theta,A,B) \frac{k}{p} \Gamma_k \right)^{1/(k-p)}.
\]

**Corollary 4.4.** If \( \alpha \leq -1 \), then
\[T_\pi(\alpha,\beta) \subset S^*_p.\]
References