Best constant inequalities for conjugate functions

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Abstract

A survey is given of sharp forms of some classical inequalities for the conjugate function. © 1999 Elsevier Science B.V. All rights reserved.

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Introduction

Let \( F = f + i \tilde{f} \) be analytic in the unit disc \( U \) with \( \tilde{f}(0) = 0 \). In this survey, we discuss inequalities involving the functions \( F, f \) and \( \tilde{f} \) starting with work from the 1920s of M. Riesz and A. Zygmund. The emphasis is on our work during the last 15 years on different kinds of best constant inequalities involving conjugate functions. In all cases, what we have to do is to construct subharmonic minorants to certain real-valued functions in the plane. Recently, we have found a general method which will give the old results as well as new inequalities. The present survey contains a little about the history, an outline of this method and some of our main new results. The full story will be described in [12]. Let \( \| \cdot \|_p \) denote Hardy norms in the spaces \( H^p(U) \) or \( h^p(U) \) (cf. [5]).

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1. The Riesz inequalities and Cole’s theorem

If $1 < p < \infty$, it is known that

\[ \| F \|_p \leq C_p \| f \|_p, \quad (1.1) \]
\[ \| \hat{f} \|_p \leq c_p \| f \|_p. \quad (1.2) \]

These inequalities are due to Riesz (cf. [17]). The best constant in (1.2) was found in 1972 by Pichorides (cf. [16, Theorem 3.7]): he proved that

\[ c_p = \tan(\pi/(2p)), \quad 1 < p \leq 2, \quad c_p = \cot(\pi/(2p)), \quad 2 < p < \infty. \quad (1.3) \]

The best constant in (1.1) is (cf. [6, 18])

\[ C_p = (\cos(\pi/(2p)))^{-1}, \quad 1 < p \leq 2, \quad C_p = (\sin(\pi/(2p)))^{-1}, \quad 2 < p < \infty. \quad (1.4) \]

To deduce this result in the case $1 < p \leq 2$, we consider the following function $L$: here $z = \pi/(2p)$, $w = u + iv$ and $\varphi = \arg w$:

\[ L(w) = \begin{cases} |w|^p - (\cos z)^{-p}|u|^p, & \alpha < \varphi < \pi - \alpha, \\ -\tan z |w|^p \cos(p \varphi), & |\varphi| \leq \alpha. \end{cases} \]

Furthermore, we define $L$ in the remaining part of $\mathbb{C}$ by requiring that $L(w) = L(-w)$. Then it can be proved that $L$ is superharmonic in $\mathbb{C}$, and we have

\[ |w|^p - (\cos z)^{-p}|u|^p \leq L(w), \quad w \in \mathbb{C}. \quad (1.5) \]

For proofs, we refer to [6]. It follows that

\[ |F(re^{i\alpha})|^p - (\cos z)^{-p}|f(re^{i\alpha})|^p \leq L \circ F(re^{i\alpha}). \]

Integrating over $\{|z|=r\}$, applying the superharmonic mean value inequality to the right-hand member and letting $r \to 1$, we obtain

\[ \|F\|_p^p - (\cos z)^{-p}\|f\|_p^p \leq L \circ F(0) \leq 0. \]

This gives inequality (1.1) with $C_p = (\cos(\pi/(2p)))^{-1}$. For details, we refer again to [6].

An example showing that the constants $C_p$ and $c_p$ are sharp in the case $1 < p < 2$ is given by approximations to a conformal mapping of $U$ onto the sector $\{w: |\arg w| < \pi/(2p)\}$ (there is no function $F \in H^p$ for which there is equality in (1.1) or (1.2) with best constants; however, we can get as close as we like).

A related result is given by Cole (cf. [13, Theorem 8.3]):

**Theorem A.** Let $H$ be a continuous and real-valued function on $\mathbb{C}$. The following two statements are equivalent:
(ii) for all trigonometric polynomials \( f \) with \( \hat{f}(0) = 0 \), we have
\[
\int_0^{2\pi} H(f(e^{i\theta}), \hat{f}(e^{i\theta})) \, d\theta \geq 0.
\] (1.6)

An alternative way of proving (1.1) with a constant given by (1.4) in the case \( 1 < p \leq 2 \) would be to apply Cole’s theorem to the function \( H(w) = C_p |u|^p - |w|^p \). According to (1.5), this function \( H \) has the subharmonic minorant \( h = -L \). However, even if we could guess that this \( H \), with \( C_p \) given by (1.4), might have a subharmonic minorant, how are we to prove this? It is difficult to apply Cole’s theorem to concrete examples. In Section 3, we shall state our Theorem 2. It shows how starting from appropriate conformal maps, we can construct useful, explicit choices for \( H \), in such a way that we can apply Theorem A and obtain Cole’s inequality (1.6).

It is not difficult to find a conformal mapping associated with the extremal cases for the Riesz theorems. In Section 5, we shall consider inequalities where it may be more difficult to prove that our constants are best possible. In such situations, Cole’s theorem gives a possible way around these obstacles. Therefore, let us here show how this last method can be used to prove that the constant \( C_p \) given by (1.4) is best possible in inequality (1.1) in the case \( 1 < p < 2 \).

We define \( H_\varepsilon(w) = (C_p - \varepsilon)^p |u|^p - |w|^p \). If (1.1) holds with \( C_p \) replaced be \( C_p - \varepsilon \), then by Cole’s theorem we know that \( H_\varepsilon \) will have a subharmonic minorant \( V_\varepsilon \) in \( \mathbb{C} \). Let \( \Omega_\varepsilon = \{ w \in \mathbb{C} : H_\varepsilon(w) > 0, u > 0 \} \). It is clear that there exists \( \beta \) such that \( \Omega_\varepsilon = \{ w \in \mathbb{C} : |\arg w| < \beta < \pi/(2 p) \} \), that \( V_\varepsilon \) is nonpositive on \( \partial \Omega_\varepsilon \) and that \( V_\varepsilon(w) \) is majorized by \( (C_p - \varepsilon)^p |u|^p \) in \( \Omega_\varepsilon \). According to a classical Phragmén–Lindelöf theorem, \( V_\varepsilon \) is nonpositive in \( \Omega_\varepsilon \). It is now easy to see that \( V_\varepsilon \) is nonpositive in \( \mathbb{C} \). The only subharmonic functions in \( \mathbb{C} \) which are bounded above are the constant ones (cf. [14, Theorem 2.14]). On the other hand, we see that

\[
V_\varepsilon(iv) \leq H_\varepsilon(iv) = -|v|^p \to -\infty, \quad |v| \to \infty
\]

and thus \( V_\varepsilon \) cannot be constant. Hence \( H_\varepsilon \) cannot have a subharmonic minorant in \( \mathbb{C} \). Again applying Cole’s theorem, we see that \( C_p \) is the best constant in (1.1).

Remark 1. We continue with the case \( 1 < p < 2 \). The essential part of Pichorides’ proof that the best constant in (1.2) is given by \( c_p \) in (1.3) is as follows: if \( x_p = (\sin(\pi/(2 p))^{p-1}(\cos(\pi/(2 p)))^{-1} \), then the function \( h(u + iv) = x_p \Re(|u| + iv)^p \) is a subharmonic minorant of \( H(w) = c_p |u|^p - |v|^p \). We note that \( h \) is harmonic except on the imaginary axis, that \( H - h \) and its gradient vanish on \( \{ |\theta| = \pi/(2 p) \} \) and that \( c_p \) is the smallest constant such that \( H \) has a subharmonic minorant.

Let us here add that the function \( L(w) \) above has been constructed in such a way that the difference between the right- and left-hand members in (1.5) vanish to the first order on the same lines.

Remark 2. Another subharmonic proof of (1.2) can be found in [18].

Remark 3. There is an interesting proof of an inequality of Paley due to Burkholder (cf. [3]). Also here, the argument gives a sharp constant: it depends on constructing a biconcave majorant of a cleverly chosen function. These two functions can be viewed as analogues of the two functions in
2. Inequalities of Zygmund type

Combining a classical inequality of Zygmund [19] with the best constant found by Pichorides [16, Theorem 3.4], we have:

**Theorem B.** For every $C > 2/\pi$, there is a constant $A = A(C)$ such that

$$||\tilde{f}||_1 \leq C \sup_{r < 1} \int_0^{2\pi} |f(re^{i\theta})| \log^+ |f(re^{i\theta})| d\theta/(2\pi) + A.$$  

For simplicity, we shall in the sequel write expressions as in the right-hand side of this inequality as

$$C \int |f| \log^+ |f| + A.$$  

Our general method, to be described below, gives a more precise result:

**Theorem 1.** There exist absolute constants $B_0$ and $B_1$ such that

$$||\tilde{f}||_1 \leq (2/\pi) \int |f| \log(e + |f|) + B_0 \int |f| \log \log(e + |f|) + B_1 ||f||_1.$$  

The first constant $2/\pi$ is sharp. We can prove that $B_0 \leq 4/\pi$. Our examples show that $B_0 \geq 2/\pi$. Theorem 1 is the case $\alpha = 1$ of a more general result (cf. Theorem 3).

Can we prove Theorem 1 by arguing as in the proof of (1.1) given in Section 1. It is natural to try to construct a superharmonic majorant $G_1$ by studying $\Re \{ w \log w \}, \Re w > 0$, (cf. [8]). More general inequalities of this type were deduced in [10]: starting with $\Re \{ w \log w \}^\alpha$, $\Re w > 0$, we proved in the case $1 \leq \alpha \leq 2$ that

$$\int |F| \log^+ |F|^\alpha \leq (2/(\pi \alpha)) \int |f| \log^+ |f|^\alpha + R_\alpha,$$

where $2/(\pi \alpha)$ is best possible and $R_\alpha$ is an error term. Unfortunately, this error term depends not only on $|f|$ but also on $|\tilde{f}|$. To get an estimate only depending on $|f|$ which holds for all $\alpha \in (0, \infty)$, a new method is needed.

3. A general construction

We begin by constructing a subharmonic function of a particular form. Let $G = g + i\tilde{g}$ be analytic on the right half-plane and define $h(x, y) = g(|x| + iy)$. We assume that

(a) $h$ can be extended to a subharmonic function in $\mathbb{C}$.

Furthermore, if $Q_1$ is the first quadrant, we assume that $G$ and $h$ satisfy the following conditions:
(b) $G$ maps the positive real axis onto itself;
(c) $G'$ maps $Q_1$ into $Q_1$;
(d) $G''$ maps $Q_1$ into the lower half-plane;
(e) Let $\gamma = \{ z \in Q_1: g(x, y) = 0 \}$. We assume that the projection of $\gamma$ onto the $x$-axis is the entire positive real $x$-axis and that the projection of $\gamma$ onto the $y$-axis is the entire positive imaginary axis.

It follows that there exist functions $y(x)$ and $x(y)$ such that $\gamma = \{(x, y(x)), x \geq 0\} = \{(x(y), y), y \geq 0\}$. We define

$$\phi(x) = \int_0^x D_1 g(t, y(t)) \, dt,$$
$$\psi(y) = -\int_0^y D_2 g(x(t), t) \, dt,$$
where $D_1 g = \partial g / \partial x$ and $D_2 g = \partial g / \partial y$.

**Theorem 2.** For any analytic polynomial $F = f + i \tilde{f}$ with $\tilde{f}(0) = 0$, we have

$$\int \psi(|\tilde{f}|) \leq \int \phi(|f|). \quad (3.1)$$

In the proof, we use the following properties of $\phi$ and $\psi$:

(i) $\phi(x) - \psi(y)$ vanishes on $\gamma$;
(ii) $\nabla (\phi(x) - \psi(y) - g(x, y)) = 0$ on $\gamma$;
(iii) $h(x, y) \leq \phi(|x|) - \psi(|y|)$ in $\mathbb{C}$.

Since $h$ is a subharmonic minorant of $\phi(|x|) - \psi(|y|)$ in $\mathbb{C}$, Theorem 2 is an immediate consequence of Cole’s theorem.

**4. Applications of Theorem 2**

Let us first choose $G(z) = z^p$, $1 < p \leq 2$. Here $g(re^{i\theta}) = r^p \cos(p\theta)$ and $\gamma$ is the line $\arg z = \pi/(2p)$.

Theorem 2 gives us immediately inequality (1.2) with Pichorides’ best constant.

Our second example is given by

$$G(z) = \int_0^z \log^p(z + e) \, d\zeta, \quad \Re z > 0.$$

Applying Theorem 2 with $G = G_z$, we obtain

**Theorem 3.** (i) If $x > 1$, there exists an absolute constant $A$ such that

$$\int |\tilde{f}|(\log(e + |\tilde{f}|))^{x-1} \leq (2/\pi x) \int |f|((\log(e + |f|))^{x}
+ (2/\pi) \int |f|((\log(e + |f|))^{x-1} \log \log(e + |f|)
+ A \int |f|((\log(e + |f|))^{x-1}.$$
The constants $2/\pi z$ and $2/\pi$ are sharp.

(ii) If $0 < z < 1$, there exists an absolute constant $A$ such that
\[ \int |\tilde{f}|(\log(e + |\tilde{f}|))^{\lambda - 1} \leq (2/\pi z) \int |f|(\log(e + |f|))^{\lambda} + A \int |f|\log \log(e + |f|). \]

The constant $2/\pi z$ is sharp.

(iii) If $z = 1$, we refer to Theorem 1.

We note that when $G = G_0$, and $z$ is positive, the curve $\gamma$ is of the form
\[ y = (2/\pi z)x \log x(1 + o(1)), \quad x \to \infty. \]

To see that our constants are sharp, we consider examples of the form $G_0(z) = ((1 + z)/(1 - z))^p, \; 0 < p < 1$.

In the case $z = 1$ discussed in Theorem 1, we have not been able to find the best constant $B_0$ in the first error term. The reason is that in the first step, our argument gives us an error term which contains $\int |\hat{f}|(\log(e + |\hat{f}|))^{-1}$. To handle this error term, we apply Theorem 2 to the function
\[ G_0(z) = \int_0^z \log \log(e + z) \, d\zeta, \quad \Re z > 0. \]

In a certain sense, $G_0$ represents the limiting case when we let $z \to 0$ in $G_z$.

**Theorem 4.** There exists an absolute constant $A$ such that
\[ \int |\tilde{f}|(\log(e + |\tilde{f}|))^{\lambda - 1} \leq (2/\pi) \int |f|\log \log(e + |f|) + A||f||. \]

**Remark.** In a special case, there is a well-known converse of Theorem B: if $f$ is nonnegative, then (cf. [D, Theorem 4.4])
\[ \int f \log(1 + f) \leq (\pi/2) \int |\hat{f}| + 2\pi f(0) \log(1 + f(0)). \]

The hypothesis that $f$ is nonnegative is explained and substantially weakened in [9]. There is an analogous converse of Theorem 4 (cf. [11]).

We note that when $G = G_0$, the curve $\gamma$ is of the form
\[ y = (2/\pi x)\log x \log \log x(1 + o(1)), \quad x \to \infty. \]

To check that all our assumptions (a)–(e) in Section 3 hold when we choose $G = G_0$ for some nonnegative $z$, we have to study the basic mapping properties of $G_0$. These arguments are elementary but require a lot of computation. Similarly, a lot of work is required in the examples which show that our constants are best possible (cf. [12]).

We note that the curves $\gamma$ described above are the analogues of certain straight lines which arise in the proofs of the Riesz theorems, as sketched in Sections 1 and 4 above.

5. A general result on sharpness

Let $G$ be an analytic function in $D = \{\Re z > 0\}$ satisfying assumptions (a)–(e) in Section 3 and assume furthermore that $G$ is univalent in $D$. We can prove that (3.1) is best possible in the case
when \( G'(z) \) behaves like a logarithm in this sense: we assume that

\[
G'(z)/G'(|z|) \to 1, \quad z \to \infty, \quad z \in D.
\]

(5.1)

We assume also that

\[
\phi(x)/x D_1 g(x, y(x)) \to 1, \quad x \to \infty,
\]

\[
\psi(y)/y (-D_2 g(x(y), y)) \to 1, \quad y \to \infty.
\]

(5.2)

(5.3)

It is clear that these assumptions hold when \( G = G_z, \; z \geq 0 \).

**Theorem 5.** Let \( \Omega = \{(x, y) : |y| < y(x)\} \). Assume that \( D \setminus \Omega \) is not minimally thin at infinity in \( D \). Then (3.1) is best possible in the sense that for any \( \varepsilon > 0 \), no inequality of the form

\[
\int \psi(|\tilde{f}|) \leq (1 - \varepsilon) \int \phi(|f|)
\]

(5.4)

can hold for all analytic polynomials \( f + i \tilde{f} \) with \( \tilde{f}(0) = 0 \).

**Remark.** For details on minimal thinness, we refer to [1, p. 81] and [7]. A necessary and sufficient condition for the set \( D \setminus \Omega \) not to be minimally thin at infinity is that \( \int_{-\infty}^{\infty} x(|y|)/(1 + y^2) \, dy \) diverges. All domains \( D \setminus \Omega \) discussed in Section 4 satisfy this condition.

To prove Theorem 5, it suffices to show that \((1 - \varepsilon)\phi(|x|) - \psi(|y|)\) does not have a subharmonic minorant in \( \mathbb{C} \) (cf. Cole’s theorem). Therefore, let us assume that such a minorant \( u \) exists. We define

\[
\Omega_{\varepsilon} = \{(x, y) : (1 - \varepsilon)\phi(|x|) - \psi(|y|) > 0\}.
\]

Then \( u \) is nonpositive in \( \mathbb{C} \setminus \Omega_{\varepsilon} \). Furthermore, \( u(z) \) is majorized by \((1 - \varepsilon)\phi(|x|)\) in \( \Omega_{\varepsilon} \). To conclude that \( u \) is nonpositive in \( \Omega_{\varepsilon} \) and thus in \( \mathbb{C} \) which will give us a contradiction, we need a more general Phragmén–Lindelöf theorem than the classical result used in Section 1. It is possible to prove this more general theorem but the details are again rather complicated, thus we omit the proof here.

**6. Further references**

As a referee points out, Orlicz spaces of type \( L \log^2 L \) arise naturally in many settings in analysis, and the reader is urged to compare the results discussed here with recent papers on sharp inequalities for Riesz transforms in \( \mathbb{R}^n \) by Ivanie and Martin [15] and by Bañuelos and Wang [2], for instance.

**References**