Hyperbolic geometry and disks

F.W. Gehring\textsuperscript{a}, K. Hag\textsuperscript{b, *}, \textsuperscript{1}

\textsuperscript{a}University of Michigan, Ann Arbor, MI 48109, United States
\textsuperscript{b}Norwegian University of Science and Technology, N-7034 Trondheim, Norway

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Abstract
We give here a pair of characterizations for a euclidean disk $D$ which are concerned with the hyperbolic geometry in $D$ and in domains which contain $D$. © 1999 Published by Elsevier Science B.V. All rights reserved.

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1. Introduction

Ref. [4] contains many different characterizations or ways of viewing quasidisks, the images of a disk under quasiconformal self-maps of the extended complex plane. Some of these are extensions of geometric properties of euclidean disks which actually characterize disks. Others involve properties of conformal invariants or criteria for the injectivity or extension of various classes of functions, properties for which it is not immediately obvious that they yield analogous characterizations of euclidean disks.

In this paper, we give a pair of nonstandard characterizations for a euclidean disk $D$ which involve hyperbolic geometry. The first concerns a relation between the euclidean and hyperbolic geometry in $D$, more specifically an inequality between a function which involves ratios of euclidean distances and the hyperbolic distance in $D$. The second is a convexity condition for $D$ in terms of the hyperbolic geometry in domains which contain $D$.

\textsuperscript{*} Corresponding author.

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2. Euclidean bound for hyperbolic distance

We assume throughout this section that $D$ is a proper subdomain of the complex plane $C$. If $D$ is simply connected, then the hyperbolic density at $z \in D$ is given by

$$\rho_D(z) = \rho_B(g(z)) |g'(z)|,$$

where

$$\rho_B(z) = \frac{2}{1 - |z|^2}$$

and $g$ is any conformal mapping of $D$ onto the unit disk $B$. Then the hyperbolic density at $z \in D$ is given by

$$D(z) = B(g(z)) j g(0).$$

The hyperbolic distance between $z_1, z_2 \in D$ is defined by

$$h_D(z_1, z_2) = \inf_{\gamma} \int_{\gamma} \rho_D(z) |dz|,$$

where the infimum is taken over all rectifiable curves $\gamma$ which join $z_1$ and $z_2$ in $D$. There is a unique hyperbolic geodesic $\beta$ joining $z_1$ and $z_2$ in $D$ for which

$$h_D(z_1, z_2) = \int_{\beta} \rho_D(z) |dz|.$$  

If $D$ is a Jordan domain, then for each $w_1, w_2 \in \partial D$ there is a unique hyperbolic line $\gamma$ joining $w_1$ and $w_2$, i.e., a crosscut of $D$ joining $w_1, w_2$ each subarc of which is a hyperbolic geodesic in $D$.

If $D = B$, then

$$h_D(z_1, z_2) = \log \left( \frac{|1 - \bar{z}_1 z_2| + |z_1 - z_2|}{|1 - \bar{z}_1 z_2| - |z_1 - z_2|} \right)$$

for $z_1, z_2 \in D$ and

$$h_D(z_1, z_2) = \log \left( \frac{|z_1 - z_2|}{1 - |z_1|} + 1 \right) \left( \frac{|z_1 - z_2|}{1 - |z_2|} + 1 \right)$$

$$\quad = \log \left( \frac{|z_1 - z_2|}{\text{dist}(z_1, \partial D)} + 1 \right) \left( \frac{|z_1 - z_2|}{\text{dist}(z_2, \partial D)} + 1 \right)$$

if, in addition, 0 lies in the euclidean segment $[z_1, z_2]$.

We shall show how a disk $D$ can be characterized by comparing the euclidean and hyperbolic geometries in $D$. This characterization makes use of the function

$$j_D(z_1, z_2) = \log \left( \frac{|z_1 - z_2|}{\text{dist}(z_1, \partial D)} + 1 \right) \left( \frac{|z_1 - z_2|}{\text{dist}(z_2, \partial D)} + 1 \right)$$

suggested by the above formula for $h_D(z_1, z_2)$ when $D = B$ and $0 \in [z_1, z_2]$.

**Lemma 2.1.** $j_D$ is a metric in $D$.

**Proof.** It suffices to show that

$$l = j_D(z_1, z_3) \leq j_D(z_1, z_2) + j_D(z_2, z_3) = r$$
for $z_1, z_2, z_3 \in D$. For convenience of notation let
\[ d_i = \text{dist}(z_i, \partial D), \quad i = 1, 2, 3. \]
Then from the euclidean triangle inequality and the inequalities
\[ d_2 \leq |z_1 - z_2| + d_1, \quad d_2 \leq |z_2 - z_3| + d_3 \]
we obtain
\[
\exp(r) = \frac{|z_1 - z_2| + d_1}{d_1} \frac{|z_1 - z_2| + d_2}{d_2} \frac{|z_2 - z_3| + d_2}{d_2} \frac{|z_2 - z_3| + d_3}{d_3} \geq \frac{|z_1 - z_3| + d_1}{d_1} \frac{|z_1 - z_3| + d_3}{d_3} = \exp(l) \]
\[
= \frac{|z_1 - z_3| + d_1}{d_1} \frac{|z_1 - z_3| + d_3}{d_3} = \exp(l). \quad \square
\]

The following three results show how the metrics $h_D$ and $j_D$ are related.

**Lemma 2.2.** For each simply connected domain $D$,
\[ j_D(z_1, z_2) \leq 4 h_D(z_1, z_2) \]
for $z_1, z_2 \in D$.

**Lemma 2.3.** A simply connected domain $D$ is a quasidisk if and only if there exists a constant $c$ such that
\[ h_D(z_1, z_2) \leq cj_D(z_1, z_2) \]
for $z_1, z_2 \in D$.

**Lemma 2.4.** If $D$ is a disk or half-plane, then
\[ h_D(z_1, z_2) \leq j_D(z_1, z_2) \]
\[ (2.5) \]
for $z_1, z_2 \in D$.

See [5] for Lemma 2.2. Lemma 2.3 follows from results in [3] and [6]; we outline the argument in Section 4. We give the proof for Lemma 2.4 below because this result is needed in what follows.

**Proof of Lemma 2.4.** Since each half-plane can be written as the increasing union of disks, it is sufficient to consider the case where $D$ is a disk. Next since $h_D$ and $j_D$ are both invariant with respect to similarity mappings, we may further assume that $D = B$. Then,
\[ h_D(z_1, z_2) = \log \left( \frac{|1 - z_1 z_2| + |z_1 - z_2|}{|1 - z_1 z_2| - |z_1 - z_2|} \right) = \log \left( \frac{n}{d} \right) \]
and
\[ n = |1 - |z_2|^2 - z_2(\bar{z}_1 - \bar{z}_2)| + |z_1 - z_2| \leq 1 - |z_2|^2 + (1 + |z_2|)|z_1 - z_2|, \]
whence
\[ n \leq (1 - |z_2|^3) \left( \frac{|z_1 - z_2|}{\text{dist}(z_2, \partial D)} + 1 \right) . \]

Similarly,
\[ n \leq (1 - |z_1|^3) \left( \frac{|z_1 - z_2|}{\text{dist}(z_1, \partial D)} + 1 \right) . \]

Next
\[ nd = |1 - z_1 z_2|^2 - |z_1 - z_2|^2 = (1 - |z_1|^2)(1 - |z_2|^2) \]
and thus
\[ \frac{n}{d} = \frac{n^2}{nd} \leq \left( \frac{|z_1 - z_2|}{\text{dist}(z_1, \partial D)} + 1 \right) \left( \frac{|z_1 - z_2|}{\text{dist}(z_2, \partial D)} + 1 \right) . \]

**Definition 2.6.** A domain \( D \) is circularly accessible at a point \( w \in \partial D \) if there exists a disk \( B_w \subset D \) with \( w \in \partial B_w \).

**Definition 2.7.** \( w_1, w_2 \) are diametral points for a bounded domain \( D \) if \( w_1, w_2 \in \partial D \) and \( |w_1 - w_2| = \text{dia}(D) \).

We show here that inequality (2.5) essentially characterizes the simply connected domains \( D \) which are disks by establishing the following result.

**Theorem 2.8.** A bounded simply connected domain \( D \) is a disk if and only if \( D \) is circularly accessible at a pair of diametral points and
\[ h_D(z_1, z_2) \leq f_D(z_1, z_2) \]
for \( z_1, z_2 \in D \).

The proof for Theorem 2.8 is based on the following two lemmas.

**Lemma 2.9.** If \( D \) is bounded and \( f : B \to D \) is conformal, then
\[ |f(z) - f(-z)| \leq \text{dia}(D) |z| \]
for \( z \in B \). Inequality (2.10) holds with strict inequality for all \( z \in B \setminus \{0\} \) unless \( D \) is a disk.

**Proof.** We may assume by means of a preliminary similarity mapping that \( \text{dia}(D) = 2 \). Then
\[ g(z) = \frac{1}{2} (f(z) - f(-z)) \]
is analytic in \( B \) with \( g(0) = 0 \) and
\[ |g(z)| = \frac{1}{2} |f(z) - f(-z)| < \frac{1}{2} \text{dia}(D) = 1 \]
for $z \in B$. Thus,
\begin{equation}
|f(z) - f(-z)| = 2|g(z)| \leq 2|z| \tag{2.11}
\end{equation}
by the Schwarz Lemma.

Now suppose that inequality (2.11) holds with equality for some $z \in B \setminus \{0\}$. Then by the Schwarz Lemma there exists a constant $a$ with $|a| = 1$ such that
\[ f(z) - f(-z) = 2g(z) = 2az \]
for $z \in B$. Hence $f'(z) + f'(-z) = 2a$ and
\[ |f'(z)|^2 - |f'(-z)|^2 = |f'(z)|^2 - |2a - f'(z)|^2 \leq 4(|f'(z)| - 1) \]
for $z \in B$. Then
\[ 0 = \int_B |f'(z)|^2 \, dm - \int_B |f'(-z)|^2 \, dm \leq 4 \left( \int_B |f'(z)| \, dm - \int_B \, dm \right), \]
while
\[ \int_B |f'(z)|^2 \, dm = \text{meas}(D) \leq \frac{\pi}{4} \text{dia}(D)^2 = \pi \]
by the isodiametric inequality. See, for example, 9.13.8 in [1] or p. 110 in [2]. Hence,
\[ \pi^2 = \left( \int_B \, dm \right)^2 \leq \left( \int_B |f'(z)| \, dm \right)^2 \leq \left( \int_B |f'(z)|^2 \, dm \right) \left( \int_B \, dm \right) \leq \pi^2 \]
and
\[ \left( \int_B |f'(z)| \, dm \right)^2 = \left( \int_B |f'(z)|^2 \, dm \right) \left( \int_B \, dm \right) \tag{2.12} \]
by Hölder’s inequality.

Finally, (2.12) implies there exists a constant $b$ such that $|f'(z)| = b$ a.e. in $B$. Then because $f$ is analytic,
\[ f'(z) = c, \quad f(z) - f(0) = cz \]
in $B$, where $c$ is a constant, and $D = f(B)$ is a disk. \qed

**Lemma 2.13.** Suppose that $D$ is bounded, that $w_1, w_2 \in \partial D$ with $|w_1 - w_2| = \text{dia}(D)$ and that $z_1, z_2 \in D$ with
\[ |z_1 - w_1| = |z_2 - w_2| < \frac{\text{dia}(D)}{2}. \]
Then
\[ h_D(z_1, z_2) \geq \log \left( \frac{|z_1 - z_2|}{|z_1 - w_1|} + 1 \right) \left( \frac{|z_1 - z_2|}{|z_2 - w_2|} + 1 \right) \tag{2.14} \]
with equality only if $D$ is a disk.

**Proof.** We may assume that $\text{dia}(D) = 2$. Choose a conformal mapping $f : B \to D$ and $0 < r < 1$ so that $f(-r) = z_1$ and $f(r) = z_2$. Then by Lemma 2.9,
\[ 2s = |z_1 - z_2| = |f(-r) - f(r)| \leq 2r \]
with equality only if $D$ is a disk. Next by the triangle inequality,
\[ |z_1 - w_1| + 2s + |z_2 - w_2| \geq |w_1 - w_2| = 2, \]
whence
\[ |z_1 - w_1| = |z_2 - w_2| = \frac{1}{2}(|z_1 - w_1| + |z_2 - w_2|) \geq 1 - s. \]
Thus,
\[ \frac{1 + s}{1 - s} \geq \frac{|z_1 - z_2|}{|z_1 - w_1|} + 1, \quad \frac{1 + s}{1 - s} \geq \frac{|z_1 - z_2|}{|z_2 - w_2|} + 1 \]
and we obtain
\[ h_D(z_1, z_2) = h_g(-r, r) = 2 \log \left( \frac{1 + r}{1 - r} \right) \geq 2 \log \left( \frac{1 + s}{1 - s} \right) \]
\[ \geq \log \left( \frac{|z_1 - z_2|}{|z_1 - w_1|} + 1 \right) \left( \frac{|z_1 - z_2|}{|z_2 - w_2|} + 1 \right) \]
with equality only if $s = r$, in which case $D$ is a disk. \( \Box \)

**Proof of Theorem 2.8.** The necessity is an immediate consequence of Lemma 2.4 and the fact that a disk is circularly accessible at each point of its boundary.

For the sufficiency, by hypothesis we can choose points $w_1, w_2 \in \partial D$ and open disks $B_1, B_2 \subset D$ so that
\[ |w_1 - w_2| = \text{dia}(D) \quad \text{and} \quad w_j \in \partial B_j \quad (2.15) \]
for $j = 1, 2$. Let $S$ denote the open strip bounded by the lines $L_1, L_2$ which meet the closed segment $[w_1, w_2]$ at right angles at the points $w_1, w_2$, respectively. Then (2.15) implies that $B_j$ is tangent to $L_j$ at $w_j$ for $j = 1, 2$. Thus we can find
\[ z_j \in [w_1, w_2] \cap B_j \]
so that
\[ |z_1 - w_1| = |z_2 - w_2| < \text{dia}(D)/2, \quad \text{dist}(z_j, \partial D) = |z_j - w_j| \]
for $j = 1, 2$. By hypothesis and Lemma 2.13,
\[ h_D(z_1, z_2) \leq j_D(z_1, z_2) \]
\[ = \log \left( \frac{|z_1 - z_2|}{\text{dist}(z_1, \partial D)} + 1 \right) \left( \frac{|z_1 - z_2|}{\text{dist}(z_2, \partial D)} + 1 \right) \]
\[ = \log \left( \frac{|z_1 - z_2|}{|z_1 - w_1|} + 1 \right) \left( \frac{|z_1 - z_2|}{|z_2 - w_2|} + 1 \right) \leq h_D(z_1, z_2). \]
Hence, we have equality throughout and $D$ is a disk by Lemma 2.13. \( \Box \)
3. Hyperbolic convexity

Since hyperbolic distance, geodesics and lines are invariant with respect to Möbius transformations, we can define these notions when $D$ is a simply connected subdomain of the extended complex plane $\overline{C}$ of hyperbolic type, i.e. with at least two points in its complement.

A set $E \subset C$ is convex with respect to euclidean geometry in $C$ if $E \cap \beta$ is either connected or empty for each euclidean geodesic $\beta$. We extend this notion to hyperbolic geometry as follows.

**Definition 3.1.** $E \subset \overline{C}$ is a hyperbolically convex subset of a simply connected domain $D \subset \overline{C}$ of hyperbolic type if it is convex with respect to the hyperbolic geometry of $D$, i.e. if $E \cap \beta$ is either connected or empty for each hyperbolic geodesic $\beta$ in $D$.

**Definition 3.2.** $E \subset \overline{C}$ is hyperbolically convex if it is a hyperbolically convex subset of each simply connected domain $D \subset \overline{C}$ of hyperbolic type which contains it.

**Remark 3.3.** Hyperbolic convexity is preserved under Möbius transformations.

**Lemma 3.4.** A disk or half-plane is hyperbolically convex.

**Proof.** If $D$ is a disk or half-plane, then it is a hyperbolically convex subset of each simply connected domain $G \subset C$ of hyperbolic type which contains it (see p. 118 of [7]). Remark 3.3 implies that this is also true for domains $G \subset \overline{C}$. □

The following result shows that a disk or half-plane is characterized by the property of hyperbolic convexity.

**Theorem 3.5.** A simply connected domain $D \subset C$ with $\overline{D} \neq C$ is a disk or half-plane if and only if it is hyperbolically convex.

The proof of the sufficiency in Theorem 3.5 depends on the following three lemmas.

**Lemma 3.6.** A bounded hyperbolically convex set $E \subset C$ is convex in the euclidean sense.

**Proof.** Fix $z_1, z_2 \in E$ and choose $0 < r < \infty$ so that

$$E \subset D = \{z : |z - z_1| < r\}.$$  

Then $\beta = [z_1, z_2]$ is a hyperbolic geodesic in $D$ and $[z_1, z_2] \subset E$ because $E$ is hyperbolically convex. Thus $E$ is convex in the euclidean sense. □

**Corollary 3.7.** A simply connected domain $D \subset \overline{C}$ with $\overline{D} \neq \overline{C}$ is a Jordan domain if it is hyperbolically convex.
Proof. Since $\overline{D} \neq \overline{C}$, we can choose a Möbius transformation $f$ such that $f(D)$ is bounded and hyperbolically convex. Then $f(D)$ is convex in the euclidean sense and, in particular, locally connected at each point of its boundary. Hence $f(D)$ and $D$ are Jordan domains by, for example, Theorem VI.16.2 of [8]. $\square$

Lemma 3.8. Suppose that $\gamma$ is a hyperbolic line in a Jordan domain $G \subset \overline{C}$ with endpoints $w_1, w_2 \in \partial G$ and that $U_0$ is a neighborhood of a point $z_0 \in G \cap \gamma$. Then there exist neighborhoods $U_1, U_2$ of $w_1, w_2$ such that for each pair of points $z_1 \in U_1 \cap G$ and $z_2 \in U_2 \cap G$, the hyperbolic geodesic $\beta$ joining $z_1$ and $z_2$ in $G$ meets $U_0$.

Proof. By performing a preliminary conformal mapping we may assume that $G$ is the upper half-plane and that $w_1 = -1, w_2 = 1, z_0 = i$. The geodesics in $G$ are then subarcs of the half-circles with endpoints on the real axis and we may choose

$$U_1 = \{z: |z + 1| < r\}, \quad U_2 = \{z: |z - 1| < r\}$$

provided that

$$\{z: |z - i| < r\} \subset U_0. \quad \square$$

Lemma 3.9. If $\gamma$ is a circular crosscut of a Jordan domain $G \subset \overline{C}$ in which $G$ is symmetric, then $\gamma$ is a hyperbolic line in $G$.

Proof. By means of a preliminary Möbius transformation we may assume that $\gamma$ is a segment of the real axis. Then by reflection we can map $G$ conformally onto the unit disk $B$ so that $\gamma$ corresponds to the real diameter of $B$. $\square$

Proof of Theorem 3.5. The necessity follows from Lemma 3.4.

For the sufficiency, we may assume without loss of generality, that $D$ is a bounded Jordan domain. Then we can choose an open disk $B_0$ which contains $D$ such that there exist two points $w_1, w_2 \in \partial B_0 \cap \partial D$. We shall show that

$$B_0 \subset \overline{D} \subset \overline{B}_0$$

and hence that $D = B_0$ since $D$ is a Jordan domain.

For this suppose that $z_0$ is a point in $B_0$, let $\gamma$ be the open circular arc or segment through $w_1, z_0, w_2$ and let $B_1$ be the component of $B_0 \setminus \gamma$ whose boundary forms an interior angle $\theta \geq \pi/2$ at $w_1$ and $w_2$. Set

$$G = B_1 \cup \gamma \cup \sigma(B_1),$$

where $\sigma$ denotes reflection in $\gamma$. Then $G$ is a Jordan domain, $\gamma$ is a hyperbolic line in $G$ by Lemma 3.9 and

$$D \subset B_0 \subset G.$$

Now if $z_0$ were not $\overline{D}$, then we could choose a neighborhood $U_0$ of $z_0$ such that

$$D \cap U_0 = \emptyset.$$
Let $U_1$ and $U_2$ be the neighborhoods of $w_1$ and $w_2$ described in Lemma 3.8 which correspond to $U_0$. Since $w_1, w_2 \in \partial D$ there exist two points

$$z_1 \in U_1 \cap D \subset U_1 \cap G, \quad z_2 \in U_2 \cap D \subset U_2 \cap G,$$

and hence by Lemma 3.8 and (3.10), a hyperbolic geodesic $\beta$ of $G$ with

$$z_1, z_2 \in \beta, \quad \beta \cap U_0 \neq \emptyset, \quad \beta \cap U_0 \cap D = \emptyset.$$

Thus $\beta \cap D$ is not connected and we have a contradiction. $\square$

4. Proof of Lemma 2.3

We show here how Lemma 2.3 follows from the following result; see Sections III.10 and III.11 in [3].

**Lemma 4.1.** A simply connected domain $D$ is a quasidisk if and only if there exist constants $a$ and $b$ such that

$$h_D(z_1, z_2) \leq a \ j_D(z_1, z_2) + b$$

for $z_1, z_2 \in D$.

The sufficiency in Lemma 2.3 is an immediate consequence of Lemma 4.1 with $a = c$ and $b = 0$. The necessity in Lemma 2.3 follows from Lemma 4.1 and the following result.

**Lemma 4.3.** If (4.2) holds for $z_1, z_2 \in D$ where $a$ and $b$ are nonnegative constants, then

$$h_D(z_1, z_2) \leq c \ j_D(z_1, z_2)$$

for $z_1, z_2 \in D$ where

$$c = \max \left( a + b, \frac{a + 1 + \sqrt{(a - 1)^2 + 4b}}{2} \right).$$

**Proof.** Choose $z_1, z_2 \in D$ with

$$d_2 = \text{dist}(z_2, \partial D) \leq \text{dist}(z_1, \partial D) = d_1$$

and let

$$t = j_D(z_1, z_2).$$

By hypothesis

$$h_D(z_1, z_2) \leq \left( a + \frac{b}{t} \right) j_D(z_1, z_2)$$

and hence

$$h_D(z_1, z_2) \leq (a + b) \ j_D(z_1, z_2) \leq c \ j_D(z_1, z_2)$$
if $t \geq 1$. If $0 < t \leq 1$, then

\[
\left( \frac{|z_1 - z_2|}{d_1} + 1 \right)^2 \leq \left( \frac{|z_1 - z_2|}{d_1} + 1 \right) \left( \frac{|z_1 - z_2|}{d_2} + 1 \right) = e'
\]

and

\[
s = \frac{|z_1 - z_2|}{d_1} \leq e'^{1/2} - 1 < 1,
\]

whence

\[
z_1, z_2 \in \{ z : |z - z_1| < d_1 \} = D' \subset D.
\]

Thus,

\[
h_D(z_1, z_2) \leq h_{D'}(z_1, z_2) = \log \left( \frac{1 + s}{1 - s} \right) \leq \log \left( \frac{e'^{1/2}}{2 - e'^{1/2}} \right) \tag{4.8}
\]

by (4.7) while

\[
\log \left( \frac{e'^{1/2}}{2 - e'^{1/2}} \right) \leq t + t' = (1 + t) f_D(z_1, z_2) \tag{4.9}
\]

since $0 < t \leq 1$. Hence we obtain

\[
h_D(z_1, z_2) \leq \min \left( a + \frac{b}{t}, 1 + t \right) f_D(z_1, z_2) \leq c_D(z_1, z_2)
\]

from (4.6), (4.8), (4.9) and the fact that

\[
\max_{0 < t < \infty} \min \left( a + \frac{b}{t}, 1 + t \right) = \frac{a + 1 + \sqrt{(a - 1)^2 + 4b}}{2}.
\]

**Remark 4.10.** If $a \geq 1$, then (4.4) holds with

\[
c = a + \max(b, \sqrt{b}).
\]

Moreover, if $D$ is a $K$-quasidisk, then (4.2) holds where $a = a(K) \to 1$ and $b = b(K) \to 1$. Hence $c = c(K) \to 1$ as $K \to 1$.

**References**