Two-point distortion for univalent functions

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Abstract

We discuss two-point distortion inequalities for (not necessarily normalized) univalent functions $f$ on the unit disk $\mathbb{D}$. By a two-point distortion inequality we mean an upper or lower bound on the Euclidean distance $|f(a) - f(b)|$ in terms of $d_\mathbb{D}(a,b)$, the hyperbolic distance between $a$ and $b$, and the quantities $(1 - |a|^2)|f'(a)|$, $(1 - |b|^2)|f'(b)|$. The expression $(1 - |z|^2)|f'(z)|$ measures the infinitesimal length distortion at $z$ when $f$ is viewed as a function from $\mathbb{D}$ with hyperbolic geometry to the complex plane $\mathbb{C}$ with Euclidean geometry. We present a brief overview of the known two-point distortion inequalities for univalent functions and obtain a new family of two-point upper bounds that refine the classical growth theorem for normalized univalent functions. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

We are concerned with two-point distortion theorems for univalent functions of the form introduced by Blatter [1]. In the following, we will let $f$ denote an arbitrary univalent function on the unit disk $\mathbb{D} = \{z: |z| < 1\}$, while $g$ will designate a normalized ($g(0) = 0$, $g'(0) = 1$) univalent function on $\mathbb{D}$. Let $S$ be the family of normalized univalent functions.

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A function \( g \in S \) satisfies the classical growth and distortion theorems:
\[
\frac{|z|}{(1 + |z|)^2} \leq |g(z)| \leq \frac{|z|}{(1 - |z|)^2},
\]
(1)
\[
\frac{1 - |z|}{(1 + |z|)^3} \leq |g'(z)| \leq \frac{1 + |z|}{(1 - |z|)^3}.
\]
(2)
These inequalities are sharp; equality holds if and only if \( g \) is a rotation of the Koebe function \( K(z) = z/(1 - z)^2 \). The growth and distortion theorems are necessary, but not sufficient, for univalence. In the interesting paper \[1\] Blatter obtained a two-point distortion theorem for univalent functions on \( \mathbb{D} \) that is necessary and sufficient for univalence. He proved that if \( f \) is univalent on \( \mathbb{D} \), then the following inequality is valid for \( p = 2 \):
\[
|f(a) - f(b)| \geq \frac{\sinh(2d\mathcal{D}(a, b))}{2[2\cosh(2pd\mathcal{D}(a, b)))]^{1/p}}[|D_1f(a)|^p + |D_1f(b)|^p]^{1/p},
\]
(3)
where
\[
d\mathcal{D}(a, b) = \tanh^{-1}\left| \frac{a - b}{1 - \bar{a}b} \right|
\]
is the hyperbolic distance between \( a \) and \( b \) and
\[
D_1f(z) = (1 - |z|^2)f'(z).
\]
Moreover, equality holds for distinct \( a, b \in \mathbb{D} \) if and only if \( f = S \circ K \circ T \), where \( S \) is a conformal automorphism of the complex plane \( \mathbb{C} \), \( K \) is the Koebe function and \( T \) is a conformal automorphism of \( \mathbb{D} \) with \( T(a), T(b) \in (-1, 1) \). In words, equality holds if and only if \( f(\mathbb{D}) \) is a slit plane and the points \( f(a), f(b) \) lie on the extension of the slit into \( f(\mathbb{D}) \). Conversely, it is elementary to show that if \( f \) is holomorphic on \( \mathbb{D} \) and satisfies (3) for any \( p > 0 \), then \( f \) is either univalent or constant. Blatter’s method of proving (3) is an interesting mix of differential geometry, differential inequalities and coefficient bounds for functions in the class \( S \).
Kim and Minda \[5\] extended the work of Blatter in such a way that a connection with the lower bound in (1) was made. They showed that inequality (3) is valid for all \( p \geq \frac{3}{2} \) for any univalent function and that the result is sharp as in Blatter’s theorem. They also observed that the right-hand side of (3) is a decreasing function of \( p \) on \([1, +\infty)\) and the limiting case \((p = \infty)\) is simply an invariant version of the lower bound in (1). In this sense (3) for \( p \geq 1 \) can be viewed as a strengthening of the lower bound in (1). Kim and Minda also showed there is a family of inequalities similar to (3) that is valid for all \( p \geq 1 \) and characterizes convex univalent functions. They observed that Blatter’s method could not establish (3) for all \( p \geq 1 \). Recently, by employing a different method, Jenkins \[4\] proved the decisive result that inequality (3) is valid for all univalent functions for \( p \geq 1 \), but not for \( 0 < p < 1 \).
At the same time Jenkins obtained an upper bound that is similar to (3). He showed that if \( f \) is univalent on \( \mathbb{D} \) and \( 0 < p < \infty \), then
\[
|f(a) - f(b)| \leq 2^{-1-1/p} \sinh(2d\mathcal{D}(a, b))[|D_1f(a)|^p + |D_1f(b)|^p]^{1/p}.
\]
(4)
Equality holds for distinct \( a, b \in \mathbb{D} \) if and only if \( f = S \circ L \circ T \), where \( S \) is a conformal automorphism of \( \mathbb{C} \), \( L(z) = z/(1 + z^2) \) and \( T \) is a conformal automorphism of \( \mathbb{D} \) with \( T(a) = -r, T(b) = r \) for
some \( r \in (0,1) \). In words, equality holds if and only if \( f(\mathbb{D}) \) is the plane slit symmetrically through the point at \( \infty \) on the line determined by \( f(a) \) and \( f(b) \). In particular, \( L \) maps \( \mathbb{D} \) onto \( \mathbb{C} \setminus ((-\infty,-1/2] \cup [1/2,\infty)) \) and equality holds for \( a = -r, b = r \in (0,1) \). The right-hand side of (4) is an increasing function of \( p \) and the limiting cases \( (p = 0 \text{ and } p = \infty) \) are both valid. The case \( p = 0 \) is

\[
|f(a) - f(b)| \leq \frac{1}{2} \sinh(2d_{\mathbb{D}}(a,b))\sqrt{|D_1 f(a)||D_1 f(b)|}
\]

while \( p = \infty \) is

\[
|f(a) - f(b)| \leq \frac{1}{2} \sinh(2d_{\mathbb{D}}(a,b))\max\{|D_1 f(a)|, |D_1 f(b)|\}.
\]

None of the upper bounds, \( 0 \leq p \leq \infty \), is an invariant form of the upper bound in (1). In this sense the family of inequalities (4) is different from (3).

On the other hand, Ma and Minda [6] established sharp upper and lower two-point distortion theorems for the class of strongly close-to-convex functions of order \( \alpha \in [0,1] \). Recall that a holomorphic function \( f \) defined on \( \mathbb{D} \) is strongly close-to-convex of order \( \alpha \) if there is a convex univalent function \( \phi \) defined on \( \mathbb{D} \) such that \( |\arg f'(z)/\phi'(z)| < \alpha \pi/2 \) for all \( z \in \mathbb{D} \). They used Blatter’s method and obtained bounds valid for all \( p \geq 1 \). Their lower bounds are analogous to (3), but their upper bounds are not parallel to (4). In this note we establish a family of upper two-point distortion inequalities for univalent functions by again using Blatter’s method.

**Theorem 1.** Suppose \( f \) is univalent in \( \mathbb{D} \). Then for \( a,b \in \mathbb{D} \) and \( p \geq 1 \)

\[
|f(a) - f(b)| \leq \frac{[2 \cosh(2pd_{\mathbb{D}}(a,b))]^{1/p} \sinh(2d_{\mathbb{D}}(a,b))}{2[1/|D_1 f(a)|^p + 1/|D_1 f(b)|^p]^{1/p}}.
\]

Equality holds for distinct \( a,b \in \mathbb{D} \) if and only if \( f = S \circ K \circ T \), where \( S \) is a conformal automorphism of \( \mathbb{C} \), \( K \) is the Koebe function and \( T \) is a conformal automorphism of \( \mathbb{D} \) with \( T(a), T(b) \in (-1,1) \).

The right-hand side of (5) is an increasing function of \( p \) and the case \( p = \infty \) is

\[
|f(a) - f(b)| \leq \frac{1}{2} \exp(2d_{\mathbb{D}}(a,b))\sinh(2d_{\mathbb{D}}(a,b))\min\{|D_1 f(a)|, |D_1 f(b)|\}.
\]

This is an invariant form of the upper bound in (1). In fact if we apply (6) to \( g \in S \) with \( a = 0 \) and \( b = z \), we obtain

\[
|g(z)| \leq \frac{|z|}{(1 - |z|)^2} \min\{1, |D_t g(z)| \} \leq \frac{|z|}{(1 - |z|)^2}.
\]

Thus for \( 1 \leq p < \infty \) the right-hand side of (5) can be regarded as a refinement of the upper bound in (1). None of the upper bounds (5) characterizes univalent functions.

There are two-point distortion theorems for univalent functions in other contexts. Upper and lower two-point distrotion theorems for bounded univalent functions are given in [7]. A lower two-point distortion theorem for spherically convex functions is contained in [8]; this is the first two-point result for meromorphic univalent functions. Two-point distortion theorems for convex biholomorphic mappings of the unit ball in \( \mathbb{C}^n \) are treated in [3].

Two-point distortion theorems for univalent functions yield sharp comparison theorems between hyperbolic and euclidean geometry on simply connected regions. We discuss these comparisons in Section 4.
2. Preliminaries

In addition to the differential operator $D_1$ we need the differential operators $D_2$ and $D_3$ given by

$$D_2 f(z) = (1 - |z|^2)^2 f''(z) - 2\overline{z}(1 - |z|^2)f(z),$$

$$D_3 f(z) = (1 - |z|^2)^3 f'''(z) - 6\overline{z}(1 - |z|^2)^2 f''(z) + 6\overline{z}^2(1 - |z|^2)f'(z).$$

These differential operators are invariant in the sense that $|D_j(S \circ f \circ T)| = |D_j f| \circ T$ ($j = 1, 2, 3$) whenever $T$ is a conformal automorphism of $\mathbb{D}$ and $S$ is a euclidean motion of $\mathbb{C}$. Two combinations of these operators for locally univalent functions occur frequently in practice:

$$Q_f(z) = \frac{D_2 f(z)}{D_1 f(z)} = (1 - |z|^2)^2 \frac{f''(z)}{f'(z)} - 2\overline{z}$$

and

$$D_3 f(z) - \frac{3}{2} \left( \frac{D_2 f(z)}{D_1 f(z)} \right)^2 = (1 - |z|^2)^3 S_f(z),$$

where

$$S_f(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2} \left( \frac{f''(z)}{f'(z)} \right)^2$$

is the Schwarzian derivative of $f$. The absolute value of these two quantities is unchanged when $f$ is replaced by $S \circ f \circ T$, where $S$ is any conformal automorphism of $\mathbb{C}$ and $T$ is any conformal automorphism of $\mathbb{D}$.

For a normalized univalent function $g$ defined on $\mathbb{D}$, $|g''(0)| \leq 4$ with equality if and only if $g$ is a rotation of $K$ [2, p. 30]. This can be formulated invariantly as

$$|Q_f(z)| \leq 4$$

for any univalent function $f$ on $\mathbb{D}$. We shall also need the Kraus–Nehari result [2, p. 263] which asserts that

$$(1 - |z|^2)^3 |S_f(z)| \leq 6$$

for any univalent function $f$ on $\mathbb{D}$.

An integral inequality that follows from two differential inequalities is essential for our proof.

Lemma 1. Suppose $v \in \mathbb{C}^2[-L,L]$, $v > 0$, $|v| \leq 4v$ and $v'' \leq 16v$. Then for any $p \geq 1$

$$\int_{-L}^{L} \frac{dx}{v(s)} \leq \frac{(2 \cosh(4pL))^{1/p} \sinh(4L)}{2[v(L)^p + v(-L)^p]^{1/p}}.$$
The hyperbolic metric $\lambda_\Omega(z)\,dz$ on a hyperbolic region $\Omega$ is determined by $\lambda_\Omega(f(z))|f'(z)| = \lambda_\Omega(z)$, where $f: \mathbb{D} \to \Omega$ is any holomorphic covering projection. It is independent of the choice of the covering projection. We note that

$$\lambda_\Omega(f(z)) = \frac{1}{|D_1 f(z)|}.$$ 

The hyperbolic distance between $A, B \in \Omega$ is defined by

$$d_\Omega(A, B) = \inf \int_\gamma \lambda_\Omega(z)|dz|,$$

where the infimum is taken over all paths $\gamma$ in $\Omega$ joining $A$ and $B$. We already have seen the explicit formula for $d_\Omega$. If $f: \mathbb{D} \to \Omega$ is any holomorphic covering projection, then $d_\Omega(f(a), f(b)) \leq d_\Omega(a, b)$. This is an identity if $f$ is a conformal mapping; that is, a conformal mapping is an isometry relative to the hyperbolic distance.

### 3. Proof of Theorem 1

Fix distinct $a, b \in \mathbb{D}$ and let $\gamma: z = z(s), -L \leq s \leq L$, be the hyperbolic geodesic arc from $a$ to $b$ parameterized by hyperbolic arclength. This means that $2L = d_\Omega(a, b)$ is the hyperbolic length of $\gamma$ and $z'(s) = (1 - |z(s)|^2)e^{ik(s)}$, where $e^{ik(s)}$ is a unit tangent for $\gamma$ at $z(s)$. At this point we only assume that $f$ is locally univalent on $\mathbb{D}$, so $f'(z) \neq 0$ for $z \in \mathbb{D}$. Set

$$v(s) = |D_1 f(z(s))|^{-1}.$$ 

It is elementary to show that

$$\frac{d}{ds}|D_1 f(z(s))| = |D_1 f(z(s))| \text{Re}\{e^{ik(s)}Q_f(z(s))\}.$$ 

Therefore,

$$v'(s) = -v(s)\text{Re}\{e^{ik(s)}Q_f(z(s))\}.$$ 

Next,

$$v''(s) = -v'(s)\text{Re}\{e^{ik(s)}Q_f(z(s))\} - v(s)\frac{d}{ds}\text{Re}\{e^{ik(s)}Q_f(z(s))\}$$

$$= v(s)\left[\text{Re}^2\{e^{ik(s)}Q_f(z(s))\} - \frac{d}{ds}\text{Re}\{e^{ik(s)}Q_f(z(s))\}\right].$$ 

A tedious calculation gives

$$\frac{d}{ds}[e^{ik(s)}Q_f(z(s))] = e^{2ik(s)}(1 - |z(s)|^2)^2s_f(z(s)) + \frac{i}{2}[e^{ik(s)}Q_f(z(s))]^2 - 2.$$ 

See [6] for a more general result; the preceding formula makes use of the fact that the hyperbolic curvature of $\gamma$ is 0. Recall that the hyperbolic curvature of $\gamma$ at $z(s)$ is

$$\kappa_h(z(s), \gamma) = (1 - |z(s)|^2)\kappa_e(z(s), \gamma) + \text{Im}\{2e^{ik(s)}z(s)\},$$
where
\[ \kappa_e(z(s), \gamma) = \frac{1}{|z'(s)|} \text{Im} \left\{ \frac{z''(s)}{z'(s)} \right\} \]
is the Euclidean curvature of \( \gamma \) at \( z(s) \). Then
\[
v''(s) = v(s)[\text{Re}\{e^{i\theta_s}Q_f(z(s))\} - \frac{1}{2}\text{Re}\{[e^{i\theta_s}Q_f(z(s))]^2 \}
- \text{Re}\{e^{2i\theta_s}(1 - |z(s)|^2)^2S_f(z(s))\} + 2] \\
= v(s)[\frac{1}{2}|Q_f(z(s))|^2 - \text{Re}\{e^{2i\theta_s}(1 - |z(s)|^2)^2S_f(z(s))\} + 2] \\
\leq v(s)[\frac{1}{2}|Q_f(z(s))|^2 + (1 - |z(s)|^2)^2|S_f(z(s))| + 2].
\]

If we now assume that \( f \) is univalent, then (7) and (8) imply \( |v'(s)| \leq 4v(s) \) and \( v''(s) \leq 16v(s) \).

Because \( f \circ \gamma \) is a path connecting \( f(a) \) to \( f(b) \),
\[
|f(a) - f(b)| \leq \int_{f \circ \gamma} |dw| = \int_{\gamma} |f'(z)| |dz| \\
= \int_{-L}^{L} |f'(z(s))|(1 - |z(s)|^2) \, ds \\
= \int_{-L}^{L} |D_f(z(s))| \, ds = \int_{-L}^{L} \frac{ds}{v(s)}
\]
with equality if and only if \( f \circ \gamma \) is the euclidean line segment joining \( f(a) \) to \( f(b) \).

The inequality in the lemma gives for \( p \geq 1 \)
\[
\int_{-L}^{L} \frac{ds}{v(s)} \leq \frac{(2 \cosh(2pd_\mathbb{D}(a,b))^\frac{1}{p} \sinh(2d_\mathbb{D}(a,b))}{2 \left[ \frac{1}{|D_f(a)|^p} + \frac{1}{|D_f(b)|^p} \right]^\frac{1}{p}}
\]
and equality implies \( |v'| = 4v \).

Together the two preceding inequalities show that (5) holds for any univalent function \( f \) on \( \mathbb{D} \) and all \( p \geq 1 \).

Now, we determine when equality holds in (5). Suppose equality holds in (5) for distinct \( a, b \in \mathbb{D} \). Then \( f \circ \gamma \) is the straight line segment from \( f(a) \) to \( f(b) \) and \( |v'| = 4v \) implies \( |Q_f(z(s))| = 4 \) on \([-L, L]\). Let \( T \) be a conformal automorphism of \( \mathbb{D} \) with \( T(a) = 0 \) and \( T(b) = \tilde{b} > 0 \).

Then \( T \) maps \( \gamma \) onto the interval \([0, \tilde{b}]\).

Next, we can determine a conformal automorphism \( S \) of \( \mathbb{C} \) and \( \tilde{b} > 0 \) so that \( S(0) = f(a), S(\tilde{b}) = f(b) \) and \( g = S^{-1} \circ f \circ T^{-1} \) has derivative 1 at the origin. Then \( g \) is a normalized univalent function which maps \([0, \tilde{b}]\) onto \([0, \tilde{b}]\), so \( g \) is real valued on \((-1, 1)\).

The condition \( |Q_f(a)| = 4 \) implies \( |Q_g(0)| = 4 \), so \( g \) must be a rotation of the Koebe function. This implies \( g(z) = K(z) \), or \( g(z) = -K(-z) \), since \( g \) is real-valued on \((-1, 1)\). In the first case, \( f = S \circ K \circ T \) as desired.
In the second case, \( f = S \circ K \circ T \), where \( \tilde{T}(z) = -T(z) \) and \( \tilde{S}(z) = S(-z) \) still have the appropriate form.

Conversely, if \( f \) has the specified form, it is straightforward to verify that equality holds.

4. Concluding remarks

Two-point distortion theorems yield comparison results between hyperbolic and Euclidean geometry on simply connected regions.

**Theorem 2.** Suppose \( \Omega \) is a simply connected hyperbolic region in the upper bound. Then for \( p \geq 1 \) and \( A, B \in \Omega \)

\[
\frac{\sinh(2d_\Omega(A, B))}{2[\cosh(2pd_\Omega(A, B))]^{1/p}} \left[ \frac{1}{\lambda_\Omega^p(A)} + \frac{1}{\lambda_\Omega^p(B)} \right]^{1/p} \leq |A - B| \leq \frac{[2\cosh(2pd_\Omega(A, B))]^{1/p} \sinh(2d_\Omega(A, B))}{2[\lambda_\Omega^p(A) + \lambda_\Omega^p(B)]^{1/p}}.
\]

Equality holds in either inequality if and only if \( \Omega \) is a slit plane and \( A, B \) lie on the extension of the slit into \( \Omega \).

**Proof.** The upper bound follows immediately from applying Theorem 1 to any conformal mapping \( f: \mathbb{D} \to \Omega \) since \( d_\Omega(f(a), f(b)) = d_\mathbb{D}(a, b) \) and \( \lambda_\Omega(f(z)) = 1/|D_1 f(z)| \). Similarly, the lower bound is obtained from Blatter’s result as improved by Kim-Minda and Jenkins.

As noted in [5] the lower bounds in Theorem 2 do not characterize simply connected regions even though the corresponding two-point distortion theorem does characterize univalent functions. We want to show that Theorem 1 does not characterize univalent functions, and the upper bound in Theorem 2 does not characterize simply connected regions. The function \( f(z) = (1 + z)/(1 - z) \), \( \delta > 0 \), is a covering projection of \( \mathbb{D} \) onto the annulus \( \mathbb{A}_\delta = \{w: \exp(-\pi\delta/2) < |w| < \exp(\pi\delta/2)\} \) and [5]

\[
|Q_f(z)| \leq 2\sqrt{1 + \delta^2},
\]

\[
(1 - |z|^2)^2 |S_f(z)| \leq 2(1 + \delta^2).
\]

Then inequalities (7) and (8) hold for \( f \) when \( 0 < \delta < \sqrt{2} \). As we noted in the proof of Theorem 1, inequality (5) is valid for any locally univalent function \( f \) that satisfies inequalities (7) and (8). The fact that inequality (5) holds for \( f \) when \( 0 < \delta < \sqrt{2} \) implies that the upper bound in Theorem 2 holds for \( \mathbb{A}_\delta \) when \( 0 < \delta < \sqrt{2} \). Given points \( A, B \in \mathbb{A}_\delta \), select \( a, b \in \mathbb{D} \) with \( f(a) = A, f(b) = B \) and \( d_\mathbb{D}(a, b) = d_{\mathbb{A}_\delta}(A, B) \). This latter equality can be achieved even though \( d_\mathbb{D}(z_1, z_2) \geq d_{\mathbb{A}_\delta}(f(z_1), f(z_2)) \) in general. This shows that the upper bound in Theorem 2 is valid for \( \mathbb{A}_\delta \) for \( 0 < \delta < \sqrt{2} \).

**References**