Perturbations of Laguerre–Hahn linear functionals

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Abstract

In this paper we shall make some perturbations in a Laguerre–Hahn linear functional, such as the addition of a Dirac Delta or the left multiplication by a polynomial. We shall study that these transformations carried out on Laguerre–Hahn linear functionals originate new Laguerre–Hahn linear functionals. We shall also analyze the class of the resulting functional.

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1. Introduction

Modifications by means of Dirac deltas have been considered by several authors from different point of view. In 1940 Krall [12] obtained three new classes of polynomials orthogonal with respect to measures which are not absolutely continuous with respect to the Lebesgue measure; the resulting polynomials satisfy a fourth-order linear differential equation. Nevai [17] considered the addition of finitely many mass points to a positive measure and he studied the asymptotic behavior of the corresponding orthogonal polynomials. In [13], Maroni and Marcellán were interested in the analysis of modifications of semiclassical functionals by adjoining arbitrary masses at any point of the real line. In that paper necessary and sufficient conditions for the quasi-definiteness of the modified functional have been given.

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More recently, in [3, 4], a generalization of the Laguerre classical polynomials has been treated, by adding a derivative of a Dirac delta in the point \( z = 0 \). In particular, the hypergeometric character of these new polynomials is showed (see also [6]).

Our aim is to analyze how the Laguerre–Hahn character for the perturbed linear functionals is preserved as well as to determine the class of such a functional (see [1, 14, 18]).

The structure of the paper is as follows: Section 2 contains some introductory results and notations concerning linear functionals. Section 3 is devoted to the characterization of the Laguerre–Hahn linear functionals and the definition of the concept of class. In Section 4 a modification of a Laguerre–Hahn linear functional is carried out by adding a Dirac delta, studying the modification of the order of the class of the new Laguerre–Hahn functional obtained in such a way. In Section 5 we carried out the study of the linear functional \( \mathcal{U} \), fulfilling \((x - c)\mathcal{U} = \mu\mathcal{V}\), where \( \mathcal{V} \) is a Laguerre–Hahn linear functional, (see [11, 15]), as well as the order of the class of the involved functionals.

The modifications mentioned in Sections 4 and 5 have been carried out in the first kind associated functional for the classical ones, by studying the modification of the order of the class and showing the equations which the new functional satisfies, as well as the Riccati differential equation which fulfils the corresponding Stieltjes function.

2. Preliminaries and notations

Let \( \mathcal{U} \) be a linear functional on the linear space \( P \) of polynomials with complex coefficients and let \( S(\mathcal{U})(z) \) be its Stieltjes function defined by

\[
S(\mathcal{U})(z) = -\sum_{n \geq 0} \frac{\mathcal{U}_n}{z^{n+1}},
\]

where \( \mathcal{U}_n = \langle \mathcal{U}, x^n \rangle \), \( n \geq 0 \), are the moments of \( \mathcal{U} \) and \( \langle \cdot, \cdot \rangle \) means the duality bracket. By a convention, we will suppose that \( \mathcal{U}_0 = 1 \).

Let \( P' \) be the algebraic dual space of \( P \) and \( \Delta \) the linear space generated by \( \{D^\nu \delta\}_{n \geq 0} \), where \( D^\nu \delta \) means the \( n \)th derivative of Dirac delta in the origin.

We consider the isomorphism \( \mathcal{F} : \Delta \rightarrow P \) given as follows (see [16]).

For \( \mathcal{U} = \sum_{n \geq 0} \mathcal{U}_n \frac{(-1)^n}{n!} D^\nu \delta, \quad \mathcal{F}(\mathcal{U})(z) = \sum_{n \geq 0} \mathcal{U}_n z^n. \)

Then,

\[
S(\mathcal{U})(z) = -z^{-1}\mathcal{F}(\mathcal{U})(z^{-1}).
\]

We introduce

\[
\langle p\mathcal{U}, q \rangle = \langle \mathcal{U}, pq \rangle \quad \text{for every polynomial } q(z).
\]

Furthermore, we define

\[
(\mathcal{U}p)(z) = \sum_{m=0}^{n} \left( \sum_{j=m}^{n} a_j \mathcal{U}_{j-m} \right) z^m, \quad p(z) = \sum_{j=0}^{n} a_j z^j,
\]
and
\[(\theta_0 p)(z) = \frac{p(z) - p(0)}{z}.\]

Thus, \(S(pU)(z) = p(z)S(U)(z) + (\theta_0 p)(z),\) where \(p(z)\) is a polynomial.

We define the functional \(x^{-1}U\) and the product of two linear functionals in the following way:
\[\langle x^{-1}U, p \rangle = \langle U, \theta_0 p \rangle, \quad \langle UV, p \rangle = \langle U, V p \rangle.\]

Then it is straightforward to prove that
(i) \(x(x^{-1}U) = U,\)
(ii) \(x^{-1}(xU) = U - U_0 \delta,\)
(iii) \(x^{-2}(x^2U) = x^{-1}(x^{-1}U) = U - U_0 \delta + U_1 D \delta.\)

**Remark.** We shall use the \(z\) variable for all those equations where the Stieltjes function appears and the \(x\) variable in the rest of the equations.

**Definition 2.1.** A linear functional \(U\) is said to be a quasi-definite or regular (see [9]) functional if there exists a sequence of monic orthogonal polynomials (MOPS), \(\{P_n\}_{n \geq 0}\) with respect to \(U\), i.e., it satisfies
(i) \(P_n(x) = x^n + \text{lower degree terms}.\)
(ii) \(\langle U, P_n P_m \rangle = k_{nm}, k_n \neq 0, n = 0, 1, 2, \ldots .\)

A MOPS \(\{P_n\}_{n \geq 0}\) with respect to a quasi-definite linear functional satisfies the following three-term recurrence relation:
\[P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \quad n \geq 0,\]
\[P_0(x) = 1, \quad P_1(x) = x - \beta_0\]
with \(\gamma_n \neq 0, n \geq 0\) and \(\gamma_0 = 1 = \langle U, P_0^2 \rangle.\)

**Proposition 2.1.** A linear functional \(U\) is quasi-definite if and only if \(\Delta_n(U) = \det[U_{i+j}]_{i,j=0}^n \neq 0,\) for all \(n \geq 0.\)

**Definition 2.2.** Let \(\{P_n\}_{n \geq 0}\) be a MOPS with respect to a quasi-definite functional \(U\). The sequence \(\{P_n^{(1)}\}_{n \geq 0}\) defined by
\[P_n^{(1)}(x) = \langle U, \frac{P_{n+1}(x) - P_{n+1}(\zeta)}{x - \zeta} \rangle, \quad n \geq 0\]
is called the associated sequence of first kind for the sequence \(\{P_n\}_{n \geq 0}.\)

**Remark.** \(\langle U, \cdot \rangle\) means the action of \(U\) over the polynomial on the \(\zeta\)-variable.

We shall note by \(U^{(1)}\) the normalized functional, \(\|U_0^{(1)}\| = 1,\) such that the sequence \(\{P_n^{(1)}\}_{n \geq 0}\) is the corresponding MOPS.
Theorem 2.1. Let $\mathcal{U}$ be a linear functional. Then
\[ \gamma_1 \mathcal{U}(1) = -x^2 \mathcal{U}^{-1}. \]

In a general way, the associated sequence of $r$th kind $r = 1, 2, 3, \ldots, \{P_n^{(r)}\}_{n \geq 0}$ is defined by the recurrence relation
\[ P_{n+2}^{(r)}(x) = (x - \beta_{n+r+1})P_{n+1}^{(r)}(x) - \gamma_{n+r+1}P_n^{(r)}(x), \quad n \geq 0, \]
\[ P_1^{(r)}(x) = x - \beta_r, \quad P_0^{(r)}(x) = 1. \]

This corresponds to a shifted perturbation in the coefficients of the three-term recurrence relation.

We give the following previous results (see [8, 14, 16] for a more comprehensive recurrence approach).

Lemma 2.2. For $p, q \in P$ and for $\mathcal{U}, \mathcal{V} \in P'$, we have
(i) $x^{-1}(p \mathcal{U}) + \langle \mathcal{U}, \theta_0 p \rangle \delta = p(x^{-1} \mathcal{U}),$
(ii) $q(\mathcal{U} \theta_0 p) - \mathcal{U} \theta_0(q^p) = -\theta_0[(p \mathcal{U}) q],$
(iii) $\theta_0(\mathcal{L} p) = \mathcal{U}(\theta_0 p),$
(iv) $\mathcal{U}(pq) = (p \mathcal{U})q + xq(\mathcal{U} \theta_0 p),$
(v) $p(\mathcal{U} \mathcal{V}) = (p \mathcal{V}) \mathcal{U} + x(\mathcal{V} \theta_0 p) \mathcal{U}.$

In terms of the Stieltjes functions:

Lemma 2.2. For $p \in P$ and for $\mathcal{U}, \mathcal{V} \in P'$, we have
(i) $S'(\mathcal{U})(z) = S(D \mathcal{U})(z),$
(ii) $S(\mathcal{U} \mathcal{V})(z) = -zS(\mathcal{U})(z)S(\mathcal{V})(z),$
(iii) $S(x^{-1} \mathcal{U})(z) = (1/z)S(\mathcal{U})(z),$
(iv) $\frac{1}{z}(\mathcal{U} \theta_0 p)(z) = -\frac{1}{z}S(\langle \mathcal{U}, \theta_0 p \rangle \delta)(\frac{1}{z}) + (\mathcal{U} \theta_0^2 p)(z).$

3. The Laguerre–Hahn class

Definition 3.1 (Alaya [1]). A linear functional $\mathcal{U}$ on the linear space $P$ is said to be of the Laguerre–Hahn class if its Stieltjes function satisfies a Riccati equation
\[ \Phi(z)S'(\mathcal{U})(z) = B(z)S^2(\mathcal{U})(z) + C(z)S(\mathcal{U})(z) + D(z), \]
where $\Phi(z), B(z)$ and $C(z)$ are polynomials with complex coefficients, and
\[ D(z) = -[(D \mathcal{U}) \theta_0 \Phi](z) + (\mathcal{U} \theta_0 C)(z) - (\mathcal{U} \theta_0^2 B)(z), \quad \Phi(z) \neq 0. \tag{1} \]

Remark. When $B(z) = 0$, the Stieltjes function satisfies a linear differential equation $\Phi(z)S'(\mathcal{U})(z) = C(z)S(\mathcal{U})(z) + D(z)$ and the corresponding polynomials are called affine Laguerre–Hahn polynomials. More precisely, they are the semiclassical polynomials (see [5, 16]).

Definition 3.2. Let $\{P_n\}_{n \geq 0}$ be a MOPS with respect to a quasi-definite linear functional $\mathcal{U}$. $\{P_n\}_{n \geq 0}$ belongs to the Laguerre–Hahn class if $\mathcal{U}$ is a Laguerre–Hahn linear functional.
Theorem 3.1 (Bouakkaz [7], Dini and Maroni [10] and Maroni [16]). Let $\mathcal{U}$ be a quasi-definite and normalized linear functional $[\mathcal{U}]_0 = 1$ and let $\{P_n\}_{n \geq 0}$ be the corresponding MOPS. The following statements are equivalent:

(i) $\mathcal{U}$ is a Laguerre–Hahn functional.

(ii) $\mathcal{U}$ verifies the functional equation $D[\Phi \mathcal{U}] + \Psi \mathcal{U} + B(x^{-1}\mathcal{U}^2) = 0$, where $\Phi$, $\Psi$, and $C$ are the polynomials defined in (1), and

$$\Psi(x) = -[\Phi'(x) + C(x)].$$

(iii) $\mathcal{U}$ satisfies the functional equation $D[x\Phi \mathcal{U}] + (x\Psi - \Phi)\mathcal{U} + B\mathcal{U}^2 = 0$ with the additional condition $\langle \mathcal{U}, \Psi \rangle + \langle \mathcal{U}^2, 0_0B \rangle = 0$ where $\Phi, \Psi$ and $B$ are the polynomials defined in (ii).

(iv) Each polynomial $P_n$, $n \geq 0$, verifies the so-called structural relation

$$\Phi(x)P_{n+1}(x) - B(x)P^{(1)}_n(x) = \sum_{j=n-s}^{n+d} \delta_{n,j} P_{\mu}(x), \quad n \geq s + 1,$$

where $\Phi$ and $B$ are the polynomials defined in (i) and $\{P^{(1)}_n\}_{n \geq 0}$ is the sequence of associated orthogonal polynomials of first kind relative to $\{P_n\}_{n \geq 0}$, where $t = \deg \Phi$, $p = \deg \Psi \geq 1$, $r = \deg B$, $s = \max(p - 1, d - 2)$ and $d = \max(t, r)$.

3.1. Determination of the order of the class

In the characterization (ii), we must notice that there does not exist uniqueness in the representation. In fact, it is enough to multiply by any polynomial both sides of the equation. On the other hand, uniqueness is obtained by imposing a minimality condition as we will discuss below.

If $\mathcal{U}$ satisfies the equation $D[\Phi \mathcal{U}] + \Psi \mathcal{U} + B(x^{-1}\mathcal{U}^2) = 0$, multiplying it by a polynomial $q(x)$, $\mathcal{U}$ satisfies $D[\Phi^* \mathcal{U}] + \Psi^* \mathcal{U} + B^*(x^{-1}\mathcal{U}^2) = 0$, where $\Phi^* = q\Phi$, $\Psi^* = q\Psi - q'\Phi$ and $B^* = qB$.

We shall associate to $\mathcal{U}$ the set of nonnegative numbers $h(\mathcal{U}) = \{\max(p - 1, d - 2)\}$, being $d = \max(t, r)$, where $t = \deg \Phi^*$, $p = \deg \Psi^*$ and $r = \deg B^*$, among all the possible choices of $\Phi^*$, $\Psi^*$ and $B^*$ (whenever the quasi-definiteness is preserved).

Definition 3.2 (Bouakkaz [7]). The class of the Laguerre–Hahn functional $\mathcal{U}$ is the minimum of $h(\mathcal{U})$.

Theorem 3.2 (Marcellán and Prianes [14] and Prianes [18]). Let $\mathcal{U}$ be a quasi-definite linear functional verifying $D[\Phi \mathcal{U}] + \Psi \mathcal{U} + B(x^{-1}\mathcal{U}^2) = 0$ where $\Phi$, $\Psi$ and $B$ are the polynomials introduced in Theorem 3.1. We define $d = \max(t, r)$ and $s = \max(p - 1, d - 2)$. The Laguerre–Hahn functional $\mathcal{U}$ is said to be of class $s$ if and only if

$$\prod_{\alpha \in Z_\Phi} \{\langle \mathcal{U}, \Psi_\alpha \rangle + (\mathcal{U}^2, 0_0B_\alpha) \} \neq 0,$$

where $Z_\Phi$ is the set of zeros of $\Phi(x)$. The polynomials $\Phi_\alpha$, $\Psi_\alpha$ and $B_\alpha$ as well as the numbers $r_\alpha$ and $s_\alpha$ are defined by the expressions

$$\Phi(x) = (x - a)\Phi_\alpha(x),$$
$$\Psi(x) + \Phi_\alpha(x) = (x - a)\Psi_\alpha(x) + r_\alpha,$$
$$B(x) = (x - a)B_\alpha(x) + s_\alpha.$$
Let us establish an equivalent result to Theorem 3.2, where the condition about the class will be given in terms of the polynomials $B$, $C$ and $D$ defined in (1), using the Stieltjes function.

**Corollary 3.1.** Let $\mathcal{U}$ be a quasi-definite linear functional of Laguerre–Hahn class verifying (2). A necessary and sufficient condition for $\mathcal{U}$ to be of class $s$ is

$$\prod_{a \in \mathcal{E}_a} \{|B(a)| + |C(a)| + |D(a)|\} \neq 0$$

i.e., the polynomials $\Phi$, $B$, $C$, and $D$ are coprime.

### 4. Modification by a Dirac Delta

**Proposition 4.1** (Alaya [2], Marcellán and Prianes [13] and Prianes [18]). Let $\mathcal{U}$ be a Laguerre–Hahn linear functional and let $\mu$ and $c$ be complex arbitrary numbers with $\mu \neq 0$. Then, $\mathcal{U} = \mathcal{U} + \mu \delta_c$ is a Laguerre–Hahn linear functional.

**Proof.** Let $\mathcal{U}$ be a Laguerre–Hahn functional fulfilling (2). Substituting $\mathcal{U}$ by $\mathcal{U} - \mu \delta_c$ in this equation, we have

$$D(\Phi \mathcal{U}) + \Psi \mathcal{U} + B(x^{-1}\mathcal{U}^2) - \mu D(\Phi \delta_c) - \mu \Psi \delta_c - 2\mu B[x^{-1}(\mathcal{U} \delta_c)] + \mu^2 B(x^{-1} \delta_c^2) = 0.$$  \hfill (3)

On the other hand,

$$\langle B[x^{-1}(\mathcal{U} \delta_c)], p(x) \rangle = \langle \mathcal{U}, \theta_c(x \theta_c B) p \rangle = \langle B(x - c)^{-1} \mathcal{U}, p \rangle.$$  \hfill (4)

From this $B[x^{-1}(\mathcal{U} \delta_c)] = B(x - c)^{-1} \mathcal{U}$.

Analogously,

$$\langle B(x^{-1} \delta_c^2), p(x) \rangle = \lim_{x \to c} \left[ \frac{(B(x) p(x) - B(0) p(0))}{x - c} \right] = \langle B'(c) \delta_c - B(c) \delta'_c, p \rangle$$

from where

$$B(x^{-1} \delta_c^2) = B'(c) \delta_c - B(c) \delta'_c.$$  \hfill (5)

In a similar way, we obtain

$$D(\Phi \delta_c) = \Phi(c) \delta'_c \quad \text{and} \quad \Psi \delta_c = \Psi(c) \delta'_c.$$  \hfill (6)

Substituting (4)–(6) in (3),

$$D(\Phi \mathcal{U}) + [\Psi - 2\mu B(x - c)^{-1}] \mathcal{U} + B(x^{-1} \mathcal{U}^2) = \mu[\Phi(c) + \mu B(c)] \delta'_c + \mu[\Psi(c) - \mu B'(c)] \delta_c.$$  \hfill (7)

Multiplying (7) by $(x - c)$ and taking into account that $(x - c) \delta'_c = -\delta_c$ and $(x - c) \delta_c = 0,

$$D[(x - c) \Phi \mathcal{U}] + [(x - c) \Psi - \Phi - 2\mu B] \mathcal{U} + B(x - c)(x^{-1} \mathcal{U}^2) = -\mu[\Phi(c) + \mu B(c)] \delta_c.$$  \hfill (8)

We shall distinguish two situations,

(i) $\Phi(c) + \mu B(c) = 0$, $\mathcal{U}$ is a Laguerre–Hahn functional, fulfilling the equation

$$D[(x - c) \Phi \mathcal{U}] + [(x - c) \Psi - \Phi - 2\mu B] \mathcal{U} + B(x - c)(x^{-1} \mathcal{U}^2) = 0,$$
\( \Phi(c) + \mu B(c) \neq 0. \) Then, multiplying (8) by \((x - c)\)
\[ D[(x - c)^2 \Phi \mathcal{U}] + (x - c)[(x - c)\Psi - 2\Phi - 2\mu B]\mathcal{U} + (x - c)^2 B(x^{-1}\mathcal{U})^2 = 0 \]
and the proposition holds.

**Remark.** \( \delta, p(x) = 0, [xp(x)], \) (see [10]). The proof of the proposition can also be carried out with the help of the Stieltjes function in the following way:

Let \( S = S(\mathcal{U})(z) \) be the Stieltjes function corresponding to \( \mathcal{U} \). This function verifies
\[ \Phi(z)S' = B(z)S^2 + C(z)S + D(z). \] (9)

Let \( \overline{S} = S(\overline{\mathcal{U}})(z) \) be the corresponding Stieltjes function for \( \overline{\mathcal{U}} \).

From \( \langle \overline{\mathcal{U}}, x^n \rangle = \langle \mathcal{U} + \mu \delta, x^n \rangle = \mathcal{U}_n + \mu c^n, n \geq 0 \), it follows that \( S = \overline{S} + \mu/(z - c) \).

Substituting in (9),
\[ (z - c)^2 \Phi \overline{S} = (z - c)^2 BS\overline{S} + (z - c)[2\mu B + (z - c)C]\overline{S} \]
\[ + [\mu \Phi + \mu^2 B + \mu(z - c)C + (z - c)^2 D]. \] (10)

In Proposition 4.4, where we shall study the order of the class of the functional \( \overline{\mathcal{U}} \), we shall see if the condition \( \Phi(c) + \mu B(c) = 0 \) is fulfilled, Eq. (10) is reducible and we can divide by \((z - c)\). So \((z - c)\Phi \overline{S} = (z - c)BS\overline{S} + [2\mu B + (z - c)C]\overline{S} + [\mu \Phi + \mu^2 B + \mu C + (z - c)D] \) holds. This agrees with the results obtained in the proof of Proposition 4.1 using the functional equations.

### 4.1. Determination of the order of the class

In the following, let us suppose that \( \mathcal{U} \) is a Laguerre–Hahn linear functional of class \( s \).

**Proposition 4.2.** Let \( \mathcal{U} \) be a Laguerre–Hahn linear functional of class \( s \), and \( \overline{\mathcal{U}} = \mathcal{U} + \mu \delta \), \( \mu \neq 0 \). Then \( \overline{\mathcal{U}} \) is a Laguerre–Hahn linear functional of class \( \overline{s} \) such that \( s - 2 < \overline{s} < s + 2 \).

**Proof.** We shall use the following notation:
\[ \Phi(x) = \sum_{i=0}^{s+2} d_i x^i, \quad \Psi(x) = \sum_{i=0}^{s+1} c_i x^i \quad \text{and} \quad B(x) = \sum_{i=0}^{s+2} b_i x^i. \] (11)

Let \( D(\Phi^* \overline{\mathcal{U}}) + \Psi^* \overline{\mathcal{U}} + B^*(x^{-1}\overline{\mathcal{U}}^2) = 0 \) be the equation which fulfills \( \overline{\mathcal{U}} \) where
\[ \Phi^* = (x - c)^2 \Phi, \quad \Psi^* = (x - c)[(x - c)\Psi - 2\Phi - 2\mu B] \quad \text{and} \quad B^* = (x - c)^2 B. \] (12)

Then \( \deg \Phi^* = t^* \leq s + 4 \), \( \deg \Psi^* = p^* \leq s + 3 \), and \( \deg B^* = r^* \leq s + 4 \).

Thus \( d^* = \max(t^*, r^*) \leq s + 4 \) and \( \overline{s} = \max(p^* - 1, d^* - 2) \leq s + 2 \).

On the other hand, since \( \mathcal{U} = \overline{\mathcal{U}} - \mu \delta \), then \( s \leq \overline{s} + 2 \).

**Proposition 4.3.** Let \( \overline{\mathcal{U}} \) be a Laguerre–Hahn functional satisfying Eq. (12). Then for every zero \( a \) of \( \Phi^* \) different from \( c \), Eq. (12) is irreducible.
Proof. Following Theorem 3.2, we shall write

\[ \Phi^*_a(x) = (x - c)^2 \Phi_a(x), \quad r^*_a = (c - a)^2 r_a - 2\mu(a - c)s_a, \]
\[ \Psi^*_a(x) = (x - c)^2 \Psi_a(x) - 2(x - c) \Phi_a(x) - 2\mu(x - c)B_a(x) + (x + a - 2c)r_a - 2\mu s_a, \]
\[ B^*_a(x) = (x - c)^2 B_a(x) + (x + a - 2c)s_a, \quad s^*_a = (c - a)^2 s_a. \]

If \( s_a \neq 0 \) then \( s^*_a \neq 0 \) and (12) remains irreducible.
If \( s_a = 0 \), and \( r_a \neq 0 \), then \( r^*_a \neq 0 \) and (12) is irreducible.
If \( s_a = r_a = 0 \), then \( \langle \mathcal{W}, \Psi^*_a \rangle + \langle \mathcal{W^2}, \theta_0 B^*_a \rangle = (a - c)^2 \left[ \langle \mathcal{W}, \Psi_a \rangle + \langle \mathcal{W^2}, \theta_0 B_a \rangle \right] \neq 0 \) and the result follows. \( \square \)

Proposition 4.4. Given \( \mathcal{W} = \mathcal{U} + \mu \delta_c \), let \( \bar{s}, s \) be the class of \( \mathcal{W} \) and \( \mathcal{U} \), respectively. Then we have

\[ \Phi(c) + \mu B(c) \neq 0 \Rightarrow \bar{s} = s + 2, \]
\[ \Phi(c) + \mu B(c) = 0 \Rightarrow \begin{cases} B(c) \neq 0 \Rightarrow \bar{s} = s + 1, \\ B(c) = 0 \Rightarrow \begin{cases} \mu B'(c) + \Phi'(c) + C(c) \neq 0 \Rightarrow \bar{s} = s + 1, \\ \mu B'(c) + \Phi'(c) + C(c) = 0 \Rightarrow [1] \end{cases} \end{cases} \]
\[ [1] \Rightarrow \begin{cases} 2\mu B'(c) + C(c) \neq 0 \Rightarrow \bar{s} = s, \\ 2\mu B'(c) + C(c) = 0 \Rightarrow \begin{cases} \frac{1}{2} \mu^2 B''(c) + \frac{1}{2} \mu \Phi''(c) + \mu C'(c) + D(c) \neq 0 \Rightarrow \bar{s} = s, \\ \frac{1}{2} \mu^2 B''(c) + \frac{1}{2} \mu \Phi''(c) + \mu C'(c) + D(c) = 0 \Rightarrow [2] \end{cases} \end{cases} \]
\[ [2] \Rightarrow \begin{cases} \Phi'(c) \neq 0 \Rightarrow \bar{s} = s - 1, \\ \Phi'(c) = 0 \Rightarrow \begin{cases} \mu B'(c) + C'(c) \neq 0 \Rightarrow \bar{s} = s - 1, \\ \mu B'(c) + C'(c) = 0 \Rightarrow [3] \end{cases} \end{cases} \]
\[ [3] \Rightarrow \begin{cases} \frac{\mu^2}{3!} B'''(c) + \frac{\mu}{3!} \Phi'''(c) + \frac{\mu}{2} C'''(c) + D'(c) \neq 0 \Rightarrow \bar{s} = s - 1, \\ \frac{\mu^2}{3!} B'''(c) + \frac{\mu}{3!} \Phi'''(c) + \frac{\mu}{2} C'''(c) + D'(c) = 0 \Rightarrow \bar{s} = s - 2, \end{cases} \]

where the polynomials \( \Phi, B, C \) and \( D \) are defined in (1).

Proof. We shall use the following notation:

\[ \Phi_{c,t}(z) = (z - c)\Phi_{c,(t+1)}(z) + t \Phi_{c,(t+1)}, \quad \Phi_{c,0} = \Phi, \]
\[ B_{c,t}(z) = (z - c)B_{c,(t+1)}(z) + s_{c,(t+1)}, \quad B_{c,0} = B, \]
\[ C_{c,t}(z) = (z - c)C_{c,(t+1)}(z) + q_{c,(t+1)}, \quad C_{c,0} = C, \]
\[ D_{c,t}(z) = (z - c)D_{c,(t+1)}(z) + p_{c,(t+1)}, \quad D_{c,0} = D. \]

In general, \( \bar{s} = S(\mathcal{W})(z) \) satisfies Eq. (10) with \( \bar{s} = s + 2 \).
If $\Phi(c) + \mu B(c) = 0$ holds, the previous equation is divisible by $(z - c)$ and thus the order of the class of $\mathcal{U}$ decreases in one unit. In fact,

$$(z - c)\Phi S = (z - c)B S^2 + [2\mu B + (z - c)C]S + [\mu \Phi + \mu^2 B + \mu C + (z - c)D]$$

and then $\bar{s} = s + 1$. If $B(c) = 0$ and $[\Phi + \mu B + \mu C](c) = 0$ this is equivalent to $B(c) = 0$ and $\mu B'(c) + \Phi'(c) + C(c) = 0$. Thus the previous equation can be divided by $(z - c)$ and the class of $\mathcal{U}$ decreases.

$$\Phi S = B S^2 + [2\mu B + \mu C]S + [\mu \Phi + \mu^2 B + \mu C + D]$$

and then $\bar{s} = s$.

Assume $\Phi(c) = B(c) = 0$ (conditions already satisfied); $[2\mu B + C](c) = 0$ and

$$[\mu \Phi + \mu^2 B + \mu C + D](c) = 0 \iff 2\mu B'(c) + C(c) = 0$$

and

$$\frac{1}{2} \frac{d}{dc} B''(c) + \frac{1}{2} \frac{d}{dc} \Phi'(c) + \frac{d}{dc} = 0.$$

Then the class of $\mathcal{U}$ decreases.

$$\Phi S = B S^2 + [2\mu B + \mu C]S + [\mu \Phi + \mu^2 B + \mu C + D].$$

Finally,

$$\Phi(c) = 0,$$

$$B(c) = 0,$$

$$[2\mu B + C](c) = 0,$$

$$[\mu \Phi + \mu^2 B + \mu C + D](c) = 0,$$

then

$$\Phi'(c) = 0,$$

$$B'(c) = 0 \quad \text{(already required)}$$

$$\mu B''(c) + C'(c) = 0,$$

$$\frac{d}{dc} B'' + \frac{d}{dc} \Phi' + \frac{d}{dc} = 0.$$

$S$ satisfies

$$\Phi S = B S^2 + [2\mu B + C]S + [\mu \Phi + \mu^2 B + \mu C + D].$$

Thus, $\bar{s} = s - 2$.

This result is a more complete description than the presented in [1], Theorem 4.1.5.

4.2. Examples

In the next examples we shall describe the equations which $\mathcal{U} = \mathcal{U}(1) + \mu \delta_c$ satisfies where $\mathcal{U}(1)$ is the associated functional of the first kind for the classical polynomials, as well as the Riccati equations which the corresponding Stieltjes function, $\mathcal{S}(z) = S(\mathcal{U})(z)$, satisfies.

Because $\mathcal{U}(1)$ is a Laguerre–Hahn linear functional, it fulfils a distributional differential equation and its corresponding Stieltjes function a Riccati differential equation.
Table 1

<table>
<thead>
<tr>
<th>( \mathcal{U}^{(1)} )</th>
<th>( \Phi(z) )</th>
<th>( \Psi(z) )</th>
<th>( B(z) )</th>
<th>( C(z) )</th>
<th>( D(z) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hermite</td>
<td>( 1 )</td>
<td>( 2z )</td>
<td>( -1 )</td>
<td>( -2z )</td>
<td>( -2 )</td>
</tr>
<tr>
<td>Laguerre</td>
<td>( z )</td>
<td>( z - x - 3 )</td>
<td>( -x - 1 )</td>
<td>( -x + x + 2)</td>
<td>( -1 )</td>
</tr>
<tr>
<td>Jacobi</td>
<td>( z^2 - 1 )</td>
<td>( -(x + \beta + 4)z - \frac{z^2 - \beta^2}{x + \beta + 1} )</td>
<td>( \frac{4(\beta + 1)(x + \beta)}{(x + \beta)^2} )</td>
<td>( (x + \beta + 2)z - \frac{z^2 - \beta^2}{x + \beta + 1} )</td>
<td>( x + \beta + 3 )</td>
</tr>
<tr>
<td>Bessel</td>
<td>( z^2 )</td>
<td>( -2[(x + 1)z + 1 - x^{-1}] )</td>
<td>( \frac{-2z - 1}{x(2x + 1)} )</td>
<td>( 2(zx + 1 - x^{-1}) )</td>
<td>( 2x + 1 )</td>
</tr>
</tbody>
</table>

The polynomials \( \Phi, \Psi, B, C \) and \( D \) are obtained from the following table (according to Definition 3.1. and Theorem 3.1):

### 4.3. Hermite polynomials

Let \( \mathcal{U}^{(1)} \) be the linear functional corresponding to the associated polynomials of the first kind for the Hermite polynomials and \( \overline{\mathcal{U}} = \mathcal{U}^{(1)} + \mu \delta_c, \mu \neq 0 \).

(i) If \( \mu \neq 1 \), \( \Phi(c) + \mu B(c) \neq 0 \). \( \overline{\mathcal{U}} \) fulfills the equation
\[
D[(x - c)^2 \overline{\mathcal{U}}] + 2(x - c)[(x - c)x + \mu - 1] \overline{\mathcal{U}} - (x - c)^2(x^{-1} \overline{\mathcal{U}}^2) = 0.
\]

The class of the functional \( \overline{\mathcal{U}} \) is \( \overline{s} = 2 \) and \( \overline{S}(z) \) satisfies the equation
\[
(z - c)^2 \overline{S} = -(z - c)^2 \overline{S}^2 + [ - 2z^2 + 4cz^2 - 2(\mu + c^2)z + 2\mu c] \overline{S} + [ - 2(1 + \mu)z^2 + 2c(2 + \mu)z + \mu - \mu^2 - 2c^2].
\]

(ii) If \( \mu = 1 \), \( \Phi(c) + \mu B(c) = 0 \). Since \( B(c) \neq 0 \), \( \overline{\mathcal{U}} \) satisfies the equation
\[
D[(x - c) \overline{\mathcal{U}}] + [2(x - c)x + 1] \overline{\mathcal{U}} - (x - c)(x^{-1} \overline{\mathcal{U}}^2) = 0.
\]

The class of the functional \( \overline{\mathcal{U}} \) is \( \overline{s} = 1 \) and \( \overline{S}(z) \) verifies the equation
\[
(z - c) \overline{S} = -(z - c)^2 \overline{S}^2 - 2[(z - c)z + 1] \overline{S} - 2(2z + c).
\]

### 4.4. Laguerre polynomials

Let \( \mathcal{U}^{(1)} \) be the linear functional corresponding to the associated polynomials of the first kind for the Laguerre polynomials and \( \overline{\mathcal{U}} = \mathcal{U}^{(1)} + \mu \delta_c, \mu \neq 0 \).

(i) If \( c \neq \mu(z + 1) \) then \( \Phi(c) + \mu B(c) \neq 0 \). \( \overline{\mathcal{U}} \) fulfills the equation
\[
D[(x - c)^2 \overline{\mathcal{U}}] + (x - c)[(x - c)(x - 3) + 2\mu(x + 1) - 2x] \overline{\mathcal{U}} - (x + 1)(x - c)^2(x^{-1} \overline{\mathcal{U}}^2) = 0.
\]

The class of the functional \( \overline{\mathcal{U}} \) is \( \overline{s} = 2 \) and \( \overline{S}(z) \) satisfies the equation
\[
z(z - c)^2 \overline{S} = -(z + 1)(z - c)^2 \overline{S}^2 + (z - c)[ - 2\mu(z + 1) + (-z + x + 2)(z - c)] \overline{S} + [\mu z - \mu^2(z + 1) + \mu(z - c)(-z + x + 2) - (z - c)^2].
\]
(ii) If \( c = \mu(x + 1) \), \( \Phi(c) + \mu B(c) = 0 \), but \( B(c) = -(x + 1) \neq 0 \), \( (x \neq -1) \). Then \( \mathcal{W} \) fulfills the equation

\[
D[(x-c)^2\mathcal{W}] + [2c - (x-c)(-x + x + 2)]\mathcal{W} - (x + 1)(x-c)(x^{-1}\mathcal{W}) = 0.
\]

The class of the functional \( \mathcal{W} \) is \( \bar{s} = 1 \) and \( \bar{S}(z) \) satisfies the equation

\[
z(z-c)\bar{S} = -(z + 1)(z - c)\bar{S}^2 + [(z - c)(-z + x + 2) - 2c]\bar{S}
+ \left[ \frac{3c}{x+1} + \frac{-z + x}{x + 1} \right] (z - c).
\]

4.5. Jacobi polynomials

Let \( \mathcal{W}^{(1)} \) be the linear functional corresponding to the associated polynomials of the first kind for the Jacobi polynomials and \( \mathcal{W} = \mathcal{W}^{(1)} + \mu \delta_c, \mu \neq 0 \).

(i) If

\[
\mu \neq \frac{(1 - c^2)(x + \beta + 3)(x + \beta + 2)^2}{4(x + 1)(\beta + 1)(x + \beta + 1)},
\]
then \( \Phi(c) + \mu B(c) \neq 0 \) and \( \bar{s} = 2 \). \( \bar{S} \) and \( \mathcal{W} \) satisfy, respectively,

\[
(z - c)^2(z^2 - 1)\bar{S} = (z - c)^2 \frac{4(x + 1)(\beta + 1)(x + \beta + 1)}{(x + \beta + 3)(x + \beta + 2)^2} \bar{S}^2 + \mu \frac{4(x + 1)(\beta + 1)(x + \beta + 1)}{(x + \beta + 3)(x + \beta + 2)^2}
\]

\[
+ (z - c) \left[ 2\mu \frac{4(x + 1)(\beta + 1)(x + \beta + 1)}{(z - c)(x + \beta + 3)(x + \beta + 2)^2} \right] \bar{S}
\]

\[
+ (z - c) \left[ (x + \beta + 2)z - \frac{x^2 - \beta^2}{(x + \beta + 2)} \right] \bar{S}
\]

\[
+ \mu(z^2 - 1) + \mu(z - c) \left[ (x + \beta + 2)z - \frac{x^2 - \beta^2}{(x + \beta + 2)} \right]
\]

\[
+ (z - c)^2(x + \beta + 3),
\]

\[
D[(x-c)^2(x^2 - 1)\mathcal{W}] + (x - c) \frac{4(x + 1)(\beta + 1)(x + \beta + 1)}{(x + \beta + 3)(x + \beta + 2)^2} (x^{-1}\mathcal{W})
\]

\[
+ (x - c) \left[ (x - c) \frac{-(x + \beta + 4)x + \frac{x^2 - \beta^2}{(x + \beta + 2)}}{2} \mathcal{W} - (x^2 - 1) \right] \mathcal{W} = 0.
\]

(ii) If

\[
\mu = \frac{(1 - c^2)(x + \beta + 3)(x + \beta + 2)^2}{4(x + 1)(\beta + 1)(x + \beta + 1)},
\]
then \( \Phi(c) + \mu B(c) = 0 \) begin \( B(c) \neq 0 \) \( (x + \beta = -1 \) leads to the semiclassical case), and \( \bar{s} = 1 \). \( \bar{S} \) and \( \mathcal{W} \) satisfy, respectively,

\[
(z - c)^2(z^2 - 1)\bar{S} = (z - c) \frac{4(x + 1)(\beta + 1)(x + \beta + 1)}{(x + \beta + 3)(x + \beta + 2)^2} \bar{S}^2 + (z - c)^2(x + \beta + 3)
\]
\[
\begin{align*}
&+ \frac{(1 - c^2)(\alpha + \beta + 3)(\alpha + \beta + 2)^2}{4(\alpha + 1)(\beta + 1)(\alpha + \beta + 1)} (z - c) \left( (\alpha + \beta + 2)z - \frac{x^2 - \beta^2}{x + \beta + 2} \right) \\
&+ \left[ 2\mu \frac{4(\alpha + 1)(\beta + 1)(\alpha + \beta + 1)}{(\alpha + \beta + 3)(\alpha + \beta + 2)^2} \\ &+ (z - c) \left( (\alpha + \beta + 2)z - \frac{x^2 - \beta^2}{x + \beta + 2} \right) \right] \mathcal{S} \\
&+ (z + c) \frac{(1 - c^2)(\alpha + \beta + 3)(\alpha + \beta + 2)^2}{4(\alpha + 1)(\beta + 1)(\alpha + \beta + 1)},
\end{align*}
\]

\[
D[(x - c)(x^2 - 1)\mathcal{W}] + (x - c) \frac{4(\alpha + 1)(\beta + 1)(\alpha + \beta + 1)}{(\alpha + \beta + 3)(\alpha + \beta + 2)^2} (x^{-1}\mathcal{W}^2) 
- \left[ 3x^2 - 2cx + 2c^2 - 3 - (x - c) \left( (\alpha + \beta + 2)x - \frac{x^2 - \beta^2}{x + \beta + 2} \right) \right] \mathcal{W} = 0.
\]

### 4.6. Bessel polynomials

Let \(\mathcal{W}^{(1)}\) be the linear functional corresponding to the associated polynomials of the first kind for the Jacobi polynomials and \(\mathcal{W} = \mathcal{W}^{(1)} + \mu \delta_c, \mu \neq 0\).

(i) If
\[
\mu \neq \frac{c^2x^2(2x + 1)}{2x - 1}
\]
then \(\Phi(c) + \mu B(c) \neq 0\) and \(\bar{s} = 2\). \(\mathcal{S}\) and \(\mathcal{W}\) fulfil
\[
z^2(z - c)^2\mathcal{S} = (z - c) \left[ -2\mu \frac{(1 - 2x)}{x^2(2x + 1)} + 2(z - c)(x + 1 - x^{-1}) \right] \mathcal{S} \\
+ \mu \mathcal{S}^2 + \mu^2 \frac{(1 - 2x)}{x^2(2x + 1)} + 2\mu(z - c)(x + 1 - x^{-1}) + (z - c)^2(2x + 1) \\
+ (z - c)^2 \frac{(1 - 2x)}{x^2(2x + 1)} \mathcal{S}^2,
\]

\[
D[x^2(x - c)^2\mathcal{W}] + (x - c) \left[ -2(x - c)((\alpha + 1)x + 1 - x^{-1}) - 2\mu \frac{(2x - 1)}{x^2(2x + 1)} - 2x^2 \right] \mathcal{W} \\
+ (x - c)^2 \frac{(1 - 2x)}{x^2(2x + 1)} (x^{-1}\mathcal{W}^2) = 0.
\]

(ii) If \(\mu = c^2x^2(2x + 1)/(2x - 1)\), then \(\Phi(c) + \mu B(c) = 0\) being \(B(c) \neq 0\) (\(x = \frac{1}{2}\) leads to the semiclassical case), and \(\mathcal{S} = 1\). \(\mathcal{S}\) and \(\mathcal{W}\) satisfy, respectively,
\[
z(z - c)^2\mathcal{S} = (z - c) \frac{(1 - 2x)}{x^2(2x + 1)} \mathcal{S}^2 + [ -2c^2 + 2(z - c)(x + 1 - x^{-1})] \mathcal{S} \\
+ \left[ (z + c) \frac{c^2x^2(2x + 1)}{(2x - 1)} + 2 \frac{c^2x^2(2x + 1)}{(2x - 1)} (x + 1 - x^{-1}) + (z - c)(2x + 1) \right],
\]

\[
\mathcal{W} = \mu \delta_c + \mathcal{W}^{(1)}.
\]
\[ D[x(x-c)\mathcal{U}] + [-2x - c + 2c^2 - 2(x-c)(x+1)x+1-x^{-1})] \mathcal{U} + \{(x-c)(1-2x)\}\mathcal{U} = 0. \]

5. Study of the functional \( \mathcal{U} \) such that \((x-c)\mathcal{U} = \mu \psi, \mu \text{ and } c \in C, \) where the \( \psi \) is a Laguerre–Hahn functional

Proposition 5.1. Let \( \mathcal{U} \) and \( \psi \) be two linear functionals related by \((x-c)\mathcal{U} = \mu \psi \). Then if \( \psi \) is a Laguerre–Hahn functional, \( \mathcal{U} \) is also a Laguerre–Hahn functional and conversely.

Proof. Let \( \psi \) be a Laguerre–Hahn functional such that the corresponding Stieltjes function \( S = S(\psi)(z) \) satisfies the equation

\[ \Phi S' = B S^2 + C S + D. \]  \hspace{1cm} (13)

Let \( S(z) = S(\mathcal{U})(z) \) be the Stieltjes function relative to the functional \( \mathcal{U} \). From

\[ \mu \psi_n = \mu \langle \psi, x^n \rangle = \mathcal{U}_{n+1} - c \mathcal{U}_n \]

we get \( S(z) = (1/\mu)[(z-c)S(z) + 1] \). Substituting in (13) we obtain

\[ (z-c)\Phi S' = (z-c)^2 B S^2 + [-\mu \Phi + 2(z-c)B + \mu(z-c)C]S + [B + \mu C + \mu^2 D]. \]  \hspace{1cm} (14)

Moreover, \( \mathcal{U} \) satisfies the equation

\[ D[\mu(x-c)\Phi \mathcal{U}] + (x-c)[\mu \psi - 2B] \mathcal{U} + (x-c)^2 B(x^{-1} \mathcal{U}^2) = 0 \]  \hspace{1cm} (15)

with \( \psi = -\Phi' - C \).

Conversely, from the relation between \( S \) and \( S \) we deduce that if \( \mathcal{U} \) is a Laguerre–Hahn functional, i.e., \( \mathcal{U} \) verifies the Riccati equation

\[ \Phi \cdot S = B S^2 + C \mathcal{S} + D, \]

then \( \psi = (1/\mu)(x-c)\mathcal{U} \) is also a Laguerre–Hahn functional and for the corresponding Stieltjes function \( S = S(\psi)(z) \) the equation

\[ \mu(z-c)\Phi S' = \mu^2 B S^2 + [-2\mu B + \mu(z-c)C + \mu \Phi] S + [\Phi + B - (z-c)C + (z-c)^2 D], \]

holds.

Furthermore, the functional \( \mathcal{U} \) satisfies the distributional equation

\[ D[\mu(x-c)\Phi \mathcal{U}] + \mu[2B + (x-c)C] \psi + \mu^2 B(x^{-1} \psi^2) = 0 \]  \hspace{1cm} (16)

with \( \psi = -(\Phi' + C) \).

5.1. Determination of the order of the class

In the following, we will assume that \( \psi \) is a Laguerre–Hahn functional of class \( s \).

Proposition 5.2. Let \( \psi \) be a Laguerre–Hahn linear functional of class \( s \) and \((x-c)\mathcal{U} = \mu \psi \). Then \( \mathcal{U} \) is a Laguerre–Hahn linear functional of class \( s \), such that \( s - 1 \leq \mathcal{S} \leq s + 2 \).
Proof. Let $\Phi, \Psi$ and $B$ be as in Proposition 4.2 and let $D(\Phi^* \Psi) + \Psi^* \Psi + B(x^{-1} \Psi^2) = 0$ be the equation which fulfills $\Psi$ where $\Phi^* = \mu(x-c)\Phi, \Psi^* = (x-c)\mu\Psi - 2B$ and $B^* = (x-c)^2B$ according to Eq. (15). Thus,
\[
\deg \Phi^* = t^* \leq s + 3, \quad \deg \Psi^* = p^* \leq s + 3, \quad \deg B^* = r^* \leq s + 4
\]
and
\[
d^* = \max(t^*, r^*) \leq s + 4 \quad \text{and} \quad \overline{s} = \max(p^* - 1, d^* - 2) \leq s + 2.
\]
On the other hand, if $\Psi$ is a Laguerre–Hahn functional of class $\overline{s}$ such that
\[
D(\Phi^* \Psi) + \Psi^* \Psi + B(x^{-1} \Psi^2) = 0, \quad \Psi = -(\Phi' + C)
\]
then $\Psi$ satisfies the equation
\[
D(\Phi^* \Psi) + \Psi^* \Psi + B(x^{-1} \Psi^2) = 0, \quad \Phi = \mu(x-c)\Phi, \quad \Psi = \mu(2B + (x-c)\Psi), \quad B = \mu^2B,
\]
and $\deg \Phi = t \leq \overline{s} + 3; \deg \Psi = p \leq \overline{s} + 2; \deg B^* = r \leq \overline{s} + 2$.
Thus $d = \max(t, r) \leq s + 3$ and $s = \max(p - 1, d - 2) \leq \overline{s} + 1$.

Proposition 5.3. Let $\Psi$ be a Laguerre–Hahn functional satisfying Eq. (15). For every zero $a$ of $\Phi^* = \mu(x-c)\Phi$ different from $c$, Eq. (14) is irreducible.

Proof. Since $\Psi$ is a Laguerre–Hahn functional of class $s$, $S(\Psi)(z)$ satisfies Eq. (13), where the polynomials $\Phi, B, C$ and $D$ are coprime. Let $\Phi^*$ and $B^*$ as in the Proposition 5.2 and
\[
C^*(z) = -\mu \Phi + 2(z-c)B + \mu(z-c)C \quad \text{and} \quad D^*(z) = B + \mu C + \mu^2 D.
\]
(i) If $B(a) \neq 0$ then $B^*(a) \neq 0$.
(ii) If $B(a) = 0$ and $C(a) \neq 0$, then $C^*(a) \neq 0$.
(iii) If $B(a) = C(a) = 0$, then $D^*(a) \neq 0$ from where $|B^*(a)| + |C^*| + |D^*(a)| \neq 0$.

Proposition 5.4. Let $(x-c)\Psi = \mu \Psi'$ and let $\overline{s}$ and $s$ be the class of $\Psi$ and $\Psi'$, respectively. Then $\Phi(c) \neq 0 \Rightarrow \overline{s} = s + 2$.

$\Phi(c) = 0 \Rightarrow$
\[
\{ B(c) + \mu C(c) + \mu^2 D(c) \neq 0 \Rightarrow \overline{s} = s + 2, \quad \} \quad \text{[1]}
\]
\[
\{ B(c) + \mu C(c) + \mu^2 D(c) = 0 \Rightarrow \} \quad \text{[1]}
\]
\[
\begin{align*}
-\mu \Phi'(c) + 2B(c) + \mu C(c) \neq 0 & \Rightarrow \overline{s} = s + 1, \\
-\mu \Phi'(c) + 2B(c) + \mu C(c) = 0 & \Rightarrow \}
\end{align*}
\]
\[
\text{[2]}
\]
\[
\begin{align*}
\Phi'(c) \neq 0 & \Rightarrow \overline{s} = s, \\
\} \quad \text{[2]}
\end{align*}
\]
\[
\begin{align*}
B(c) \neq 0 & \Rightarrow \overline{s} = s, \\
B(c) = 0 & \Rightarrow \}
\end{align*}
\]
\[
\text{[3]}
\]
\[
\begin{align*}
B'(c) + \mu C'(c) + \mu^2 D'(c) \neq 0 & \Rightarrow \overline{s} = s, \\
B'(c) + \mu C'(c) + \mu^2 D'(c) = 0 & \Rightarrow \}
\end{align*}
\]
where the polynomials $\Phi, B, C$ and $D$ are defined in (13).

**Proof.** We will use the same notation as in Proposition 4.4.

If $\Phi(c) \neq 0$, $\overline{S}(z) = S(\mathcal{U})(z)$ satisfies Eq. (14). Then $\overline{s} = s + 2$.

If $\Phi(c) = 0$ and $B(c) + \mu C(c) + \mu^2 D(c) = 0$, the previous equation is divisible by $(z - c)$. Thus,

$$\mu \overline{S} = (z - c)BS^2 + \left[-\mu \Phi(c) + 2B + \mu C\right]S + \left[2c,1 + \mu C,1 + \mu^2 D,1\right]$$

and $\overline{s} = s + 1$.

If $-\mu \Phi(c) + 2B(c) + \mu C(c) = 0$ and $B(c) + \mu C(c) + \mu^2 D(c) = 0$, then

$$-\mu \Phi'(c) + 2B(c) + \mu C(c) = 0$$

and $B'(c) + \mu C'(c) + \mu^2 D'(c) = 0$.

Dividing by $(z - c)$,

$$\mu \Phi(c)S = BS^2 + \left[-\mu \Phi(c) + 2B + \mu C\right]S + \left[2c,1 + \mu C,1 + \mu^2 D,1\right]$$

and then $\overline{s} = s$.

If \begin{align*}
\Phi(c) & = 0 \Rightarrow \Phi'(c) = 0, \\
B(c) & = 0,
\end{align*}

we can divide again by $(z - c)$. So,

$$\mu \Phi(c)S = Bc,1S^2 + \left[-\mu \Phi(c) + 2B + \mu C\right]S + \left[2c,1 + \mu C,1 + \mu^2 D,1\right]$$

and $\overline{s} = s - 1$.

This result gives a more descriptive analysis than that presented in [1], Lemma 4.2.6.

5.2. Examples

In these examples we shall describe the equations which $\mathcal{U}$ satisfies, when $(x - c)\mathcal{U} = \mu \mathcal{V}$ and $\mathcal{V}$ is the associated functional of the first kind for the classical polynomials, as well as the Riccati equation which satisfies the corresponding Stieltjes function $\mathcal{S}(z) = S(\mathcal{U})(z)$.

Because $\mathcal{V}$ is a Laguerre–Hahn functional, it fulfils a distributional differential equation and $S(z) = S(\mathcal{U}(1))(z)$, a Riccati equation.

The polynomials $\Phi, \mathcal{V}, B, C$ and $D$ are obtained from the Table 1.

5.3. Hermite polynomials

$$\Phi(c) \neq 0 \Rightarrow \mathcal{U}$$

fulfils the equation

$$D[\mu(x - c)\mathcal{U}] + 2(x - c)(\mu x + 1)\mathcal{U} - (x - c)^2(x^{-1}\mathcal{U}^2) = 0.$$ 

The class $\overline{s}$ of the linear functional $\mathcal{U}$ is $\overline{s} = 2$. $\overline{S}(z)$ satisfies the equation

$$\mu(z - c)\overline{S} = -(z - c)^2\overline{S}^2 - [\mu + 2(z - c)(\mu z + 1)]\overline{S} - [1 + 2\mu z + 2\mu^2].$$ 

5.4. Laguerre polynomials

(i) If $c \neq 0$, $\Phi(c) \neq 0$ and $\overline{s} = 2$. $\overline{\mathcal{U}}$ and $\overline{S}(z)$ satisfy, respectively,

$$D[\mu x(x - c)\mathcal{U}] + (x - c)[\mu(x - x - 3) + 2(x + 1)]\mathcal{U} - (x + 1)(x - c)^2(x^{-1}\mathcal{U}^2) = 0.$$ 

(ii)
\[ \mu(z - c)S' = -(\alpha + 1)(z - c)^2S^2 - [\mu z + 2(\alpha + 1)(z - c) + \mu(z - \alpha - 2)(z - c)]S' \\
+ [- \mu^2 - (\alpha + 1)\mu(-z + \alpha + 2)]. \]

(ii) If \( c = 0 \) and \( B(c) + \mu C(c) + \mu^2 D(c) = -(\alpha + 1) + \mu(\alpha + 2) - \mu^2 \neq 0 \), then \( s = 2 \). Moreover,
\[
D[\mu z^2\mathcal{U}] + x[\mu(x - x - 3) + 2(\alpha + 1)]\mathcal{U} - (\alpha + 1)x^2(x^{-1}\mathcal{U}^2) = 0,
\]
\[
\mu z^2 S' = -(\alpha + 1)z^2 S^2 + [ - \mu z - 2(\alpha + 1)z + \mu z(-z + \alpha + 2)]S' \\
+ [- (\alpha + 1)\mu(-z + \alpha + 2) - \mu^2].
\]

(iii) If \( c = 0 \), \( B(c) + \mu C(c) + \mu^2 D(c) = -(\alpha + 1) + \mu(\alpha + 2) - \mu^2 = 0 \) and
\[ -\mu \Phi'(c) + 2B(c) + \mu C(c) = -\mu - 2(\alpha + 1) + \mu(\alpha + 2) \neq 0. \]
Thus \( s = 1 \). Furthermore,
\[
D[\mu x \mathcal{U}] + [2(\alpha + 1) - \mu(-x + \alpha + 2)]\mathcal{U} - (\alpha + 1)x(x^{-1}\mathcal{U}^2) = 0,
\]
\[
\mu z S' = -(\alpha + 1)z S^2 + [ - \mu - 2(\alpha + 1) + \mu(-z + \alpha + 2)]S - \mu.
\]

(iv) If \( c = 0 \), \( B(c) + \mu C(c) + \mu^2 D(c) = -(\alpha + 1) + \mu(\alpha + 2) - \mu^2 = 0 \)
\[ -\mu \Phi'(c) + 2B(c) + \mu C(c) = -\mu - 2(\alpha + 1) + \mu(\alpha + 2) = 0,
\]
\[ B'(c) + \mu C'(c) + \mu^2 D'(c) = -\mu \neq 0. \]
Then \( s = 1. \) (\( \mu = 0 \) is excluded because of the quasi-definiteness condition). \( \mathcal{U} \) and \( S(z) \) satisfy, respectively,
\[
D[\mu x \mathcal{U}] + \mu(x - 1)\mathcal{U} - (x + 1)x(x^{-1}\mathcal{U}^2) = 0,
\]
\[
\mu z S' = -(\alpha + 1)z S^2 - \mu z S - \mu.
\]

5.5. Jacobi polynomials

(i) If \( c \neq \pm 1, \Phi(c) = c^2 - 1 \neq 0 \). Then \( s = 2 \). Moreover,
\[
D[\mu(x - c)^2(x^2 - 1)\mathcal{U}] + (x - c) \left[ \mu \left[-(\alpha + \beta + 4)x + \frac{x^2 - \beta^2}{(x + \beta + 2)} \right] \\
- \frac{8(\alpha + 1)(\beta + 1)(x + \beta + 1)}{(x + \beta + 3)(x + \beta + 2)^2} \right] \\
+ (x - c)^2 \Delta(\alpha + 1)(\beta + 1)(x + \beta + 1)(x^{-1}\mathcal{U}^2) = 0,
\]
\[
\mu(z - c)(z^2 - 1)S = (z - c)^2 \frac{4(\alpha + 1)(\beta + 1)(\alpha + \beta + 1)}{(\alpha + \beta + 3)(\alpha + \beta + 2)^2} S^2 + \frac{4(\alpha + 1)(\beta + 1)(\alpha + \beta + 1)}{(\alpha + \beta + 3)(\alpha + \beta + 2)^2} (z - c)^2 + \mu(z - c) \frac{8(\alpha + 1)(\beta + 1)(\alpha + \beta + 1)}{(\alpha + \beta + 3)(\alpha + \beta + 2)^2}
\]
\[
+ \left[ -\mu(z^2 - 1) + (z - c) \frac{8(\alpha + 1)(\beta + 1)(\alpha + \beta + 1)}{(\alpha + \beta + 3)(\alpha + \beta + 2)^2} \right] S
\]
\[
+ \mu(z - c) \left[ (\alpha + \beta + 2)z - \frac{\alpha^2 - \beta^2}{(\alpha + \beta + 2)} \right] S
\]
\[
+ \mu \left[ (\alpha + \beta + 2)z - \frac{\alpha^2 - \beta^2}{(\alpha + \beta + 2)} \right] + \mu^2(\alpha + \beta + 3),
\]

(ii) If \(c = \pm 1\) and

\[
B(c) + \mu C(c) + \mu^2 D(c) = \frac{4(\alpha + 1)(\beta + 1)(\alpha + \beta + 1)}{(\alpha + \beta + 3)(\alpha + \beta + 2)^2}
\]
\[
+ \mu \left[ (\alpha + \beta + 2)(\pm 1) - \frac{\alpha^2 - \beta^2}{(\alpha + \beta + 2)} \right] + \mu^2(\alpha + \beta + 3) \neq 0.
\]

Then \(\bar{s} = 2\).

\[
D[\mu(x - (\pm 1))(x^2 - 1)U] + (x - (\pm 1))^2 \frac{4(\alpha + 1)(\beta + 1)(\alpha + \beta + 1)}{(\alpha + \beta + 3)(\alpha + \beta + 2)^2} (x^{-1}U^2)
\]
\[
- \left[ 2(x - (\pm 1)) \frac{4(\alpha + 1)(\beta + 1)(\alpha + \beta + 1)}{(\alpha + \beta + 3)(\alpha + \beta + 2)^2} + \mu(x - (\pm 1)) \right]
\]
\[
\times \left[ (\alpha + \beta + 4)x - \frac{\alpha^2 - \beta^2}{(\alpha + \beta + 2)} \right] U = 0,
\]

\[
\mu(z - (\pm 1))(z^2 - 1)S = (z - (\pm 1))^2 \frac{4(\alpha + 1)(\beta + 1)(\alpha + \beta + 1)}{(\alpha + \beta + 3)(\alpha + \beta + 2)^2} S^2
\]
\[
+ \frac{4(\alpha + 1)(\beta + 1)(\alpha + \beta + 1)}{(\alpha + \beta + 3)(\alpha + \beta + 2)^2}
\]
\[
+ \mu(z - (\pm 1)) \left[ -\frac{(z^2 - 1)}{(z - (\pm 1))} + \frac{8(\alpha + 1)(\beta + 1)(\alpha + \beta + 1)}{(\alpha + \beta + 3)(\alpha + \beta + 2)^2} \right]
\]
\[
+ \left[ (\alpha + \beta + 2)z - \frac{\alpha^2 - \beta^2}{(\alpha + \beta + 2)} \right] S
\]
\[
+ \mu \left[ (\alpha + \beta + 2)z - \frac{\alpha^2 - \beta^2}{(\alpha + \beta + 2)} \right] + \mu^2(\alpha + \beta + 3).
\]
(iii) If $c = \pm 1$, $B(c) + \mu C(c) + \mu^2 D(c) = 0$ and

$$-\mu \Phi'(c) + 2B(c) + \mu C(c) = -2(\pm 1)\mu + \frac{8(x + 1)(\beta + 1)(x + \beta + 1)}{(x + \beta + 3)(x + \beta + 2)^2} + \mu \left[(x + \beta + 2)(\pm 1) - \frac{x^2 - \beta^2}{(x + \beta + 2)}\right] \neq 0.$$ 

Then $s = 1$.

Furthermore,

$$D[\mu(x^2 - 1)\Psi] + \frac{8(x + 1)(\beta + 1)(x + \beta + 1)}{(x + \beta + 3)(x + \beta + 2)^2} + \mu \left[-(x + \beta - 3)x + \frac{x^2 - \beta^2}{(x + \beta + 2)} \pm 1\right] \Psi$$

$$+ (x - (\pm 1)) \frac{4(x + 1)(\beta + 1)(x + \beta + 1)}{(x + \beta + 3)(x + \beta + 2)^2} (x^{-1} \Psi^2) = 0,$$

$$\mu(x^2 - 1)S = (z - (\pm 1)) \frac{4(x + 1)(\beta + 1)(x + \beta + 1)}{(x + \beta + 3)(x + \beta + 2)^2} S + \frac{8(x + 1)(\beta + 1)(x + \beta + 1)}{(x + \beta + 3)(x + \beta + 2)^2} S + \mu(x + \beta + 2).$$

(iv) If $c = \pm 1$, $B(c) + \mu C(c) + \mu^2 D(c) = 0$, $-\mu \Phi'(c) + 2B(c) + \mu C(c) = 0$, and $B'(c) + \mu C'(c) + \mu^2 D'(c) = 0(\pm 1)$. Then $s = 1$.

$$D[\mu(x^2 - 1)\Psi] - \mu[2x + (x + \beta + 1)(x - (\pm 1))] \Psi$$

$$+ (x - (\pm 1)) \frac{4(x + 1)(\beta + 1)(x + \beta + 1)}{(x + \beta + 3)(x + \beta + 2)^2} (x^{-1} \Psi^2) = 0,$$

$$\mu(x^2 - 1)S = (z - (\pm 1)) \frac{4(x + 1)(\beta + 1)(x + \beta + 1)}{(x + \beta + 3)(x + \beta + 2)^2} S$$

$$+ \mu(x + \beta + 1)(z - (\pm 1)S + \mu(x + \beta + 2).$$

5.6. Bessel polynomials

(i) If $c \neq 0$, then $\Phi(c) \neq 0$ and $s = 2$.

$$D[\mu z^2(x - c)\Psi] - 2(x - c) \left[\frac{2(x - 1)}{x^2(2x + 1)} + \mu[(x + 1)x + 1 - x^{-1}]\right] \Psi$$

$$- (x - c)^2 \frac{2(x - 1)}{x^2(2x + 1)} (x^{-1} \Psi^2) = 0,$$

$$\mu z^2(z - c)S = -(z - c)^2 \frac{2(x - 1)}{x^2(2x + 1)} S^2 + \left[2\mu(xz + 1 - x^{-1}) + \mu^2(2x + 1) - \frac{2(x - 1)}{x^2(2x + 1)}\right]$$

$$+ \left[-\mu z^2 - 2(z - c) \frac{2(x - 1)}{x^2(2x + 1)} + 2\mu(z - c)(xz + 1 - x^{-1})\right] S.$$
(ii) If $c = 0$ and $B(c) + \mu C(c) + \mu^2 D(c) = (-2(\alpha - 1)/x^2(2\alpha + 1)) + 2\mu(1 - x^{-1}) + \mu^2(2\alpha + 1) \neq 0$, then $\bar{s} = 2$.

\[
D[\mu x^3 \mathcal{U}] - 2x \left[ \frac{2(\alpha - 1)}{x^2(2\alpha + 1)} + \mu [(\alpha + 1)x + 1 - x^{-1}] \right] \mathcal{U} - x^2 \frac{2(\alpha - 1)}{x^2(2\alpha + 1)} (x^{-1} \mathcal{U}^2) = 0,
\]

\[
\mu^2 \mathcal{S} = -z^2 \frac{2(\alpha - 1)}{x^2(2\alpha + 1)} \mathcal{S}^2 + \left[ 2\mu z(xz + 1 - x^{-1}) - \mu z^2 - 2z^2 \frac{2(\alpha - 1)}{x^2(2\alpha + 1)} \right] \mathcal{S}
+ \left[ -2(\alpha - 1) + 2\mu (xz + 1 - x^{-1}) + \mu^2(2\alpha + 1) \right].
\]

(iii) If $c = 0$, $B(c) + \mu C(c) + \mu^2 D(c) = 0$, and $-\mu \Phi'(c) + 2B(c) + \mu C(c) = -2(2(\alpha - 1)/x^2(2\alpha + 1)) + 2\mu(1 - x^{-1}) \neq 0$, thus $\bar{s} = 1$,

\[
D[\mu x^2 \mathcal{U}] - \frac{-4(\alpha - 1)}{x^2(2\alpha + 1)} \mathcal{U} - x \frac{2(\alpha - 1)}{x^2(2\alpha + 1)} (x^{-1} \mathcal{U}^2) = 0,
\]

\[
\mu^2 \mathcal{S} = -z \frac{2(\alpha - 1)}{x^2(2\alpha + 1)} \mathcal{S}^2 + \left[ 2\mu (xz + 1 - x^{-1}) - \mu z - \frac{4(\alpha - 1)}{x^2(2\alpha + 1)} \right] \mathcal{S} + 2\mu z.
\]

(iv) If $c = 0$, $B(c) + \mu C(c) + \mu^2 D(c) = 0$, $-\mu \Phi'(c) + 2B(c) + \mu C(c) = 0$, and $B'(c) + \mu C'(c) + \mu^2 D'(c) = 2\mu z \neq 0$, then $\bar{s} = 1$,

\[
D[\mu x^2 \mathcal{U}] - [\mu x(2\alpha + 1)] \mathcal{U} - x \frac{2(\alpha - 1)}{x^2(2\alpha + 1)} (x^{-1} \mathcal{U}^2) = 0,
\]

\[
\mu^2 \mathcal{S} = -z \frac{2(\alpha - 1)}{x^2(2\alpha + 1)} \mathcal{S}^2 + \mu z(2\alpha + 1) \mathcal{S} + 2\mu z.
\]

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