Convergence regions with bounded convex complements
for continued fractions $K(1/b_n)$

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Dedicated to Haakon Waadeland on the occasion of his 70th birthday

Abstract

In this paper we resurrect some twin convergence region results of Thron from the 1940s for continued fractions $K(1/b_n)$ and derive from them best simple convergence region results for these continued fractions. Our principal contribution is an elementary proof of what we call the Uniform Circle Theorem. This theorem says that a certain one-parameter family of regions which are complements of open disks containing the origin is a family of best uniform convergence regions for continued fractions $K(1/b_n)$ and, moreover, it contains a sharp useful estimate for the speed of convergence. We apply this theorem to obtain a new convergence result for variable element continued fractions of Stieltjes type, which we call the Limacon Theorem. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

In a sequence of two papers [10,11] in the 1940s, Thron studied twin convergence region problems for continued fractions (c.f.s)

$$\frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3} + \cdots.$$ (1.1)

We have been aware of the impressive results in these two papers for a long time and have often felt that the very general characterization lemma and convergence theorem in [11], especially, were worthy of further study. Time and page length restrictions for the preparation of this manuscript have influenced us to concentrate our efforts on simple convergence region results that follow from, or are fostered by, Thron’s cited twin convergence region papers. Computer graphics tools were a valuable aid in our investigations of the behavior of the boundaries of some of the convergence
regions that can be derived from the results in [11]. In order to make ourselves more understandable throughout the remainder of this paper, we proceed now to give some definitions. By a region we shall mean an open connected set in the extended complex plane plus a subset (possibly empty) of its boundary. A region $E$ (a pair of regions $E_1$ and $E_2$) is (are) called a simple convergence region (twin convergence regions) for continued fractions (1.1) if conditions (a) and (b) ((a) and (c)) below

\begin{align}
(1.2)\quad & (a) \quad \sum |b_n| = \infty, \\
& (b) \quad b_n \in E, \\
& (c) \quad b_{2n-1} \in E_1, \quad b_{2n} \in E_2, \quad n \geq 1,
\end{align}

when satisfied, ensure the convergence of (1.1). A simple convergence region $E$ for c.f.s (1.1) is called best if there does not exist a simple convergence region $E'$ for these c.f.s such that $E \subset E'$ and $E \neq E'$. Twin convergence regions $E_1$ and $E_2$ for c.f.s (1.1) are said to be best if there does not exist twin convergence regions $E'_1$ and $E'_2$ such that $E'_1 \subset E_1$, $E'_2 \subset E_2$, where at least one of the containments is proper. A simple convergence region $E$ (twin convergence regions $E_1$ and $E_2$) is (are) said to be uniform if there exists a sequence $\{f_n\}$ of positive numbers depending only on $E$ (or $E_1$ and $E_2$) with $\lim f_n = f$ is finite and

\[ |f_n - f| \leq \varepsilon_n, \quad n = 1, 2, \ldots, \]

where $f_n$ is the $n$th approximant of (1.1).

Thron was concerned with pairs of regions $E_1$, $E_2$ defined by

\begin{align}
(1.3)\quad & z = re^{i\theta} \in E_1 \quad \text{if} \quad r \geq f(\theta), \quad 0 < m_1 < f(\theta) < m_2, \\
& z = re^{i\theta} \in E_2 \quad \text{if} \quad r \geq g(\theta), \quad 0 < m_3 < g(\theta) < m_4,
\end{align}

where $f(\theta)$ and $g(\theta)$ are periodic continuous functions of period $2\pi$ and the $m_i$ are constants. He proved in [10] that

A necessary condition for $E_1$ and $E_2$ to be twin convergence regions for c.f.s of the form (1.1) is that

\[ f(\theta)g(\pi - \theta) \geq 4 \]

for all real $\theta$.

Thus he was naturally led to his investigation of regions whose defining functions $f$ and $g$ satisfy

\[ f(\theta)g(\pi - \theta) \equiv 4. \quad (1.4) \]

Here we wish to see what Thron’s results in [10,11] lead us to when we set $f = g$, $m_1 = m_3$, and $m_2 = m_4$ in (1.3) and (1.4). For a given continuous periodic function $f : \mathbb{R} \rightarrow \mathbb{R}$ of period $2\pi$ with the property that there exist constants $m_1$ and $m_2$ such that $0 < m_1 < f(\theta) < m_2$ for all $\theta \in \mathbb{R}$, let us define the region $E[f]$ associated with $f$ by

\[ E[f] = \{z = re^{i\theta} : r \geq f(\theta)\}. \quad (1.5) \]

The important value region result in [10] allows us to deduce the following lemma:
Lemma 1.1. If the complement $C \setminus E[f]$ of a region $E[f]$ given by (1.5) is convex and
\[ f(\theta)f(\pi - \theta) \equiv 4, \] (1.6)
then $f_n \in 2/E[f]$ for all $n \geq 1$ if $b_n \in E[f]$ for all $n \geq 1$, where
\[ f_n = \frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_n} \]
is the $n$th approximant of (1.1).

Thron’s characterization Lemma 2.1 in [11] with an added correction and modified to fit the simple
region case becomes:

Lemma 1.2. There exist constants $m_1$ and $m_2$ and a continuous $2\pi$-periodic function $f: \mathbb{R} \to \mathbb{R}$
such that
\[ 0 < m_1 < f(\theta) < m_2, \quad f(\theta)f(\pi - \theta) = 4 \]
for all $\theta \in \mathbb{R}$, and $C \setminus E[f]$ is convex if and only if
\[ f(\theta) = 2 \exp \int_0^\theta \tan \alpha(\psi) \, d\psi, \]
where $\alpha: \mathbb{R} \to \mathbb{R}$ is a continuous $2\pi$-periodic function that satisfies the conditions
\[ |\alpha(\theta)| < \frac{\pi}{2} - \epsilon, \quad |\alpha(\theta) - \alpha(\phi)| \leq |\theta - \phi|, \] (1.7)
\[ \alpha(\pi - \theta) = \alpha(\theta), \quad \int_{-\pi}^{\pi} \tan \alpha(\psi) \, d\psi = 0 \] (1.8)
for some fixed $\epsilon > 0$.

The integral condition in (1.8) was not present in Thron’s original version of this lemma, but it
is easily seen that it needs to be there in order for $f$ to satisfy the desired periodicity condition.
For example, the derived function $f$ is not periodic of period $2\pi$ if
\[ \alpha(\psi) = \arctan(1 + \sin \psi). \]
The first condition in (1.8) is there to guarantee that $f$ has property (1.6).

2. Best simple convergence regions

The main convergence theorem in [11], modified to fit our setting and with the integral periodicity
condition added by us, follows.
Theorem 2.1. Let \( z(\theta) \) be a continuous function of period \( 2\pi \) which satisfies the conditions
\[
|z(\theta)| < \frac{\pi}{2} - \varepsilon, \quad |z(\theta) - z(\phi)| < (1 - \eta)|\theta - \phi|, \quad \theta \neq \phi
\]  
(2.1)
\[
z(\pi - \theta) = z(\theta), \quad \int_{-\pi}^{\pi} \tan z(\psi) \, d\psi = 0,
\]  
(2.2)
where \( \varepsilon > 0 \) and \( \eta > 0 \). Let
\[
f(\theta) := 2 \exp \int_{\pi/2}^{\theta} \tan z(\psi) \, d\psi.
\]  
(2.3)
Then the region \( E[f] \) defined by
\[
E[f] = \{ z = re^{i\theta} : r \geq f(\theta), \ \theta \in \mathbb{R} \}
\]
is a best simple convergence region for continued fractions (1.1).

The function-theoretic proof given by Thron for this theorem depends on a very sophisticated application of the Stieltjes-Vitali convergence extension theorem. We give the following corollary of Theorem 2.1 to show some interesting and specific families of best simple convergence regions for the c.f.s under consideration.

Corollary 2.2. For each fixed real number \( c \) satisfying the conditions indicated below let
\[
B_k[c] = \{ z = re^{i\theta} : r \geq f_k(c,t), \ 0 \leq t \leq 2\pi \}, \quad k = 0,1,\ldots,6,
\]
where
\[
f_0(c,t) = 2 \exp(\text{Arcsinh}(-c \cos t)), \quad -\infty < c < \infty,
\]
\[
f_1(c,t) = 2 \exp(-c \cos t), \quad -1 < c < 1,
\]
\[
f_2(c,t) = 2 \exp(-c(\cos^3 t)/3), \quad -1 < c < 1,
\]
\[
f_3(c,t) = 2 \exp(\text{Arccsc}(-c \cos t)), \quad -1/\sqrt{2} < c < 1/\sqrt{2},
\]
\[
f_4(c,t) = 2 \exp(\text{Arctanh}(-c \cos t)), \quad -(-1 + \sqrt{5})/2 < c < (-1 + \sqrt{5})/2,
\]
\[
f_5(c,t) = 2 \exp(\text{Arctan}(-c(\cos^3 t)/3)), \quad -(9 - 3\sqrt{5})/2 < c < (9 - 3\sqrt{5})/2,
\]
\[
f_6(c,t) = 2 \exp(\text{Arcsinh}(-c \text{Arctan}((\cos t)/\sqrt{c}))), \quad 0 < c < \infty.
\]
Then the regions \( B_k[c] \), \( k = 0,1,\ldots,6 \), are best simple convergence regions for c.f.s (1.1).

The most tedious part of justifying this corollary is in showing that the Lipschitz condition in (2.9) in Theorem 2.1 is satisfied for the indicated choices of the \( c \)'s in each case. The following graphical array is intended to illustrate the boundary behavior of the variety of best convergence regions for c.f.s (1.1) that are given in Corollary 2.2 (see Fig. 1).

The dotted curves are the circular boundaries of the regions \( B_0[a] \), where each \( a \) is chosen so that the boundary of \( B_0[a] \) is tangent to the boundary of the corresponding \( B_k[c] \) at its intersection points with the real axis. From this array and the many other examples we have considered it is
easy to derive the notion that the regions under consideration with convex noncircular holes cannot "deviate" much from well chosen ones with circular holes. The bestness property, however, does guarantee that the one type cannot be embedded in the other. A corollary given in [11] of the main theorem in that work and one that follows from Theorem 2.1 above is
Corollary 2.3. Let $c$ be an arbitrary real number. Then the continued fraction (1.1) converges if for all $n \geq 1$

$$|b_n + 2c| \geq 2\sqrt{1+c^2}.$$ 

In the original statement of this corollary in [11] the $c$ was replaced by $-c/2$. It is not difficult to verify that the region determined by $c$ in Corollary 2.3 is the same as the region $B_0[c]$ in Corollary 2.2. Our recent attempts to learn more about the convergence behavior of Stieltjes type and other variable element continued fractions and our related attempts to decide whether or not the regions in Theorem 2.1 are uniform convergence regions have led to our renewed interest in Corollary 2.3. This corollary is also stated in [4, p. 92], and there it is pointed out on p. 98 that it is also a corollary of a 1965 result of Hillam and Thron (Theorem 2 in [3], Theorem 4.37 in [4], or Theorem 12 in [7]). The limit point-limit circle method of proof used to prove the latter theorem is not conducive to establishing uniform convergence or to estimating the speed of convergence. Through our attempts to make a thorough search of the literature for continued fraction results that might relate to our work here, we discovered a claim by Sweezy and Thron in the Introduction of [9] that Hillam [2] gave an elementary proof of Corollary 2.3 in his 1962 Ph.D. thesis. However, it was pointed out in [9] that certain crucial estimates in Hillam’s proof did not lend well to the study of the speed of convergence of the continued fractions involved. Though we have since obtained a copy of [2], we had not seen the mentioned proof therein (which has not been published elsewhere) prior to our writing of this paper, its scrutiny by referees, and its acceptance for publication. We have found also that Sweezy and Thron [9] and Field and Jones [1] were able to obtain truncation error estimates for the convergence of c.f.s (1.1) for cases where its elements are confined to certain circle bounded subregions of a typical region in Corollary 2.3, bounded away from the boundary of the containing region. Our investigations of late have convinced us that Corollary 2.3 deserves a higher status than it has been given in the past, especially because of the potential of the uniform version of it for studying the convergence behavior of continued fractions with variable elements. Thus we upgrade it to a theorem in the next section and give a proof of our own that clearly establishes that the convergence regions involved are uniform convergence regions and provides a sharp useful truncation error formula.

3. Uniform circle theorem and Stieltjes fractions

Before we state and prove our main results of this section, we shall need some additional notation. Let us define

$$t_n(z) = \frac{1}{b_n + z},$$

and

$$T_1(z) = t_1(z), \quad T_n(z) = T_{n-1}(t_n(z)), \quad n \geq 2. \quad (3.2)$$

Then if we denote the $n$th numerator, denominator, and approximant of (1.1) by $A_n$, $B_n$, and $f_n$, respectively, we can write

$$\frac{A_n}{B_n} = f_n = T_n(0) = \frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_n}, \quad (3.3)$$
where in general
\[ T_n(z) = \frac{A_{n-1}z + A_n}{B_{n-1}z + B_n}. \] (3.4)

The quantities \( A_n \) and \( B_n \) satisfy the following recurrence relations and determinant formula:
\[ A_0 = 0, \quad A_1 = 1, \quad A_n = b_n A_{n-1} + A_{n-2}, \quad n \geq 2 \]
\[ B_0 = 1, \quad B_1 = b_1, \quad B_n = b_n B_{n-1} + B_{n-2} \]
\[ A_{n-1}B_n - A_nB_{n-1} = (-1)^n. \] (3.5)

The following theorem should prove to be very useful, especially because of the geometric simplicity of its hypotheses and its conclusions which lend well to numerical and function theoretic investigations. It shows that the best convergence regions of Corollary 2.3 are actually uniform convergence regions and it gives an estimate for the speed of convergence that cannot be obtained from the method of proof used in [11] or from the proof given for the Hillam–Thron Theorem. It is noteworthy to us that not all best convergence regions for c.f.s are uniform convergence regions. Thron [12, p. 340] proved that the regions defined by
\[ |a_{2n-1}| \leq r, \quad |a_{2n}| \geq 2(r - \cos \arg a_{2n}), \quad r > 1 \]
are best twin convergence regions for c.f.s \( K(a_n/1) \) which are not uniform. It suffices to change the condition for \( a_{2n} \) to
\[ |a_{2n}| \geq 2(r - \cos \arg a_{2n}) + \epsilon, \quad \epsilon > 0 \]
to obtain uniformity.

**Theorem 3.1** (Uniform Circle Theorem). Let \( c \) be a real number, \( r = \sqrt{1 + c^2} \), and \( B[c] \) be the closed unbounded region defined by
\[ B[c] := \{ z : |z + 2c| \geq 2r \} \]
\[ = \{ z : |z| \geq 2 \exp(Arcsinh(-c \cos \arg z)) \}. \] (3.6)

Then \( B[c] \) is a best uniform simple convergence region for continued fractions
\[ \frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3} + \cdots . \] (3.7)

Furthermore, if \( f_n \) and \( f \) denote the \( n \)th approximant and limit of (3.7), respectively, then, for all \( n \geq 0 \),
\[ |f - c| \leq r, \quad |f_{n+1} - f| \leq 2r \left( \frac{2n + 4 + \sigma}{4 + \sigma} \right)^{-\sigma}, \quad \sigma = 1 - \frac{|c|}{r}. \] (3.8)

**Proof.** As part of our proof we will use the following fundamental lemmas. \( \Box \)

**Lemma 3.2.** Let \( \{ B_n \} \) be the sequence of denominators of the continued fraction (3.7) with \( b_n \in B[c] \) for all \( n \geq 1 \), and let \( N[a, \rho] \) denote the closed disk with center \( a \) and radius \( \rho \). Then,
\[ u_n := \frac{B_{n+1}}{B_n} \in N \left[ \frac{nc}{n+1}, \frac{nr}{n+1} \right], \quad n \geq 1. \] (3.9)
Proof. Let \( k_n := n/(n+1) \) and w.l.o.g. let \( c \geq 0 \). The underlying reason why we need only consider \( c \geq 0 \) in our arguments is that
\[
K(1(-b_n)) = -K(1/(b_n))
\]
by an equivalence transformation. Our proof will be by induction. It is easy to verify that
\[
\begin{align*}
K(1((-\frac{1}{b_1})) = -K(1/(b_1)) = -K(1/(b_1 + \frac{1}{b_1})).
\end{align*}
\]
Assume that \( u_n \in N[k_n, k_{n+1}] \) and define
\[
M[a, \rho] := \{ z : |z - a| > \rho \}.
\]
Then it follows that
\[
\begin{align*}
b_{n+1} + u_n \in M((-2 + k_n)c, (2 - k_n)r).
\end{align*}
\]
Since
\[
\begin{align*}
u_{n+1} = 1/(b_{n+1} + u_n)
\end{align*}
\]
by properties of the sequence \( \{B_n\} \), it follows easily from basic properties of the linear fractional mapping \( w = 1/z \) applied to the circle bounded region \( M((-2 + k_n)c, (2 - k_n)r) \) that
\[
\begin{align*}
u_{n+1} \in N \left[ \frac{(-2 + k_n)c}{(-2 + k_n)c^2 + (2 - k_n)r}, \frac{(2 - k_n)r}{(2 - k_n)c^2 + (2 - k_n)r} \right] = N[k_{n+1}, k_{n+1} + 1]
\end{align*}
\]
and our proof is complete. \( \square \)

The following product estimate is a modification of a similar estimate that we gave in [5] and a bit sharper for our upcoming application than the one that can be derived from a result in [6].

Lemma 3.3. If
\[
P_a = \prod_{k=1}^{n} \left( 1 - \frac{a}{k+b} \right),
\]
where \( 0 < a < 1 + b \), then
\[
P_a \leq \left( \frac{n + 1 + b}{1 + b} \right)^{-a}, \quad n \geq 0.
\]

Proof. The proof of this lemma follows easily from the valid relations below under our conditions on \( a, b \) and \( k \)
\[
\begin{align*}
1 - \frac{a}{k+b} & = \left( 1 + \frac{1}{k+b} \right)^{-a} - a(1 + a) \int_{0}^{1/(k+b)} \left( 1 - t + \frac{1}{k+b} \right)^{-a-2} t \, dt \\
& \leq \left( 1 + \frac{1}{k+b} \right)^{-a}
\end{align*}
\]
after computing the product from 1 to \( n \) on both sides of the last inequality.

Now we are ready to prove the theorem. Let
\[
h_n := \frac{B_n}{B_{n-1}} = \frac{1}{u_n}.
\]
From (3.4) we derive
\[ T_n(z) = \frac{A_{n-1}}{B_{n-1}} + \frac{(-1)^{n-1}}{B_{n-1}^2(z + h_n)}. \]  
(3.10)

Employing (3.10) we obtain
\[ T_n(N[c, r]) = \frac{A_{n-1}}{B_{n-1}} + N \left[ \frac{(-1)^{n-1}(c + h_n)}{B_{n-1}^2(|c + h_n|^2 - r^2)}, \frac{r}{|B_{n-1}^2(|c + h_n|^2 - r^2)|} \right]. \]  
(3.11)

Let \( R_n \) denote the radius of the disk \( T_n(N[c, r]) \). Then from (3.11) we have that
\[ R_n = \frac{r}{|B_{n-1}^2(|c + h_n|^2 - r^2)|}. \]  
(3.12)

Using (3.12) we obtain
\[ H_n := \frac{R_n}{R_{n+1}} = \frac{|u_n|^2(|c + h_n|^2 - r^2)}{|c + h_n|^2 - r^2} = \frac{|cu_n + 1|^2 - |u_n|^2r^2}{|c + b_{n+1} + u_n|^2 - r^2}. \]  
(3.13)

By Lemma 3.2, \( u_n \in N[k_n c, k_n r] \). So, \( u_n = k_n c + rdK_a \exp(is) \) for some \( d \) and \( s \) satisfying \( 0 \leq d \leq 1 \) and \( 0 \leq s \leq 2\pi \). For convenience of notation in our upcoming calculations, let us set \( k_n = a \) for the time being. Also, \( b_{n+1} \) can be expressed as
\[ b_{n+1} = -2c + 2br \exp(it), \quad b \geq 1, \quad 0 \leq t \leq 2\pi. \]

After substituting for \( u_n \) and \( b_{n+1} \) in (3.13), we have
\[ H_n = \frac{|1 + ac^2 + acrd \exp(is)|^2 - r^2|ac + ard \exp(is)|^2}{|c(1 - a) + 2br \exp(it) + ard \exp(is)|^2 - r^2}. \]  
(3.14)

We set
\[ w = | - c(1 - a) + ard \exp(is)| \]  
(3.15)
and use the easily verified relations
\[ | - c(1 - a) + 2br \exp(it) + ard \exp(is)| \geq |2br - | - c(1 - a) + ard \exp(is)|| \geq r + (r - c)(1 - a) \]
and (3.14) to obtain
\[ H_n \leq \frac{|1 + ac^2 + acrd \exp(is)|^2 - r^2|ac + ard \exp(is)|^2}{(2br - | - c(1 - a) + ard \exp(is)|)^2 - r^2} \]
\[ \leq \frac{|1 + ac^2 + acrd \exp(is)|^2 - r^2|ac + ard \exp(is)|^2}{(2r - | - c(1 - a) + ard \exp(is)|)^2 - r^2} \]
\[ = \frac{r + w}{3r - w}. \]  
(3.16)
It is quite clear that the function of \( w \) in (3.16) is an increasing function of \( w \) under our constraints, so we have from (3.15) and (3.16) that
\[
H_n \leq \frac{r + \max w}{3r - \max w} = \frac{r + c(1 - a) + ar}{3r - c(1 - a) - ar} = \frac{n + (r + c)/(2r)}{n + (3r - c)/(2r)} = Q_n,
\]
where
\[
Q_n = 1 - \frac{\sigma}{n + 1 + \sigma/2}, \quad \sigma = 1 - c/r.
\]

Thus by choosing \( a = \sigma \) and \( b = 1 + \sigma/2 \) in Lemma 3.3 it follows from the lemma and (3.18) that
\[
\prod_{k=1}^{n} Q_k \leq \left( \frac{2n + 4 + \sigma}{4 + \sigma} \right)^{-\sigma}.
\]

Using (3.19), the fact that \( R_1 \leq r \), and inequality (3.17) we arrive at
\[
R_{n+1} = R_1 \prod_{k=1}^{n} H_k \leq r \prod_{k=1}^{n} Q_k \leq r \left( \frac{2n + 4 + \sigma}{4 + \sigma} \right)^{-\sigma},
\]
where
\[
\sigma = 1 - c/r, \quad (c \geq 0).
\]

Thus we can say that
\[
|f_{n+1} - f| \leq 2r \left( \frac{2n + 4 + \sigma}{4 + \sigma} \right)^{-\sigma}, \quad n \geq 0,
\]
which not only shows that \( B[c] \) is a uniform convergence region but also gives us, because of the nature of our proof, a sharp estimate of the speed of convergence of c.f. (1.1). We have assumed that \( c \geq 0 \) throughout our proof, but by the same reasoning that we gave in the proof of Lemma 3.2 we could also have chosen \( c \) to be negative. In this case the \( c \) in the formula for \( \sigma \) gets replaced by \( |c| \). This completes our proof of the Uniform Circle Theorem.

**Theorem 3.4 (Limacon Theorem).** Let
\[
\frac{1}{b_1z} + \frac{1}{b_2} + \frac{1}{b_3z} + \frac{1}{b_4} + \cdots
\]
be a Stieltjes type continued fraction such that

\[ b_n \geq b > 0, \quad n = 1, 2, 3, \ldots, \]  

and let \( c \geq 0 \) be a real number and \( r = \sqrt{1 + c^2} \). The c.f. (3.23) converges uniformly to a function \( f(z) \) satisfying

\[ |\sqrt{z} f(z) - c| \leq r \]

for all \( z \in B(b, c) = \{ z = \rho e^{it}; \ \rho \geq g(t) \} \), where

\[ g(t) = (4/b^2) \exp(2 \text{Arcsinh}(-c \cos(t/2))), \quad -\pi \leq t \leq \pi. \]  

is the polar equation of the inner loop of a limacon. The \( n \)th approximant \( f_n(z) \) of (3.23) satisfies

\[ |(f_{n+1}(z) - f(z))\sqrt{z}| \leq 2r \left( \frac{2n + 4 + \sigma}{4 + \sigma} \right)^{-\sigma}, \quad \sigma = 1 - c/r, \ n \geq 0. \]  

The function \( f \) is analytic on \( \text{Int} B(b, c) \) for each real \( c \geq 0 \). Furthermore, since the loop (3.25) shrinks to the closed interval \([-4/b^2, 0]\) as \( c \to +\infty \), it follows that \( f \) is actually analytic on the cut plane \( \mathbb{C} \setminus [-4/b^2, 0] \).

\textbf{Proof.} Here we give a brief sketch as to why Theorem 3.4 is essentially a corollary of Theorem 3.1. In what follows, \( \sqrt{z} \) stands for that branch of \( \sqrt{z} \) whose values lie in the closed right half-plane. For \( z \in B(b, c) \) the c.f.

\[ f(z) = \frac{1}{\sqrt{z}} \sum_{n=1}^{\infty} \frac{1}{b_n \sqrt{z}} \]  

is equivalent to the c.f. (3.23) and defines the function \( f \) where it converges. It is not difficult to see that Theorem 3.1 guarantees the uniform convergence of (3.27) if \( \sqrt{z} \in \Omega \), where

\[ \Omega = \{ z: \Re (z) \geq 0 \} \cap \{ z: \Re (z + 2c/b) \geq 2r/b \}. \]
It is well known that a circle containing the origin in its interior maps into a limacon with an inner loop under the mapping \( w = z^2 \). So using this fact it can be readily verified that the circular arc in the right half-plane of the boundary of \( \Omega \) maps into the loop defined by (3.25) under \( w \). Also, \( w \) takes each ray in \( \Omega \) on the imaginary axis into the interval \((-\infty, 4(r-c)^2/b^2]\). Thus (1.1) converges uniformly on compact subsets of \( B(b,c) \) and its approximants are analytic functions of \( z \), so \( f \) is analytic on \( \text{Int } B(b,c) \). We close with a plot illustrating the boundary behavior of \( B(b,c) \) (see Fig. 2).

4. For further reading

[8,13]

References

[12] W.J. Thron, Zwillingkonvergenzbiete für Kettenbrüche \( 1 + K(a_n/1) \), deren eines die Kreisscheibe \( |a_{n-1}| \leq \rho^2 \) ist, Math. Z. 70 (1959) 310–344.