On the boundary integral formulation of the plane theory of
elasticity with applications (analytical aspects)

M.S. Abou-Dina*, A.F. Ghaleb
Department of Mathematics, Faculty of Science, Cairo University, Giza, Egypt

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Abstract

A formulation of the plane strain problem of the theory of elasticity in stresses, for simply connected domains, is
carried out in terms of real functions within the frame of what is known as the boundary integral method. Special
attention is devoted to the problem of determination of the arbitrary constants appearing in the solution, in view of work
in progress where numerical techniques are used. Relying on some mathematical results formulated in the appendix, simple
applications concerning the first and the second fundamental problems for the circle and for the ellipse are given, which
show the correctness of the formulation and the necessity of recurring to numerical techniques, once the geometry of
the problem or the type of boundary conditions deviates from being simple. Following parts of the present work are
devoted to the numerical treatment of the obtained system of equations, as well as to the theories of thermoelasticity and
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1. Introduction

The plane problem of linear elasticity has been the subject of extensive investigations and the
permanent focus of attention of researchers, as an alternative or indirect approach to tackle
the more realistic three-dimensional cases. A large class of problems has found its solution within
the framework of this theory using various techniques, a thorough review of which is available in the
celebrated book of Muskhelishvili [1]. Although powerful, the analytical methods become extremely
tedious to handle once the boundary of the elastic medium departs from being of simple geometrical
shape, in which case the semi-analytical and the numerical methods seem to be the only remaining
alternative. In some approaches, known as boundary element techniques, the problem is first reduced
to that of determining certain unknown functions on the boundary instead of solving directly in

* Corresponding author.

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the bulk, then using these results to solve the problem completely. Such boundary methods usually lead to boundary integral equations, the solution of which, together with the so-called fundamental solutions, provide the complete solution of the problem in closed form [2–5]. In the complex variables formulation of the problem, one gets the so-called Cauchy-type integrals. Formulation in terms of real variables is also possible, and this is the context within which the present work should be situated.

The method explained hereafter may be considered as an extension of a boundary integral technique proposed by one of the authors (M.S.A.) to solve boundary-value problems for electromagnetic current sheets involving Laplace’s equation [6]. It also complements the approach proposed by Constanda [7] to solve problems of pure elasticity in displacements. The present method is well adapted to numerical schemes and has proved its efficiency in handling some problems of electromagnetism with complex geometries, otherwise quite difficult to deal with analytically.

Relying on the well-known complex representation of the general solution of the plane theory of linear elasticity for the plane deformation of isotropic media and on the representation of harmonic functions by boundary integrals, but dealing exclusively with real functions, an algorithm is developed for the determination of Airy’s stress function by solving a system of four boundary integral relations, from which the complete solution of the problem can be obtained. As an illustration, the method is applied to solve the two fundamental problems of the theory of elasticity for the circle and for the ellipse. However simple these problems may be, their solution allows to test the method and to discuss some of its characteristic features. The obtained results in these applications clearly indicate the complexity of the analytical approach for nonsimple geometries of the boundary and, therefore, the necessity of recurring to numerical techniques. Forthcoming publications are intended to deal with the numerical aspects of the method and to generalize it to plane problems of thermoelasticity and of thermo-electromagnetoelectricity.

The present paper consists of six sections and an appendix. Section 1 is the introduction. Section 2 is devoted to the formulation of the problem and to the basic equations. Section 3 contains a discussion of the degree of arbitrariness of the solution. Sections 4 and 5 deal with the two fundamental problems for the circle and for the ellipse, respectively. Section 6 contains the conclusions. The appendix includes certain mathematical relations and proofs justifying the expansions used throughout the text.

2. Formulation of the problem and basic equations

This section contains well-known results, formulated in terms of real functions, in a convenient way for later use.

Let D be a two-dimensional, simply connected region occupied by an elastic medium and bounded by a closed contour C, at each point of which the unit outwards normal n is uniquely defined. Let s be the arc length as measured on C in the positive sense from a fixed point Q₀ to a general point Q on C and let τ be the unit vector tangent to C at Q in the sense of increase of s. Finally, let (x, y) denote a pair of orthogonal Cartesian coordinates in the plane of D, with origin O in D.

Airy’s stress function \( U(x, y) \) may be expressed in terms of three harmonic functions \( \phi(x, y) \), \( \phi_\tau(x, y) \) and \( \psi(x, y) \) in the form [1]

\[
U(x, y) = x\phi(x, y) + y\phi_\tau(x, y) + \psi(x, y),
\]  

(1)
where superscript "c" stands for the conjugate function. These three functions are taken to belong to the class $C^2(D) \cap C^1(D)$, where $D$ denotes the closure of $D$.

The only nonvanishing stress components in the plane are expressed in terms of $U$ by the relations:

$$\sigma_{xx} = \frac{\partial^2 U}{\partial y^2} = x\frac{\partial^2 \phi}{\partial y^2} + 2\frac{\partial \phi}{\partial y} + y\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \psi}{\partial y^2},$$

(2)

$$\sigma_{yy} = \frac{\partial^2 U}{\partial x^2} = x\frac{\partial^2 \phi}{\partial x^2} + 2\frac{\partial \phi}{\partial x} + y\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x^2},$$

(3)

$$\sigma_{xy} = -\frac{\partial^2 U}{\partial x \partial y} = -x\frac{\partial^2 \phi}{\partial x \partial y} - y\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x \partial y},$$

(4)

from which one obtains

$$\sigma_{xx} + \sigma_{yy} = 4\frac{\partial \phi}{\partial x} = 4\frac{\partial \phi^c}{\partial y}. $$

(5)

Thus, $\partial \phi / \partial x$ and $\partial \phi^c / \partial y$ must be univalued functions.

The mechanical displacement vector $u = (u, v)$ is given by

$$2\mu u = \frac{\partial U}{\partial x} + 4(1 - \sigma)\phi = (3 - 4\sigma)\phi - x\frac{\partial \phi}{\partial x} - y\frac{\partial \phi^c}{\partial x},$$

(6)

and

$$2\mu v = -\frac{\partial U}{\partial y} + 4(1 - \sigma)\phi^c = (3 - 4\sigma)\phi^c - x\frac{\partial \phi^c}{\partial y} - y\frac{\partial \phi}{\partial y},$$

(7)

where $\lambda$ and $\mu$ are Lamé’s coefficients and $\sigma = \lambda/(2(\lambda + \mu))$ is Poisson’s ratio.

The boundary conditions to be satisfied on $C$ are the conditions of continuity of

(i) the stress vector components for the first fundamental problem, where given external forces are applied on $C$

(ii) the displacement vector components for the second fundamental problem, where given displacements are specified on $C$.

To discuss the implications of the field equations and boundary conditions, let the parametric equations of $C$ be

$$x = x(s) \quad \text{and} \quad y = y(s)$$

(8)

and denote $dx/ds$ and $dy/ds$ by $\dot{x}$ and $\dot{y}$, respectively, for convenience. According to the definition of $s$, the unit vectors $\tau$ and $n$ have the components

$$\tau = (\dot{x}, \dot{y}) \quad \text{and} \quad n = (\ddot{y}, -\dot{x})$$

(9)

and the following relations hold

$$\frac{\partial U}{\partial s} = \frac{\partial U}{\partial x} \dot{x} + \frac{\partial U}{\partial y} \dot{y} \quad \text{and} \quad \frac{\partial U}{\partial n} = \frac{\partial U}{\partial x} \dot{y} - \frac{\partial U}{\partial y} \dot{x}. $$

(10)

The stress vector components $\sigma_{nx}$ and $\sigma_{ny}$ on $C$ are related to $U$ by the relations

$$\sigma_{nx} = -\frac{d}{ds} \left( \frac{\partial U}{\partial y} \right) \quad \text{and} \quad \sigma_{ny} = -\frac{d}{ds} \left( \frac{\partial U}{\partial x} \right). $$

(11)
The following boundary relations put in evidence the physical meaning of Airy’s stress function and its first-order partial derivatives:

\[ \frac{\partial U}{\partial x}(s) - \frac{\partial U}{\partial x}(0) = - \int_0^s \sigma_n(s') \, ds' = -Y(s) \quad \text{say} \]  
(12)

and

\[ \frac{\partial U}{\partial y}(s) - \frac{\partial U}{\partial y}(0) = \int_0^s \sigma_m(s') \, ds' = X(s) \quad \text{say}. \]  
(13)

From the continuity of the stress vector on \( C \), it follows that \( X(s) \) and \( Y(s) \) are also the components of the resultant external force per unit length applied on that section of \( C \) lying between \( Q_0 \) and \( Q \), as \( C \) is described in the positive sense. In what follows, the components of the external force per unit length of the boundary in the directions of \( x \) and \( y \) increasing will be denoted \( f_x \) and \( f_y \), respectively.

For the first fundamental problem, by integration of \( \partial U/\partial s \) along \( C \) between \( Q_0 \) and \( Q \) and use of (12), (13) and the first of Eqs. (10), one gets

\[ U(s) - U(0) - \frac{\partial U}{\partial x}(0)[x(s) - x(0)] - \frac{\partial U}{\partial y}(0)[y(s) - y(0)] = \int_0^s \{-f_x(s')[y(s') - y(s)] + f_y(s')[x(s') - x(s)]\} \, ds' = M(s) \quad \text{say}. \]  
(14)

\( M(s) \) represents the total moment of the external forces applied on that section of \( C \) lying between \( Q \) and \( Q_0 \), calculated with respect to point \( Q \).

One may also get, along with (14), a second boundary relation by integrating \( \partial U/\partial n \) along \( C \) between \( Q_0 \) and \( Q \) in two different ways using Eqs. (1), (10) and (11):

\[ (x(s)\phi^x(s) - x(0)\phi^x(0)) - (y(s)\phi(s) - y(0)\phi(0)) + (\psi(s) - \psi(0)) + 2 \int_0^s (\dot{x}(s')\phi^x(s') - \dot{x}(s')\phi^x(s')) \, ds' - \frac{\partial U}{\partial x}(0)[y(s) - y(0)] + \frac{\partial U}{\partial y}(0)[x(s) - x(0)] = \int_0^s \{f_x(s')[x(s') - x(s)] + f_y(s')[y(s') - y(s)]\} \, ds' = W(s) \quad \text{say}. \]  
(15)

For the second fundamental problem, Eqs. (14) and (15) may be rewritten in the following forms:

\[ U(s) - U(0) - 4(1 - \sigma) \int_0^s [\phi(s')\dot{x}(s') + \phi^x(s')\dot{y}(s')] \, ds' = -2\mu \int_0^s [u_x(s') \dot{x}(s') + v(s') \dot{y}(s')] \, ds' = -2\mu \int_0^s u_x(s') \, ds' \]  
(16)

and

\[ (x(s)\phi^x(s) - x(0)\phi^x(0)) - (y(s)\phi(s) - y(0)\phi(0)) + (\psi(s) - \psi(0)) + 2 \int_0^s (\dot{x}(s')\phi(s') - \dot{x}(s')\phi(s')) \, ds' - 4(1 - \sigma) \int_0^s [\phi(s')\dot{y}(s') - \phi^x(s')\dot{x}(s')] \, ds' = -2\mu \int_0^s [u_x(s') \dot{y}(s') - v(s') \dot{x}(s')] \, ds' = -2\mu \int_0^s u_x(s') \, ds', \]  
(17)

where \( u_x \) and \( u_n \) denote, respectively, the tangential and normal components of the displacement vector.
In practice, when numerical techniques are used for the solution of the problem, it may be useful to replace boundary conditions (14)–(17) by their versions obtained by derivation w.r.t. the parameter $s$.

Boundary relations (14) and (15) (or (16) and (17)) are complemented by two additional boundary relations obtained from the general integral representation of harmonic functions (transformed with the aid of Cauchy–Riemann conditions and integration by parts), i.e.

\[
\begin{aligned}
f(x, y) &= \frac{1}{2\pi} \oint_C \left( f(s') \frac{\partial}{\partial n'} \ln R + f^c(s') \frac{\partial}{\partial s'} \ln R \right) ds',
\end{aligned}
\]

where $f$ stands for any of the two functions $\phi$ and $\psi$, $R$ is the distance between the general point $(x, y)$ in $D$ and the current integration point on $C$. For the considered class of functions, one may take in Eq. (18) the limit as point $(x, y)$ tends to the boundary $C$. If $f$ is taken once as $\phi$ and a second time as $\psi$, one gets from (18) the following two boundary integral equations

\[
\begin{aligned}
\phi(s) &= \frac{1}{\pi} \oint_C \left( \phi(s') \frac{\partial}{\partial n'} \ln R + \phi^c(s') \frac{\partial}{\partial s'} \ln R \right) ds',
\end{aligned}
\]

and

\[
\begin{aligned}
\psi(s) &= \frac{1}{\pi} \oint_C \left( \psi(s') \frac{\partial}{\partial n'} \ln R + \psi^c(s') \frac{\partial}{\partial s'} \ln R \right) ds'.
\end{aligned}
\]

Up to this point, we are in possession of four boundary integral relations given by Eqs. (14), (15) (or (16) and (17)), (19) and (20), sufficient for the determination of the boundary values of functions $\phi, \phi^c, \psi$ and $\psi^c$ under proper restrictions discussed in the next section. The values of these functions in the bulk, and hence the complete solution of the problem, may be obtained, in a second stage, from (18).

3. On the degree of arbitrariness of the solution

The following transformations:

\[
\begin{aligned}
\phi_1 &= \phi + \beta y + \gamma, \quad \phi^c_1 = \phi^c - \beta x + \delta, \\
\psi_1 &= \psi + lx + my + n, \quad \psi^c_1 = \psi^c - mx + ly + r
\end{aligned}
\]

are the most general transformations that preserve the stressed state. One easily verifies that

\[
\begin{aligned}
\frac{\partial U_1}{\partial x} &= \frac{\partial U}{\partial x} + (\gamma + l), \quad \frac{\partial U_1}{\partial y} = \frac{\partial U}{\partial y} + (\delta + m), \quad U_1 = U + (\gamma + l)x + (\delta + m)y + n.
\end{aligned}
\]

As to the displacements, they are related by the relations

\[
\begin{aligned}
u_1 = u + \frac{4(1 - \sigma)\beta}{2\mu} y - \frac{l}{2\mu} + \frac{(3 - 4\sigma)\gamma}{2\mu}, \quad v_1 = v - \frac{4(1 - \sigma)\beta}{2\mu} x - \frac{m}{2\mu} + \frac{(3 - 4\sigma)\delta}{2\mu}.
\end{aligned}
\]

The differences between the two sets of displacements represent a rigid body motion.

The above-mentioned transformations include seven arbitrary constants. A closer look at the transformation formulae for $U$, $u$ and $v$, however, shows that the number of arbitrary constants may in fact be reduced to three without loss of generality. This can be achieved, for instance, by deleting the linear parts in the expressions for $\psi$ and $\psi^c$. Thus, there will be at most three arbitrary constants
in the solution of the general plane strain problems of the theory of elasticity, to be determined
by eliminating the possible rigid body motion. A discussion of the conditions imposed on the solu-
tion to eliminate arbitrariness may be found in [1, pp. 139–144]. For our later purposes, however,
and guided by concrete examples solved by the proposed method, we shall keep all seven arbitrary
constants and impose the following conditions on the solutions:

(1) + (2) Conditions for eliminating the rigid body translation

These are two conditions, to be applied only for the first fundamental problem. Following [1], we
require that the displacement at point O vanishes, i.e.

\[ u(0,0) = v(0,0) = 0. \]

From (6) and (7), these two conditions may be replaced by the equivalent conditions

\[
(3 - 4\sigma)\phi(0,0) - \frac{\partial \psi}{\partial x}(0,0) = (3 - 4\sigma)\phi(0,0) - \frac{\partial \psi}{\partial y}(0,0) = 0.
\]

In terms of the boundary values of the unknown functions, these two conditions are written respec-
tively, as

\[
\oint_C \left\{ (3 - 4\sigma) \left[ \phi(s') \frac{\partial}{\partial n'} \ln R_0 + \phi'(s') \frac{\partial}{\partial s'} \ln R_0 \right] + \left[ \psi(s') \frac{\partial}{\partial n'} \frac{x(s')}{R_0^2} + \psi'(s') \frac{\partial}{\partial s'} \frac{x(s')}{R_0^2} \right] \right\} ds' = 0
\]

(22)

and

\[
\oint_C \left\{ (3 - 4\sigma) \left[ \phi'(s') \frac{\partial}{\partial n'} \ln R_0 - \phi(s') \frac{\partial}{\partial s'} \ln R_0 \right] + \left[ \psi(s') \frac{\partial}{\partial n'} \frac{y(s')}{R_0^2} + \psi'(s') \frac{\partial}{\partial s'} \frac{y(s')}{R_0^2} \right] \right\} ds' = 0,
\]

(23)

where

\[ R_0 = \left\{ [x(s')]^2 + [y(s')]^2 \right\}^{1/2}. \]

(3) Condition for eliminating the solid body rotation

This condition, like the first two, is applied only for the first fundamental problem. We shall
require that

\[ \frac{\partial \phi}{\partial y}(0,0) = 0, \]

which may be rewritten as

\[
\oint_C \left[ \phi(s') \frac{\partial}{\partial n'} \frac{y(s')}{R_0^2} - \phi'(s') \frac{\partial}{\partial s'} \frac{y(s')}{R_0^2} \right] \, ds' = 0.
\]

(24)

(4) + (5) + (6) + (7) Additional simplifying conditions

We shall adopt the following four conditions at point \( Q_0 \) (\( s = 0 \)) of the boundary \( C \), which allow
some simplifications of the formulae and, at the same time, do not have any physical implications
on the solution:

\[ U(0) = \frac{\partial U}{\partial x}(0) = \frac{\partial U}{\partial y}(0) = x(0)\phi'(0) - y(0)\phi(0) + \psi'(0) = 0. \]

(25)
It is to be noticed that boundary relations (16) and (17) for the second fundamental problem do not involve explicitly the first derivatives of $U$. Therefore, the simplifying conditions of vanishing of these derivatives at the boundary point $s = 0$ have to be taken into account separately in this problem. This is normally carried out after finding $U(x, y)$ (up to at most three arbitrary constants) inside the region occupied by the body. One may, however, apply these two conditions directly on the boundary, as
\[
\frac{\partial U}{\partial s}(0) = \frac{\partial U}{\partial n}(0) = 0,
\]
or, equivalently:
\[
x(0)\frac{\partial \phi}{\partial s}(0) + y(0)\frac{\partial \phi}{\partial s}(0) + \frac{\partial \psi}{\partial s}(0) + \phi(0)\dot{x}(0) + \phi^c \ddot{y}(0) = 0,
\]
\[
x(0)\frac{\partial \phi^c}{\partial s}(0) - y(0)\frac{\partial \phi}{\partial s}(0) + \frac{\partial \psi}{\partial s}(0) + \phi(0)\dot{y}(0) - \phi^c \dot{x}(0) = 0.
\]
Also, some of the imposed conditions (22)–(25), in concrete problems with certain physical and geometrical symmetries, may be automatically satisfied.

In what follows, we shall test the above-mentioned method on four problems, concerning the two fundamental problems for the circle and for the ellipse.

4. The circular cylinder

4.1. The first fundamental problem

Consider a medium bounded by a circle of radius $a$ centered at the origin of coordinates and let a uniform pressure $P_0$ be applied to the circle. The parametric equations of the circle are
\[
x(s) = a \cos \theta, \quad y(s) = a \sin \theta, \quad -\pi < \theta \leq \pi,
\]
with $\theta = s/a$.

Owing to the symmetry of the problem and to the result of the appendix, one may write down the following Fourier expansions for the unknown functions:
\[
\phi(\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta, \quad \phi^c(\theta) = b_0 + \sum_{n=1}^{\infty} a_n \sin n\theta,
\]
\[
\psi(\theta) = c_0 + \sum_{n=1}^{\infty} c_n \cos n\theta, \quad \psi^c(\theta) = d_0 + \sum_{n=1}^{\infty} c_n \sin n\theta.
\]

These expansions identically satisfy the pair of boundary integral equations (19) and (20). The coefficients appearing in the formulae are to be determined from the boundary equations (14) and (15), together with condition (22)–(25).

On the boundary,
\[
\sigma_{xx} = -P_0 \cos \theta, \quad \sigma_{yy} = -P_0 \sin \theta.
\]
Taking this into account and substituting the above-mentioned expansions into formulae (14), (15) and the simplifying conditions (25), one gets

\[
\begin{align*}
  c_0 + aa_0 \cos \theta + ab_0 \sin \theta + \sum_{n=1}^{\infty} aa_n \cos(n-1)\theta + \sum_{n=1}^{\infty} c_n \cos n\theta &= a^2 P_0 (\cos \theta - 1), \\
  2ab_0 + d_0 + 2aa_1 - ab_0 \cos \theta + aa_0 \sin \theta + \sum_{n=1}^{\infty} aa_n sin(n-1)\theta \\
  + \sum_{n=1}^{\infty} c_n \sin n\theta + \sum_{n=1}^{\infty} \frac{2da_n}{n-1} \sin(n-1)\theta &= a^2 P_0 (\sin \theta - \theta).
\end{align*}
\]

The orthogonality property of the trigonometric functions applied to these last two equations reduces the expressions for the boundary values of the unknown functions to:

\[
\begin{align*}
  \phi(\theta) &= a_0 - \frac{1}{2}aP_0 \cos \theta, \quad \phi'(\theta) = \frac{1}{2}aP_0 \sin \theta, \\
  \psi(\theta) &= -\frac{1}{2}a^2 P_0 + a(aP_0 - a_0) \cos \theta, \quad \psi'(\theta) = a(aP_0 - a_0) \sin \theta,
\end{align*}
\]

where \(a_0\) is an arbitrary constant to be determined from conditions (22)–(24). Insertion of these boundary expressions into the corresponding relations (18) yields unique expressions (up to the arbitrary constant \(a_0\)) of the unknown functions inside the circle in the form

\[
\begin{align*}
  \phi(x, y) &= a_0 - \frac{1}{2}P_0 x, \quad \phi'(x, y) = \frac{1}{2}P_0 y, \\
  \psi(x, y) &= -\frac{1}{2}a^2 P_0 + a(aP_0 - a_0) x, \quad \psi'(x, y) = (aP_0 - a_0) y.
\end{align*}
\]

These expressions automatically satisfy conditions (23) and (24). Finally, condition (22) gives the value of \(a_0\) as

\[
a_0 = \frac{aP_0}{4(1 - \sigma)}.
\]

Hence the solution for the stress function

\[
U(x, y) = -\frac{1}{2} P_0 [(x - a)^2 + y^2].
\]

In fact, it is not difficult to see that this will be the solution for any simply connected, smooth boundary subjected to a uniform pressure.

### 4.2. The second fundamental problem

For the same medium of the previous section, let the boundary conditions be of the form:

\[
u(\theta) = \varepsilon \cos \theta, \quad v(\theta) = \varepsilon \sin \theta, \quad -\pi < \theta \leq \pi,
\]

with \(\theta = \delta/a\). This boundary condition corresponds to a (small) radial extension or contraction of the boundary points of amplitude \(|\delta|\).

Using the same expressions as above for the boundary values of the unknown functions, boundary conditions (16) and (17), together with the first and the last of the additional simplifying conditions (25) yield, respectively:

\[
\begin{align*}
  c_0 + aa_0 \cos \theta + ab_0 \sin \theta + \sum_{n=1}^{\infty} aa_n \cos(n-1)\theta + \sum_{n=1}^{\infty} c_n \cos n\theta
\end{align*}
\]
\[ = 4a(1 - \sigma) \left[ a_0 \cos \theta + b_0 \sin \theta - \sum_{n=2}^{\infty} \frac{a_n}{n-1} \cos((n-1)\theta) - a_0 + \sum_{n=2}^{\infty} \frac{a_n}{n-1} \right], \]

\[ = d_0 + ab_0 \cos \theta - aa_0 \sin \theta + \sum_{n=1}^{\infty} a_n cos(n-1)\theta + \sum_{n=1}^{\infty} c_n \sin n\theta \]

\[-2a(1 - 2\sigma) \left[ a_0 \sin \theta - b_0 \cos \theta - a_1 \theta + \sum_{n=2}^{\infty} \frac{a_n}{n-1} \sin((n-1)\theta) + b_0 \right] = -2a\mu\varepsilon \theta. \]

Using the orthogonality and the univaluedness properties of the trigonometric functions and after some simplifications, one obtains the following expressions for the boundary values of the unknown functions:

\[ \phi(\theta) = a_0 + \frac{\mu \varepsilon}{1 - 2\sigma} \cos \theta, \quad \phi^\prime(\theta) = \frac{\mu \varepsilon}{1 - 2\sigma} \sin \theta, \]

\[ \psi(\theta) = -\frac{\mu \varepsilon a}{1 - 2\sigma} - 4a_0(1 - \sigma) + aa_0(3 - 4\sigma) \cos \theta, \quad \psi^\prime(\theta) = aa_0(3 - 4\sigma) \sin \theta, \]

where \( a_0 \) is an arbitrary constant to be determined from the two remaining additional conditions.

From these expressions, one gets the following formulae for the unknown functions inside the medium as

\[ \phi(x, y) = a_0 + \frac{\mu \varepsilon}{a(1 - 2\sigma)} x, \quad \phi^\prime(x, y) = \frac{\mu \varepsilon}{a(1 - 2\sigma)} y, \]

\[ \psi(x, y) = -\frac{\mu \varepsilon a}{1 - 2\sigma} - 4a_0(1 - \sigma) + a_0(3 - 4\sigma) x, \quad \psi^\prime(x, y) = a_0(3 - 4\sigma) y. \]

These expressions automatically satisfy the additional condition of vanishing of \( \partial U/\partial y \) at the boundary points \( s = 0 \). The vanishing of \( \partial U/\partial x \) at the same point yields the value of the coefficient \( a_0 \) as

\[ a_0 = -\frac{\mu \varepsilon}{2(1 - \sigma)(1 - 2\sigma)}. \]

Hence the solution for the Airy’s stress function

\[ U(x, y) = \frac{\mu \varepsilon}{a(1 - 2\sigma)} [(x - a)^2 + y^2]. \]

It is interesting to note that the stress functions for the first- and second-fundamental problems have the same form, except for a multiplicative constant. We may thus infer that our expression for the stress function also solves a class of mixed boundary value problems for simply connected, smooth contours, part of which is under uniform pressure \( P_0 \), and the remaining part, which is a part of a circle of radius \( a \), is subjected to a uniform radial contraction \( \varepsilon \) given by

\[ \varepsilon = \frac{1}{2\mu} a(1 - 2\sigma) P_0. \]
5. The elliptic cylinder

5.1. The first fundamental problem

Consider an elliptic cylinder of major and minor semi-axes \(a\) and \(b\), respectively, subjected to a uniform pressure \(P_0\).

Although we expect the solution to be the same as for the circular cylinder, we give this application for more illustration of the used technique for a different geometry. Let the parametric representation of the ellipse be

\[ x(\theta) = a \cos \theta, \quad y(\theta) = b \sin \theta, \quad -\pi < \theta \leq \pi. \]

The components of the applied force in the \(x\)- and \(y\)-directions are given by the relations:

\[ \frac{ds}{d\theta} \sigma_x = -P_0 b \cos \theta, \quad \frac{ds}{d\theta} \sigma_y = -P_0 a \sin \theta. \]

For convenience, the boundary integral relations (19) and (20) satisfied by the unknown functions \(\phi(s), \psi(s)\) and their complex conjugates may be rewritten in a unified operator form as

\[ F(\theta) = [\tilde{K}_1 F](\theta) + [\tilde{K}_2 F^c](\theta) + [\tilde{K}_3 F^c](\theta), \]

where \(F\) stands for \(\phi\) or \(\psi\) and \(\tilde{K}_i\) \((i = 1, 2, 3)\) are integral operators defined on the same interval as \(\theta\) with respective kernels

\[ K_1(\theta, \theta') = \frac{ab}{\pi(a^2 + b^2)} \frac{1}{1 - \delta \cos(\theta' + \theta)}, \]

\[ K_2(\theta, \theta') = \frac{1}{2\pi} \frac{\sin(\theta' - \theta)}{1 - \cos(\theta' - \theta)}, \]

\[ K_3(\theta, \theta') = \frac{\delta}{2\pi} \frac{\sin(\theta' + \theta)}{1 - \delta \cos(\theta' + \theta)}, \]

where

\[ \delta = \frac{a^2 - b^2}{a^2 + b^2}. \]

Substituting for \(f_x\) and \(f_y\) into the boundary conditions (14) and (15), one gets

\[ a \cos \theta \phi(\theta) + b \sin \theta \phi'(\theta) + \psi(\theta) = -\frac{1}{4} P_0 [3a^2 + b^2 - 4a^2 \cos \theta + (a^2 - b^2) \cos 2\theta], \]

\[ a \cos \theta \phi'(\theta) - b \sin \theta \phi(\theta) + \psi'(\theta) + 2 \int_0^\theta [b \cos \theta' \phi'(\theta') + a \sin \theta' \phi'(\theta')] d\theta' = a b P_0 [-\theta + \sin \theta]. \]

In view of the appendix, the restrictions of the unknown functions to the boundary satisfying relations (26) may be expanded in terms of the trigonometric functions as

\[ \phi(\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta, \quad \phi'(\theta) = b_0 + \sum_{n=1}^{\infty} b_n a_n \sin n\theta, \]

\[ \psi(\theta) = c_0 + \sum_{n=1}^{\infty} c_n \cos n\theta, \quad \psi'(\theta) = d_0 + \sum_{n=1}^{\infty} d_n c_n \sin n\theta, \]
with
\[\varepsilon_n = \frac{(a + b)^n - (a - b)^n}{(a + b)^n + (a - b)^n}.\]

Substituting these expressions into boundary conditions (14) and (15) and using the simplifying conditions at \(\theta = 0\), one finally obtains the following two relations valid for the values of angle \(\theta\) in the interval \((-\pi, \pi)\):

\[
[2c_0 + (a + b\varepsilon_1)a_1] + [2aa_0 + (a + b\varepsilon_2)a_2 + 2c_1]\cos \theta + [aa_1 + (a + b\varepsilon_3)a_3
- b\varepsilon_1a_1 + 2c_2]\cos 2\theta + \sum_{n=3}^{\infty} [aa_{n-1} + aa_{n+1} + b\varepsilon_{n+1}a_{n+1} - b\varepsilon_{n-1}a_{n-1} + 2c_n]\cos n\theta
= -\frac{1}{2}P_0[(3a^2 + b^2) - 4a^2 \cos \theta + (a^2 - b^2) \cos 2\theta],
\]

\[
(d_0 + 2ab_0) + (b + a\varepsilon_1)a_1\theta - ab_0 \cos \theta + [ba_0 + \frac{1}{2}(b + a\varepsilon_2)a_2 + c_1\varepsilon_1]\sin \theta
+ \sum_{n=2}^{\infty} (\frac{1}{n} - \frac{1}{n})(a\varepsilon_{n-1} - b)a_{n-1} + (\frac{1}{n} + \frac{1}{n})(a\varepsilon_{n+1} + b)a_{n+1} + c_n\varepsilon_n]\sin n\theta
= abP_0(-\theta + \sin \theta).
\]

These relations yield the results
\[
\phi(\theta) = a_0 - \frac{1}{2}aP_0 \cos \theta, \quad \phi^c(\theta) = -\frac{1}{2}bP_0 \cos \theta,
\]

\[
\psi(\theta) = -\frac{1}{2}a^2P_0 + a(aP_0 - a_0) \cos \theta, \quad \psi^c(\theta) = b(aP_0 - a_0) \sin \theta,
\]

where \(a_0\) is an arbitrary constant to be determined by the application of the conditions, at the origin, eliminating the rigid body motion. It is worth noting that these expressions reduce to the corresponding ones for the circular cylinder on setting \(b = a\).

It is easily verified that the functions \(\phi(x, y), \phi^c(x, y), \psi(x, y)\) and \(\psi^c(x, y)\), which take the above boundary values and satisfy the conditions at the origin, may be expressed as

\[
\phi(x, y) = \frac{aP_0}{4(1 - \sigma)} - \frac{1}{2}P_0x, \quad \phi^c(x, y) = -\frac{1}{2}P_0y,
\]

\[
\psi(x, y) = -\frac{1}{2}a^2P_0 + \frac{(3 - 4\sigma)}{4(1 - \sigma)}aP_0x, \quad \psi^c(x, y) = \left(aP_0 - \frac{aP_0}{4(1 - \sigma)}\right)y.
\]

Hence the solution for the stress function
\[U(x, y) = -\frac{1}{2}P_0[(x - a)^2 + y^2].\]

As expected, this is the same solution as for the case of the circular cylinder since, in both cases, a state of hydrostatic pressure prevails everywhere in the region, and there is no shear.

It is to be noted that, for general boundary pressure distributions, the expressions for the unknown functions inside the domain are normally obtained using relation (18), replacing the function \(f(x, y)\) by either \(\phi(x, y)\) or \(\psi(x, y)\). This procedure, however, is expected to lead to difficult integrations, since the used system of Cartesian coordinates is not the natural one for elliptic boundaries. One, then, may follow an alternative procedure invoking the general harmonics in two-dimensional elliptic coordinates \((\xi, \theta)\) defined in terms of \((x, y)\) by

\[
x = A \cosh \xi \cos \theta, \quad y = A \sinh \xi \sin \theta,
\]
where \( A = \sqrt{a^2 - b^2} \), and the elliptic boundary is given by \( \zeta = \xi_0 \), with \( \xi_0 = \cosh^{-1}(a/A) = \sinh^{-1}(b/A) \).

Our unknown functions are expressed as linear combinations of these harmonics with arbitrary coefficients to be determined by imposing the already obtained boundary expressions. This procedure will be followed in dealing with the second fundamental problem for the elliptic cylinder.

### 5.2. The second fundamental problem

Let the elliptic cylinder introduced in the preceding subsection be subjected to a (small) uniform extension (or contraction) \( \varepsilon \) normal to the boundary.

The boundary displacement components are expressed as

\[
\begin{align*}
  u &= \varepsilon \frac{dy}{ds}, \\
  v &= -\varepsilon \frac{dx}{ds},
\end{align*}
\]

hence

\[
u_t = 0 \quad \text{and} \quad u_n = \varepsilon.
\]

The expansions used in the preceding subsection for the restrictions of the unknown functions to the boundary are still valid in the present case. The two boundary conditions (16) and (17) yield

\[
\begin{align*}
  [c_0 + \frac{1}{2}(a + b\xi_1)a_1] + \sum_{n=1}^{\infty} \left[ \frac{1}{2}(1 + \delta_{n,1})(a - b\xi_{n-1})a_{n-1} + \frac{1}{2}(a + b\xi_{n+1})a_{n+1} + c_n \right] & \cos n\theta \\
  + 2(1 - \sigma) \int_0^\theta \left[ -2bb_0 \cos \theta' + \sum_{n=1}^\infty \left\{ \alpha(1 + \delta_{n,1}) - b\xi_{n-1} \right\} a_{n-1} \\
  - [a + b\xi_{n+1}]a_{n+1} \right] \sin n\theta' \, d\theta' &= 0,
\end{align*}
\]

\[
\sum_{n=1}^{\theta} \left[ \frac{1}{2}(1 + \delta_{n,1})(b - a\xi_{n-1})a_{n-1} - \frac{1}{2}(b + a\xi_{n+1})a_{n+1} - \varepsilon_0 c_n \right] \sin n\theta \\
  + (1 - 2\sigma) \int_0^\theta \left[ (b + a\xi_1)a_1 + \sum_{n=1}^\infty \left[ (1 + \delta_{n,1})(b - a\xi_{n-1})a_{n-1} + (b + a\xi_{n+1})a_{n+1} \right] \cos n\theta' \right] \, d\theta' \\
  = 2\mu\varepsilon \int_0^\theta \sqrt{a^2 \sin^2 \theta' + b^2 \cos^2 \theta'} \, d\theta',
\]

where \( \delta_{m,n} \) are the Kronecker delta symbols and \( \varepsilon_0 = 0 \). These two relations lead to

\[
\begin{align*}
  b_0 &= d_0 = 0, \quad aa_0 + c_0 + \sum_{n=1}^{\infty} (aa_n + c_n) = 0, \\
  a_1 &= \frac{2\mu\varepsilon}{\pi(1 - 2\sigma)(b + a\xi_1)} \int_0^\pi \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta} \, d\theta, \\
  \text{and, for } n \geq 1, \\
  c_n &= \frac{1}{2n} [A_n a_{n-1} - B_n a_{n+1}]
\end{align*}
\]
and the recurrence relation

\[ a_{n+1} = P_{n+1} a_{n-1} + Q_{n+1}, \]

with

\[ P_{n+1} = C_n + A_n e_n, \quad Q_{n+1} = \frac{8 \mu e}{\pi(D_n + B_n)e_n} \int_0^\pi \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta} \cos n \theta d\theta, \]

\[ A_n = -(1 + \delta_{n,1})(a - b e_{n-1})(n - 4 + 4\sigma), \quad B_n = (a + b e_{n+1})(n + 4 - 4\sigma), \]

\[ C_n = -(b - a e_{n-1})(n + 4 - \delta_{n,1} - 4\sigma), \quad D_n = (b + a e_{n+1})(2 - n - 4\sigma). \]

The solution for the recurrence relation for \( a_n \) has been obtained after lengthy calculations, the details of which have been omitted for conciseness. The following expressions for the coefficients \( a_{2k} \) and \( a_{2k+1} \) for \( k \geq 1 \), which may be directly verified using mathematical induction, were finally obtained:

\[ a_{2k} = R_k a_0 + S_k, \]

\[ a_{2k+1} = \left( \prod_{i=1}^k P_{2i+1} \right) a_1 + \sum_{j=1}^{k-1} \left( Q_{2j+1} \prod_{i=j+1}^k P_{2i+1} \right) + Q_{2k+1}, \]

with

\[ R_k = \left( \prod_{i=1}^k P_{2i} \right), \quad S_k = \sum_{j=1}^{k-1} \left( Q_{2j} \prod_{i=j+1}^k P_{2i} \right) + Q_{2k}. \]

Also,

\[ c_{2k} = \frac{1}{4k} \left[ \left\{ \frac{A_{2k}}{P_{2k+1}} - B_{2k} \right\} \left( \prod_{i=1}^k P_{2i+1} \right) a_1 + \sum_{j=1}^{k-1} \left( Q_{2j+1} \prod_{i=j+1}^k P_{2i+1} \right) + Q_{2k+1} \right] - \frac{A_{2k} Q_{2k+1}}{P_{2k+1}}, \]

\[ c_{2k+1} = U_{k+1} a_0 + V_{k+1}, \]

with

\[ U_{k+1} = \frac{1}{2(2k + 1)} \left[ \frac{A_{2k+1}}{P_{2k+1}} - B_{2k+1} \right] \left( \prod_{i=1}^{k+1} P_{2i} \right), \]

\[ V_{k+1} = \frac{1}{2(2k + 1)} \left[ \frac{A_{2k+1}}{P_{2k+2}} - B_{2k+2} \right] \left( \sum_{j=1}^{k+1} \left( Q_{2j} \prod_{i=j+1}^{k+1} P_{2i} \right) + Q_{2k+2} \right] - \frac{A_{2k+1} Q_{2k+2}}{P_{2k+2}}. \]

Thus, the coefficients \( a_{2k+1} \) and \( c_{2k} \) are completely determined, while \( a_{2k} \) and \( c_{2k+1} \) are given in terms of the arbitrary constant \( a_0 \) to be determined from the remaining conditions.

The above results lead to the following expression for the coefficient \( c_0 \) in terms of \( a_0 \):

\[ c_0 = - \left[ \left\{ a + \sum_{k=1}^{\infty} (a R_k + U_k) \right\} a_0 + \sum_{k=1}^{\infty} \left( a S_k + V_k + a a_{2k-1} + c_{2k} \right) \right]. \]
Relation (26) to be verified at $\theta = 0$ is automatically satisfied, while relation (27) leads to the following expression for the coefficient $a_0$:

$$a_0 = \sum_{k=1}^{\infty} \left[ (2k - 1)c_{2k-1} + \frac{a a_{2k-1}}{2k} + b V_k + \frac{b a_{2k-1}}{2k} \right] + \sum_{k=1}^{\infty} \left[ (2k - 1)c_{2k-1} + \frac{a a_{2k-1}}{2k} \right].$$

This completes the determination of the restrictions of the unknown functions to the boundary $C$. By application of the above-mentioned procedure, one readily obtains the expressions for these functions inside the domain $D$ in the form:

$$\phi(x, y) = a_0 + \sum_{n=1}^{\infty} a_n \left[ \cosh n \zeta \cos \theta + \sinh n \zeta \sin \theta \right] \cos n \theta,$$

$$\phi^\xi(x, y) = \sum_{n=1}^{\infty} a_n \left[ \sinh n \zeta \cos \theta + \cosh n \zeta \sin \theta \right] \sin n \theta,$$

$$\psi(x, y) = c_0 + \sum_{n=1}^{\infty} c_n \left[ \cosh n \zeta \cos \theta + \sinh n \zeta \sin \theta \right] \cos n \theta,$$

$$\psi^\xi(x, y) = \sum_{n=1}^{\infty} c_n \left[ \sinh n \zeta \cos \theta + \cosh n \zeta \sin \theta \right] \sin n \theta,$$

whence the stress function

$$U(x, y) = \sqrt{a^2 - b^2} \left[ \cosh \zeta \cos \theta \phi(x, y) + \sinh \zeta \sin \theta \phi^\xi(x, y) \right] + \psi(x, y).$$

The case of the circular cylinder may be recovered from this by passing to the limit as $b \rightarrow a$ appropriately.

6. Conclusions

In this paper, we discussed the boundary integral representation technique applied to the first and the second fundamental problems of the plane strain Theory of Elasticity for simply connected regions. The same procedure can be applied to study problems with mixed boundary conditions and in multiply connected regions, with the necessary modifications. It appears from the worked examples for the circle and for the ellipse that the method provides analytical solutions in closed form for problems with simple geometry. Even though, as shown in the fourth application, the obtained analytical solution may be difficult to handle, a fact that points out at the necessity of turning to numerical treatments of the method. Within these treatments, the resulting boundary integrals involving the unknown functions are discretized and the problem ultimately reduces to the solution of a linear system of algebraic equations. Subsequent numerical integrations complete the determination of the stresses and displacements inside the body. As the used approach requires the existence of a unique normal to the boundary at each of its points, the presence of corner points on the boundary must be taken care of through an appropriate smoothing technique. This will be dealt with in a second part of the work.
Appendix

In this appendix, we give the justification for using the expansions in terms of the trigonometric functions to express the restrictions of the unknown functions of the problem to the boundary. This will be done only for the ellipse, the case of the circular cylinder being obtained as a special case.

The functions $\phi(\theta)$ and $\phi'(\theta)$ (or $\psi(\theta)$ and $\psi'(\theta)$) are related by a boundary integral relation of the form:

$$F(\theta) = [\tilde{K}_1 F](\theta) + [\tilde{K}_2 F](\theta) + [\tilde{K}_3 F](\theta),$$

where $F$ stands for $\phi$ or $\psi$ and $\tilde{K}_i$ ($i = 1, 2, 3$) are integral operators defined on the same interval as $\theta$ with respective kernels

$$K_1(\theta, \theta') = \frac{ab}{\pi(a^2 + b^2)} \frac{1}{1 - \delta \cos(\theta + \theta')},$$

$$K_2(\theta, \theta') = \frac{1}{2\pi} \frac{\sin(\theta' - \theta)}{1 - \cos(\theta' - \theta)},$$

$$K_3(\theta, \theta') = \frac{\delta}{2\pi} \frac{\sin(\theta' + \theta)}{1 - \delta \cos(\theta' + \theta)},$$

where

$$\delta = \frac{a^2 - b^2}{a^2 + b^2}.$$

Now, it may be directly verified that

$$\tilde{K}_1(\theta, \theta') \cos n\theta' = \tilde{\lambda}_{1,n} \cos n\theta, \quad \tilde{K}_1(\theta, \theta') \sin n\theta' = -\tilde{\lambda}_{1,n} \sin n\theta,$$

$$\tilde{K}_2(\theta, \theta') \cos n\theta' = -\tilde{\lambda}_{2,n} \sin n\theta, \quad \tilde{K}_2(\theta, \theta') \sin n\theta' = \tilde{\lambda}_{2,n} \cos n\theta,$$

$$\tilde{K}_3(\theta, \theta') \cos n\theta' = \tilde{\lambda}_{3,n} \sin n\theta, \quad \tilde{K}_3(\theta, \theta') \sin n\theta' = \tilde{\lambda}_{3,n} \cos n\theta,$$

where

$$\tilde{\lambda}_{1,n} = \frac{ab}{\pi(a^2 + b^2)} \int_{-\pi}^{\pi} \frac{\cos n\phi}{1 - \delta \cos \phi} d\phi = \left(\frac{a - b}{a + b}\right)^n,$$

$$\tilde{\lambda}_{2,n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin \phi \sin n\phi}{1 - \cos \phi} d\phi = 1,$$

$$\tilde{\lambda}_{3,n} = \frac{\delta}{2\pi} \int_{-\pi}^{\pi} \frac{\sin \phi \sin n\phi}{1 - \delta \cos \phi} d\phi = \left(\frac{a - b}{a + b}\right)^n.$$

Hence, for $n = 0, 1, 2, \ldots$, the sets of trigonometric functions $\{\cos n\theta, \sin n\theta\}$ are closed under the application of any of the above mentioned integral operators. This allows the expansion of the function $f(\theta)$ solving the considered integral equation as

$$f(\theta) = A_0 + \sum_{n=1}^{\infty} [A_n \cos n\theta + B_n \sin n\theta],$$
where $A_0, A_n$ and $B_n \ (n = 1, 2, \ldots)$ are arbitrary constant coefficients. The function $f^c(\theta)$ has a similar expansion with a corresponding set of coefficients. In the case of symmetry, one sets
\[
\phi(\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta, \quad \phi^c(\theta) = b_0 + \sum_{n=1}^{\infty} b_n \sin n\theta,
\]
\[
\psi(\theta) = c_0 + \sum_{n=1}^{\infty} c_n \cos n\theta, \quad \psi^c(\theta) = d_0 + \sum_{n=1}^{\infty} d_n \sin n\theta,
\]
where $c_n \ (n = 1, 2, \ldots)$ are certain coefficients determined by inserting these expansions into the corresponding integral relations and using the results of the appendix, as
\[
\epsilon_n = \frac{1 - \lambda_{1,n} \lambda_{2,n} + \lambda_{3,n}}{(a + b)^n + (a - b)^n}
\]
In the case of circular contours, one may verify that $\epsilon_n = 1$.

References