Finite element method for elliptic problems with edge singularities

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Dedicated to the memory of Professor Pierre Grisvard, an outstanding teacher of singularities who died prematurely.

Abstract

We consider tangentially regular solution of the Dirichlet problem for an homogeneous strongly elliptic operator with constant coefficients, on an infinite vertical polyhedral cylinder based on a bounded polygonal domain in the horizontal $xy$-plane. The usual complex blocks of singularities in the non-tensor product singular decomposition of the solution are made more explicit by a suitable choice of the regularizing kernel. This permits to design a well-posed semi-discrete singular function method (SFM), which differs from the usual SFM in that the dimension of the space of trial and test functions is infinite. Partial Fourier transform in the $z$-direction (of edges) enables us to overcome the difficulty of an infinite dimension and to obtain optimal orders of convergence in various norms for the semi-discrete solution. Asymptotic error estimates are also proved for the coefficients of singularities. For practical computations, an optimally convergent full-discretization approach, which consists in coupling truncated Fourier series in the $z$-direction with the SFM in the $xy$-plane, is implemented. Other good (though not optimal) schemes, which are based on a tensor product form of singularities are investigated. As an illustration of the results, we consider the Laplace operator. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

This extended version of an earlier ‘Note’ [31] deals with the finite element method (FEM) for elliptic boundary value problems on nonsmooth three-dimensional domains. We are mainly

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interested in the quality of the approximations, namely the accuracy and the rates of convergence.

It is indeed well-known that the convergence of the classical FEM in such situations is poor. In fact, the effect of rough geometries is an important loss in the expected optimal regularity of the exact solution.

Two-dimensional problems or three-dimensional ones on domains with smooth conical corners enjoy an interesting property: The solution is decomposable into a regular part and a finite linear combination of explicit intrinsic singular functions (see [22,25] for instance), whose coefficients depend continuously upon the data and are representable by global formulae [11,34]. This permits to design special strategies for restoring the optimal order of convergence of the FEM. The most typical approaches are:

- the mesh refinement method (the meshsize is suitably adapted near the corners, see [6]);
- the singular function method (the grid is uniform but the space of trial and test functions is enriched by the singular functions, cf. [41]),
- the dual singular function method (based on global representation formulae of the coefficients of singularities, see [9,38]).

For a discussion and/or a survey on all these strategies, we refer the reader to [7,8].

Regarding three-dimensional problems with both vertex and edge singularities, polyhedrons for example, the situation is more difficult. The solution still admits a singular decomposition, but its exploitation in constructive methods is not easy since the edge and edge-vertex singular functions are complex blocks generated in a specific way from associated two-dimensional singularities using regularizing kernels and pseudo-differential operators (cf. [16] and Section 2 below). Nevertheless, using specific mixed (in the vertex and edge directions) weighted Sobolev spaces, the mesh refinement method can be extended to this situation (see for example [2–5,27–30]). For a contribution towards the dual singular function method, we mention [33]. For axisymmetric (or prismatic) domains with edge singularities, using the explicit form of the edge singularities, Heinrich had considered in [24] the refined Fourier-finite element method for the Poisson problem (i.e. combining the approximate Fourier method in the rotational angle with the refined finite element method in the meridian plane) with optimal order of convergence.

This paper is primarily concerned with an extension of the singular function method to boundary value problems on domains with edge singularities. We present two types of semi-discrete and fully discrete schemes. The first method (Theorems 4.9–4.11 and 4.13), based on a non-tensor product singular decomposition of the solution, seems not to work for domains with both vertex and edge singularities. On the contrary, the second method (Theorems 4.15 and 4.16) may be extended to polyhedrons thanks to the tensor product nature of singularities established in [21,40] (see also Remark 4.17 below).

The paper is organized as follows. In the next section, we define the elliptic problem to be considered and we specify the singular decomposition of its solution. Section 3 is devoted to the discrete function spaces we need. The approximation of the elliptic problem is analyzed in Section 4 with emphasis on the following aspects: optimally (Section 4.1) and improved (Sections 4.3 and 4.4) convergent schemes, computation of discrete solutions (Section 4.2). An illustrative example of our results is finally considered in Section 5.
2. Edge behaviour of solutions of boundary value problems

The smoothness of solutions of boundary value problems of order $2m$, with $m \in \mathbb{N}^*$, is such a technical subject that we find it relevant to give in this section a self-contained summary of the results (and notation) we need. Moreover, since the numerical approach presented in the paper is based on the explicit form of the singular functions, we spend some space to specify these singularities. Our basic reference is [16].

As mentioned in the introduction, the kind of difficulties occurring here (see what follows Lemma 2.3 until getting formulae (2.32)–(2.33)) leads us to restricting our analysis to a domain with only edges. More precisely, we consider the polyhedral cylinder

$$Q := \Omega \times \mathbb{R}$$

based on the polygonal bounded domain $\Omega \subset \mathbb{R}^2$ with boundary $\Gamma$. For the sake of simplicity, we suppose that only one vertex $O$ of $\Omega$ induces singularities of the considered problem. The general case follows, as usual, by superposition. By translation and rotation, we may always suppose that $O$ is the origin of the plane $xy$. Furthermore, we assume that near $O$, the domain $\Omega$ coincides with the infinite sector $G$ of opening $\omega \in ]0,2\pi[$, defined in polar coordinates $(r,\theta)$ of center $O$ by

$$G = \{re^{i\theta} : r > 0, \; 0 < \theta < \omega\}.$$  

The edge variable will be denoted by $z$.

2.1. Functions spaces

On the cylinder $Q$, we shall consider the usual Sobolev spaces $H^k(Q)$, $k \in \mathbb{R}$, with norm and semi-norm denoted by $\|\cdot\|_{k,Q}$ and $|\cdot|_{k,Q}$, respectively (see [22]). $H^0_0(Q)$ is the closure in $H^k(Q)$ of $\mathcal{D}(Q)$, the space of $C^\infty$ functions with compact support in $Q$.

In terms of the partial Fourier transform (with respect to the edge variable $z$)

$$\hat{w}(\xi) = \hat{w}(x,y,\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-iz\xi} w(x,y,z) \, dz,$$

we have (see [16, pp. 224,238]):

$$w \in H^k(Q) \quad \text{if and only if} \quad \left( \int_{\mathbb{R}} \|\hat{w}(\xi)\|^2_{k,|\xi|+1} \, d\xi \right)^{1/2} < +\infty,$$  \hspace{1cm} (2.2)

the norm $\|\cdot\|_{k,Q}$ being equivalent to the usual norm $\|\cdot\|_{k,Q}$. In (2.2), the symbol $\|\cdot\|_{n,\alpha,\rho}$ corresponding to an integer $n \geq 0$ and to a real number $\rho > 0$ denotes the norm

$$\|v\|_{n,\alpha,\rho} := \left( \sum_{s=0}^{n} \rho^{2s} \left|v_{|n-s,\Omega}\right|^2 \right)^{1/2}$$  \hspace{1cm} (2.3)

on the Sobolev space $H^n(\Omega)$. Very often, we express this by writing $H^m(\Omega,\rho)$. The space $H^{-m}(\Omega,\rho)$ is defined by duality in [16, p. 224,237], where the case of non integer order is also studied.
2.2. The differential operator

Let

\[ L = L(D_{xy}, D_z) = \sum_{|\xi|=2m} a_\xi D_{xy}^{\xi} D_z^{\xi} = \sum_{|\xi|=2m} a_\xi D_{xy}^{\xi} D_{z}^{\xi}, \quad (2.4) \]

be a homogeneous strongly elliptic operator of order \(2m, m \in \mathbb{N}^*\), with constant coefficients on \(Q\) (with the convention \(D_z = \frac{1}{i} \frac{\partial}{\partial z}\)). Thus the operator \(L\) is associated with the continuous bilinear form

\[ a(v, w) := \langle Lv, w \rangle_{H^{−m}(Q) \times H_0^m(Q)}, \quad v, w \in H_0^m(Q), \quad (2.5) \]

which is \(H_0^m(Q)\)-coercive, i.e. there exists a positive constant \(c\) such that

\[ \text{Re} \, a(v, v) \geq c \|v\|_{H_0^m(Q)}^2, \quad \forall v \in H_0^m(Q). \quad (2.6) \]

By partial Fourier transform (2.1), the operator in (2.4) becomes the operator

\[ L(\xi) \equiv L(D_{xy}, \xi) = \sum_{|\xi|=2m} a_\xi^{\xi} D_{xy}^{\xi}, \quad \xi \in \mathbb{R}. \quad (2.7) \]

which is nonhomogeneous but still strongly elliptic on \(\Omega\). The associated continuous bilinear form is

\[ a_\xi(v, w) := \langle L(\xi)v, w \rangle_{H^{−m}(\Omega) \times H_0^m(\Omega)}, \quad v, w \in H_0^m(\Omega). \quad (2.8) \]

From Eq. (2.6), the following uniform inequality (which is useful for numerical purposes, see Proposition 4.3) holds:

\[ \exists c > 0 : \forall \xi \in \mathbb{R} : \text{Re} \, a_\xi(v, v) \geq c \|v\|_{m, \Omega, |\xi| + 1}^2, \quad \forall v \in H_0^m(\Omega). \quad (2.9) \]

For \(\xi \in \mathbb{R}\), the operator \(L(\xi)\) has the expansion

\[ L(\xi) = \sum_{k=0}^{2m} L_k(D_{xy}, \xi), \quad (2.10) \]

where

\[ L_k(\xi) \equiv L_k(D_{xy}, \xi) = \xi^k \sum_{|\xi'|=2m−k}^{2m} a_{(x', k)} D_{xy}^{\xi'}. \quad (2.11) \]

The operator \(L_0(D_{xy}, \xi)\) reduces, for any \(\xi \in \mathbb{R}\), to the principal part \(L_0 = L_0(D_{xy})\) of \(L(\xi)\), which we write in polar coordinates:

\[ L_0 = r^{-2m} \mathcal{L}(0, rD_r, D_\theta). \quad (2.12) \]

This principal part plays an important role in what follows. More precisely, we denote by \(\mathcal{L}(\lambda), \lambda \in \mathbb{C}\), the operator \(\mathcal{L}(\lambda, D_\theta)\) acting from \(H_0^m(\Omega)\) into \(H^{−m}(\Omega)\). Then the vector-valued function \(\lambda \to \mathcal{L}(\lambda)^{-1}\) is meromorphic and its poles generate the singular solutions of our boundary value problem. This statement will be clarified in Section 2.3 below. However, at this stage we can
anticipate that the singular functions relative to the two-dimensional operator $L_0$ belong, for a suitable integer $s > 0$, to the set

$$S'(G) := \left\{ v = r^J \sum_{j=0}^{J} (\log r)^j \varphi_j(\theta); J \in \mathbb{N}, \varphi_j \in H^m_0(0, \omega) \cap H^{s+m}(0, \omega) \right\},$$

with $\lambda \in \mathbb{C}$. Additional connected and useful notation are:

$$P'(G) := \{ v \in S'(G); v \text{ is a polynomial in } x, y \},$$

$$E'(G, L_0) := \{ v \in S'(G); L_0v \text{ is a polynomial} \},$$

$$J^s := \dim E'(G, L_0)/P'(G).$$

### 2.3. The elliptic problem

An integer $s \geq m$ and a distribution $f \in H^{s-m}(Q)$ being fixed, we are concerned with the variational solution $u \in H^m_0(Q)$ of the problem

$$L(D_{xy}, D_z)u = f \quad \text{in } Q,$$

or equivalently (see Eq. (2.5)).

$$a(u, w) = \int_Q f w \, dx, \quad \forall w \in H^m_0(Q).$$

Problem (2.15) is well-posed due to the inequality (2.6) and the Lax-Milgram lemma. But the solution has a bad behaviour near the edge. (For simplicity, we do not consider the case $s < m$). In the light of [16], this singular behaviour of $u$ is, as we sketch now, the propagation and superposition along the edge of the singularities of a two-dimensional problem on $\Omega$. The latter, obtained by performing the Fourier transform (2.1) on (2.5) reads as follows: For any $\xi \in \mathbb{R}$, the Fourier transform $\hat{u}(\xi) \equiv \hat{u}(-\cdot, \xi) \in H^m_0(\Omega, |\xi| + 1)$ is the unique variational solution of the well-posed (cf. (2.9)) equivalent problems (2.16a) and (2.16b):

$$L(D_{xy}, \xi)\hat{u}(\xi) = \hat{f}(\xi) \quad \text{in } \Omega,$$

$$a_\xi(\hat{u}(\xi), v) = \int_\Omega \hat{f}(\xi)v \, dx, \quad \forall v \in H^m_0(\Omega, |\xi| + 1).$$

### Remark 2.1. (Tangential regularity) The solution $\hat{u}(\xi)$ of Eq. (2.16) satisfies the uniform (in $\xi$) inequality

$$(1 + |\xi|)^s \| \hat{u}(\xi) \|_{m, \Omega, 1+|\xi|} \leq C \| \hat{f}(\xi) \|_{s-m, \Omega, 1+|\xi|}.$$ 

This inequality is obtained by taking $v = \hat{u}(\xi)$ in Eq. (2.16b). The left-hand side of (2.16b) is then bounded from below using (2.9), while its right-hand side is bounded from above using Cauchy-Schwarz inequality together with $\| \hat{u}(\xi) \|_{m, \Omega, 1+|\xi|} \geq (1 + |\xi|)^m \| \hat{u}(\xi) \|_{0, \Omega}$ and $\| \hat{f}(\xi) \|_{s-m, \Omega, 1+|\xi|} \geq (1 + |\xi|)^{s-m} \| \hat{f}(\xi) \|_{0, \Omega}$. The inequality (2.17) agrees with the results of [36]. It implies partly that the solution $u$ of (2.15) has the expected optimal tangential regularity i.e.

$$D^\alpha_x D^\beta_y D^\gamma_z u \in L_2(Q), \quad \forall \alpha = (x_1, x_2, x_3) \in \mathbb{N}^3 \text{ with } |\alpha| \leq s + m \text{ and } x_1 + x_2 + x_3 \leq m.$$ 

(2.18)
According to Remark 2.1, the bad behaviour of \( u \) appears a priori for those \( x \) whose length does not meet the constraints in Eq. (2.18). To describe this, we denote by \( A \) the set of poles of the function \( \mathcal{L}(\lambda)^{-1} \). The singular functions associated with \( L_0 \) and \( \lambda \in A \) are defined in \( E^i(G,L_0) \) as representatives

\[
\sigma^{i\lambda}(x,y) = \lambda x^\lambda \sum_{q=0}^Q \frac{(\log r)^q}{q!} \varphi_{q-p}(0), \quad v = 1, \ldots, J^i, \tag{2.19a}
\]
of a basis of the space \( E^i(G,L_0)/P^i(G) \) (see (2.14) and Remark 2.2 below).

Regarding the operator \( L(D_{xy}, \zeta) \) in (2.7), the natural effect of the expansion (2.10)–(2.11) is an additional integer index \( p > 0 \) together with the parameter \( \zeta \) in the corresponding singular functions:

\[
\sigma^{i\lambda}_p(\zeta) = \sigma^{i\lambda}(x,y;\zeta) \in S^{i+p}(G). \tag{2.19b}
\]

These functions are defined by induction on \( p \) and by a ‘polynomial resolution’ [11], i.e.,

\[
\sigma^{i\lambda}_0(\zeta) := \sigma^{i\lambda}, \\
L_0\sigma^{i\lambda}_p(\zeta) = -\sum_{i=0}^{p-1} L_{p-i}(\zeta)\sigma^{i\lambda}_i \quad \text{in } G. \tag{2.20}
\]

Since we are interested in the \( H^{s+m}\)-regularity, the definitions (2.13), (2.14) and formula (2.19) imply that the above functions are effectively singular if the pole \( \lambda \) belongs to the set

\[
\hat{A}(s) := \{ \lambda \in A; m - 1 < \text{Re } \lambda < s + m - 1 \}. \tag{2.21}
\]

Note that the poles in the region \( \text{Re } \lambda < m - 1 \) have no interest while, according to Theorem 1 in [26], the strip \( \text{Re } \lambda \in [m - 1, m - \frac{1}{2}] \) is free of such poles. For \( \lambda \in \hat{A}(s) \), the expansion (2.10)–(2.11) and the relation (2.20) yield, for \( L(D_{xy}, \zeta) \), the singular functions

\[
\tau^{i\lambda}(\zeta) = \tau^{i\lambda}(x,y;\zeta) = \sum_{0 \leq p \leq s + m - 1 - \text{Re } \lambda} \sigma^{i\lambda}_p(x,y;\zeta) = \sum_{0 \leq p \leq s + m - 1 - \text{Re } \lambda} \zeta^p \sigma^{i\lambda}_p(x,y),
\]

where \( \sigma^{i\lambda}_p(x,y) := \sigma^{i\lambda}(x,y;1) \). In the second expression of \( \tau^{i\lambda}(\zeta) \), the variables \( x, y \) and \( \zeta \) of \( \sigma^{i\lambda}_p(x,y;\zeta) \) have been separate; furthermore each \( \sigma^{i\lambda}_p \) belongs to \( S^{i+p}(G) \) and has therefore a structure similar to (2.19a) (with \( \lambda + p \) instead of \( \lambda \)).

**Remark 2.2** (Bourlard et al. [11]). The functions \( \sigma^{i\lambda}_p \) have different interpretations according as \( \lambda \in \hat{A}(s) \) satisfies \( \lambda \notin \mathbb{N} \) or not. In the first case, \( \sigma^{i\lambda} \) solves the local problem

\[
L_0\sigma^{i\lambda} = 0 \quad \text{in } G
\]

and \( \sigma^{i\lambda} \) may be constructed via a suitable Jordan chain of \( \mathcal{L}(\lambda) \) at \( \lambda [35] \). On the other hand, when \( \lambda \in \mathbb{N} \), \( \sigma^{i\lambda} \) results from a polynomial resolution of the type (2.20) with the Taylor expansion of \( f(\zeta) \) about 0 as right-hand side.
The material collected until now permits to specify the behaviour of the solution of (2.16) near the singular point 0 of \( \Omega \). Since its bad behaviour is localized in a neighbourhood of 0, we introduce a cut-off function

\[ \varphi = \varphi(r) \in \mathcal{D}(\mathbb{R}_+), \quad \varphi = 1 \quad \text{near} \quad 0, \quad \text{supp} \varphi \subset B(0, \eta) \cap \Omega, \]

with a fixed and small enough \( \eta > 0 \).

**Lemma 2.3** (Grisvard [22], Kondratiev [25]). Assume that the following spectral condition holds:

For any \( \lambda \in \mathbb{C} \) with \( \text{Re} \lambda = s + m - 1 \), \( \mathcal{L}(\lambda) \) is invertible, i.e. \( \lambda \notin \Lambda \)

(see (2.12) for the definition of \( \mathcal{L}(\lambda) \)).

Then the solution \( \hat{u}(\zeta) \) of (2.16) admits the singular representation

\[ \hat{u}(x, y, \zeta) = w_1(x, y, \zeta) + \varphi(r) \sum_{(\lambda, \nu) \in A(s)} b^{\lambda, \nu}_1(\zeta) e^{\lambda \cdot x}(x, y, \zeta), \]

where

\[ A(s) = \{(\lambda, \nu) : \lambda \in \tilde{A}(s), \nu = 1, \ldots, J^s\}, \]

the regular part \( w_1(\zeta) \in H^{s+m}(\Omega) \) and the coefficients \( b^{\lambda, \nu}_1(\zeta) \in \mathbb{C} \) fulfil the following estimate with a constant \( c = c(\zeta) > 0 \):

\[ \|w_1(\zeta)\|_{s+m, \Omega} + \sum_{(\lambda, \nu) \in A(s)} |b^{\lambda, \nu}_1(\zeta)| \leq c \|\hat{f}(\zeta)\|_{s-m, \Omega}. \]

The transposition of Lemma 2.3 to the three-dimensional problem (2.15) is not easy. The first simple mind idea is to consider the inverse Fourier transform of (2.24). Although this argument is formal (because \( c \) in (2.26) depends on \( \zeta \)), the next result is valid due to the rigorous proof in [32, pp. 40–41].

**Lemma 2.4.** Under the assumption (2.23), there holds, for \( u \) solution of (2.15), the expansion

\[ u(x, y, z) = u_0(x, y, z) + \varphi(r) \sum_{(\lambda, \nu) \in A(s)} \sum_{0 \leq p \leq s+m-1-\text{Re} \lambda} \frac{d^p K^{\lambda, \nu}}{dz^p}(z) \sigma^{\lambda, \nu}_p(x, y), \]

where, with the notation in (2.22), \( K^{\lambda, \nu} \in H^{s+m-1-\text{Re} \lambda-\nu}(\mathbb{R}) \) for all \( \varepsilon > 0 \) and \( u_0 \in L_2(\mathbb{R}, H^{s+m-\nu}(\Omega)) \),

for some \( \varepsilon > 0 \).

Thus, there is a loss of ‘tangential’ and ‘normal’ regularity of the function \( u_0 \) in Lemma 2.4. The price to pay for getting the expected optimal regularity \( H^{s+m}(\Omega \times \mathbb{R}) \) is, as we outline now, the use of a more complex structure of the singular functions. Precisely, a suitable regularizing kernel \( \Phi(r, z) \) will improve the smoothness of the regular part \( u_0 \), while appropriate pseudo-differential operators \( \Psi^{s, \nu, m} \) influence the smoothness of \( K^{\lambda, \nu} \) (see Section 16 of [16]).

To this end, the expansion (2.24) will be considered only for \( |\zeta| \) small enough, say \( |\zeta| \leq \delta \). In this case, the constant \( c \) in (2.26) may be taken independent of \( \zeta \). The main concern is the case...
when $|\zeta| > \delta$. We first observe that the effect of the change of variables $(x, y) \rightarrow (|\zeta|x, |\zeta|y)$ on the function

$$
\sigma_p^{\lambda}(x, y) = r^{k+p} \sum_{q=0}^{\Omega_p} \frac{(\log r)^q}{q!} \phi_{\Omega_p-q}(\theta)
$$
in (2.22) is

$$
\sigma_p^{\lambda}(|\zeta|x, |\zeta|y) = |\zeta|^{k+p} \sum_{n=0}^{\Omega_p} \frac{(\log |\zeta|)^n}{n!} \sigma_p^{\lambda, n}(x, y)
$$

(2.27a)

where we set:

$$
\sigma_p^{\lambda, n} = r^{k+p} \sum_{q=0}^{\Omega_p-n} \frac{(\log r)^q}{s!} \phi_{\Omega_p-n-s}(\theta).
$$

(2.27b)

(Notice that $\sigma_p^{\lambda, 0} = \sigma_p^{\lambda, \lambda}$. Therefore, for $|\zeta|$ large enough, writing the analogue of (2.24) with sgn $\zeta$ instead of $\zeta$ and by the change of variables $(x, y) \rightarrow (|\zeta|x, |\zeta|y)$, it follows from (2.27) and by homogeneity that

$$
\hat{u}(x, y, \zeta) = w_2(x, y, \zeta)
$$

$$
+ \phi(|\zeta|) \sum_{(\lambda, \nu) \in \Lambda(s)} b_{2}^{\lambda, \nu}(\zeta) \sum_{0 \leq p \leq s+m-1-\Re \lambda} \xi^p \sum_{n=0}^{\Omega_p} \frac{(\log |\zeta|)^n}{n!} \sigma_p^{\lambda, n}(x, y); \quad (2.28)
$$

furthermore, we have the estimate

$$
\|w_2(\zeta)\|_{s+m, \Omega, 1+|\zeta|} + \sum_{(\lambda, \nu) \in \Lambda(s)} |b^{\lambda, \nu}(\zeta)| \cdot |\zeta|^{s+m-1-\Re \lambda} \leq c \cdot \| \hat{f} \|_{s-m, \Omega, 1+|\zeta|}, \quad (2.29)
$$

with a constant $c > 0$ not depending upon $\zeta$.

To couple Eqs. (2.24) and (2.26) for $|\zeta| \leq \delta$ with Eqs. (2.28) and (2.29) for $|\zeta| > \delta$, we fix a smooth function $\chi$ on $\mathbb{R}$ such that $\chi = 1$ or 0 according as $|\zeta| > \delta$ or $|\zeta| \leq \delta/2$. We also consider the function $m(r, \zeta)$ defined by $\phi(r)$ if $|\zeta| \leq \delta$ and by $\phi(r|\zeta|)$ if $|\zeta| > \delta$ (see also Remark 2.10 for an explicit approach). We are now able to state the next result, which follows from Theorem 16.9 of [16] in our context:

**Theorem 2.5.** Under the assumption (2.23), the solution $\hat{u}(\zeta)$ of (2.16) admits the ‘better’ singular decomposition (2.30) with the estimate (2.31) below:

$$
\hat{u}(x, y, \zeta) = w_2(x, y, \zeta) + \phi \sum_{(\lambda, \nu) \in \Lambda(s)} b^{\lambda, \nu}(\zeta) Y^{\lambda, \nu}(x, y, \zeta), \quad (2.30a)
$$

with

$$
Y^{\lambda, \nu}(x, y, \zeta) := m(r, \zeta) \sum_{0 \leq p \leq s+m-1-\Re \lambda} \xi^p \left\{ \sigma_p^{\lambda, \nu}(x, y) + \chi(|\zeta|) \sum_{n=1}^{\Omega_p} \frac{(\log |\zeta|)^n}{n!} \sigma_p^{\lambda, n}(x, y) \right\}, \quad (2.30b)
$$

$$
\|w_2(\zeta)\|_{s+m, \Omega, 1+|\zeta|} + \sum_{(\lambda, \nu) \in \Lambda(s)} |b^{\lambda, \nu}(\zeta)| \cdot |\zeta|^{s+m-1-\Re \lambda} \leq c \cdot \| \hat{f} \|_{s-m, \Omega, 1+|\zeta|+1}.
$$

(2.31)

The constant $c$ does not depend on $\zeta$. 

Notation 2.6. In view of Theorem 2.5, let us fix the following useful notation:

- The kernel \( \Phi = \Phi(r, z) \) is the inverse Fourier transform of the function \( m(r, \zeta) \); \( \Phi(r, z) \) is a regularizing kernel as \( r \to 0 \) (cf. [16, p. 129]).
- Let \( \Psi_{\lambda, \nu, n} \) be the pseudo-differential operator with symbol \( \chi(|\zeta|)((\log |\zeta|)^p/n!) \).

Estimate (2.31) legitimates the application of the inverse Fourier transform to (2.30). With the above notation, this yields the main result of this section:

Theorem 2.7. Under the assumption (2.23), there exist a regular function \( u_R \in H^{s+k}(Q) \cap H_{0}^{s}(Q) \) and stress intensity functions \( K_{\lambda, \nu} \in H^{s+k+1-\text{Re } \lambda}(\mathbb{R}) \), for all \( (\lambda, \nu) \in A(s) \), such that the solution \( u \) of problem (2.15) is decomposable as

\[
u = u + \varphi \sum_{(\lambda, \nu) \in A(s)} Z^{\lambda, \nu}(K_{\lambda, \nu}).
\]

The singular function is defined as the block generated by \( \sigma_{\lambda, \nu}^{p} \):

\[
Z^{\lambda, \nu}(K_{\lambda, \nu}) := \sum_{0 \leq p \leq s+k-\text{Re } \lambda} \left\{ \left( \frac{d^p K_{\lambda, \nu}}{dz^p} \ast \Phi \right) \sigma_{\lambda, \nu}^{p} + \sum_{\nu = 1}^{Q_p} \psi_{\lambda, \nu, n} \left( \frac{d^p K_{\lambda, \nu}}{dz^p} \ast \Phi \right) \sigma_{\lambda, \nu, n}^{p} \right\},
\]

where \( \ast \) represents the convolution product in the variable \( z \). Moreover, \( u \) has the optimal tangential regularity specified in (2.18).

Corollary 2.8. There also holds the tensor product decomposition

\[
u = \tilde{u} \sum_{(\lambda, \nu) \in A(s)} \tilde{Z}^{\lambda, \nu}(K_{\lambda, \nu}),
\]

where \( K_{\lambda, \nu} \in H^{s+k+1-\text{Re } \lambda}(\mathbb{R}) \) are those from (2.32), \( \tilde{u} \in L_{2}(\mathbb{R}, H^{s+k}(\Omega)) \) and

\[
\tilde{Z}^{\lambda, \nu}(K_{\lambda, \nu}) := \sum_{0 \leq p \leq s+k-\text{Re } \lambda} \left\{ \frac{d^p K_{\lambda, \nu}}{dz^p} \sigma_{\lambda, \nu}^{p} + \sum_{\nu = 1}^{Q_p} \psi_{\lambda, \nu, n} \left( \frac{d^p K_{\lambda, \nu}}{dz^p} \right) \sigma_{\lambda, \nu, n}^{p} \right\}.
\]

Assume in addition that \( s \) is large enough that there exists in (2.32)–(2.33) at least one \( \sigma_{\nu, n}^{p} \) satisfying \( s \geq s_0 + s_1 + 1 \), where \( s_0 = \text{Re } \lambda + p \) and \( s_1 \) is the maximum of those \( \text{Re } \lambda' + p' < s_0 \) corresponding to \( \sigma_{\nu, n}^{p, p'} \). Then, \( u \) is decomposable according to (2.34), (2.35) but with \( A(s_0) \) and \( s_0 \) in lieu of \( A(s) \) and \( s \); the function \( \tilde{u} \) in this new decomposition has the regularity \( \tilde{u} \in H_{0}^{s+k+1}(Q) \), for any \( \varepsilon > 0 \), while the regularity of \( K_{\lambda, \nu} \) does not change.

Remark 2.9. Notice that the first part of Corollary 2.8 is indeed an improvement of Lemma 2.4 since the expression of the singular functions in the latter contains less terms than the sum in (2.34)–(2.35). For the Laplacian, the first part is due to [21]. Observe in particular how the absence of regularizing kernels reduces the smoothness of the regular part in (2.34). However, if \( f \in C^{\infty}(Q) \), then the stress intensity functions \( K_{\lambda, \nu} \) are of class \( C^{\infty}(\mathbb{R}) \) (cf. [21] for the Laplacian).
The function $\tilde{u}_R$ has instead the anisotropic regularity $\tilde{u}_R \in L_2(\mathbb{R}, H^{s+m}(\Omega)) \cap H^\infty(\mathbb{R}, L_2(\Omega))$ (cf. [16, p. 137]). Again for the Laplace operator, the second part of Corollary 2.8 is considered in [40, Theorem 3]. The arguments of these authors which consist in a suitable perturbation of $u_R$ in (2.32) remain valid for the proof of the global lower regularity of $\tilde{u}_R$ to be exploited in Section 4.4.

Remark 2.10. For numerical purposes (see in particular Section 4.1), the kernel $m(r, \xi)$ in (2.30b), and therefore the regularizing kernel $\Phi(r, z)$ in Theorem 2.7, is by perturbation of (2.30a) chosen in the more explicit way:

$$m(r, \xi) = e^{-r|\xi|} \sum_{\ell=0}^{r-1} \frac{(r|\xi|)^\ell}{\ell!}.$$

This is the approach considered in [23] for the particular case of the Laplace operator with data $f$ in $L_2(Q)$. Extension of this approach to general operators is done in [33].

3. Approximation spaces

The cylinder $Q$ being unbounded in the edge direction, we only discretize its polygonal basis $\Omega$. To this end, let us fix a family $(\mathcal{T}_h)_{h > 0}$ of triangulations of $\Omega$ which consist of straight elements $K$ and which satisfy the usual properties [14, p. 38]. (The latter reference is our standard one for finite elements). The family is supposed to be regular, that is, the ratios $h_K/\rho_K$ between the exterior diameter $h_K$ and the interior diameter $\rho_K$ of elements $K \in \bigcup_{h > 0} \mathcal{T}_h$ are uniformly bounded from above, the maximal meshsize $h := \max_{K \in \mathcal{T}_h} h_K$ tending to zero.

Furthermore, the triangulation $\mathcal{T}_h$ is uniform in the sense that it fulfills the so-called inverse assumption: there exists $\sigma_1 > 0$ (independent of $h$) such that

$$h \leq \sigma_1 h_K, \quad \forall K \in \mathcal{T}_h, \quad \forall h > 0.$$

With each $K \in \bigcup_{h > 0} \mathcal{T}_h$, we associate a finite element $(K, P_K, \Sigma_K)$ with the four properties below:

(i) $P_K$ is a vector space of finite dimension $X$;
(ii) $P_{s+m-1}(K) \subset P_K \subset P_{s+m-1}(K) \subset H^{s+m}(K)$ for some $s \geq s$; here, for $\ell \in \mathbb{N}$, $P_{\ell}(K)$ denotes the finite-dimensional space of polynomials of degree at most $\ell$ on $K$;
(iii) $\Sigma_K$ is a finite set of $X$ linearly independent continuous linear forms on $H^{s+m}(K)$;
(iv) the local interpolation operator $\pi_K$ acting on $H^{s+m}(K)$ and the global interpolation operator $\pi_h$ defined on $H^{s+m}(\Omega)$ by

$$\pi_h v|_K = \pi_K(v|_K), \quad \forall v \in H^{s+m}(\Omega), \quad (3.1a)$$

fulfill the compatibility condition

$$\pi_h v \in H^{s+m}_0(\Omega), \quad \forall v \in H^m(\Omega) \cap H^{s+m}(\Omega). \quad (3.1b)$$

The space $H^m_0(\Omega)$ is then approximated by the classical finite element space

$$V_h(\Omega) := \{v_h \in H^m_0(\Omega); v_h|_K \in P_K, \quad \forall K \in \mathcal{T}_h\}. \quad (3.2)$$
Notice that $V_h(\Omega)$ is indeed finite-dimensional; on the contrary, the following space which we use for the approximation of $H^s_0(Q)$ is infinite-dimensional:

$$W_h(Q) := \{ w_h \in H^s_0(Q); \hat{w}_h(\cdot, \cdot, \xi) \in V_h(\Omega) \text{ for a.e. } \xi \in \mathbb{R} \}. \tag{3.3}$$

In the Note [31], we should have observed that this space is however complete as we now show:

**Proposition 3.1.** The space $W_h(Q)$ is closed in $H^s_0(Q)$.

**Proof.** Let $(w^i)_{i \geq 1}$ be a sequence in $W_h(Q)$ which converges to $w \in H^s(\Omega)$ in the norm of $H^s_0(Q)$. From Eqs. (2.2) and (2.3), it follows that

$$\lim_{j \to \infty} \int_{\mathbb{R}} \| \hat{w}^i(\xi) - \hat{w}(\xi) \|^2_{0,\Omega} \, d\xi = 0.$$  

By Corollary 2.11 in [1] (see the proof of Theorem 2.10 in [1]), there exists a subsequence still denoted by $(\hat{w}^i)$ such that $\| \hat{w}^i(\xi) - \hat{w}(\xi) \|_{0,\Omega}$ converges to 0, for almost every $\xi \in \mathbb{R}$. Since for such $\xi$, the function $\hat{w}^i(\xi)$ belongs to the closed space $V_h(\Omega)$, the latter convergence implies that the limit $\hat{w}(\xi)$ belongs to $V_h(\Omega)$ as well. \( \square \)

Although the spaces $V_h(\Omega)$ and $W_h(\Omega)$ are well-defined, it must be understood that, at this stage, they do not have any approximation property. We have to impose further conditions. For example, in [12,14], the interpolation operators $\pi_h$ are subject to constraints. Instead of following these authors, we rather consider here an alternative approach which has the advantage of allowing interpolation of nonsmooth functions. More precisely, we assume that the space $V_h(\Omega)$ fulfills the approximation property below: there exists an operator $r_h,0$ from $L^2(\Omega)$ into $\{v_h \in H^m(\Omega); v_h|_K \in P_K, \forall K \in \mathcal{T}_h \}$ such that $r_h,0v \in V_h(\Omega)$ for any $v \in H^m_0(\Omega)$; moreover for any integers $0 \leq i \leq s + m, i \leq m$,

$$|v - r_h,0v|_{i,\Omega} \leq c h^{i-1} |v|_{s+m,\Omega}, \quad \forall v \in H^i(\Omega), \tag{3.4}$$

where $c$ denotes, here and after, various positive constants independent of the meshsize $h$ and of the real parameter $\xi$.

**Remark 3.2.** The above hypotheses on the approximation spaces $V_h(\Omega)$ can be effectively met. For example, for triangular polynomial finite elements of class $C^0$ $(m = 1)$, the operator $r_h,0$ is built in [15] by local regularization (see also [20, Appendix A] and [31]). \( \square \)

**Proposition 3.3.** The space $W_h(Q)$ has the following approximation property: to every $w \in H^{s+m}(Q) \cap H^m_0(Q)$, there corresponds a function $R_h,0w \in W_h(Q)$ such that

$$\| w - R_h,0w \|_{m,\Omega} \leq c h^s \| w \|_{s+m,\Omega}.$$  

(Interpolation in Sobolev spaces of fractional order will be mentioned in Sections 4.3 and 4.4.)

**Proof.** First of all, a direct consequence of the estimates (3.4), for $v \in H^{s+m}(\Omega)$ and $\xi \in \mathbb{R}$, is

$$\| v - r_h,0v \|_{m,\Omega,1+|\xi|} \leq c h^s \left( \sum_{n=0}^{m} (1 + |\xi|)^{2n} |v|^2_{s+m-n,\Omega} \right)^{1/2}. \tag{3.5}$$
Consider now \( w \in H^{s+m}(\Omega) \cap H^m_0(\Omega); \) for almost every \( \xi \in \mathbb{R}, \) the expression 
\[
v_\xi(x, y) := \hat{w}(x, y, \xi),
\]
defines a function \( v_\xi \) on \( \Omega \) which belongs to \( H^{s+m}(\Omega) \cap H^m_0(\Omega) \) and fulfills the estimate (3.5). The requested element \( R_{h, \xi} w \) of \( W_h(\Omega) \) is then defined by 
\[
(R_{h, \xi} w)(x, y, \xi) := r_{h, \Omega} v_\xi(x, y).
\]
The proof of the proposition is complete by combining (3.5) for \( v_\xi \) and the equivalence of norms stated in (2.2).

4. The singular function method

4.1. Optimally convergent semi-discrete FEM

To take into account the singular structure of the exact solution \( u \) specified in Theorem 2.7 and Notation 2.6, \( H^m_0(\Omega) \) is approximated by the enriched subspaces
\[
W^+_h(\Omega) := W_h(\Omega) \oplus \left\{ \phi \sum_{(\lambda, \gamma) \in \mathcal{A}(s)} Z^{\lambda, \gamma}(q^{\lambda, \gamma}); q^{\lambda, \gamma} \in H^{s+m-1-Re(\lambda)}(\mathbb{R}) \right\}.
\]
(4.1)

The semi-discrete singular function method reads as follows: find 
\[
u^+_h = u_h + \sum_{(\lambda, \gamma) \in \mathcal{A}(s)} Z^{\lambda, \gamma}(K^{\lambda, \gamma}_h) \in W^+_h(\Omega),
\]
solution of (cf. (2.15))
\[
a(u^+_h, w^+_h) = \int_{\Omega} f w^+_h \, dx, \quad \forall w^+_h \in W^+_h(\Omega).
\]
(4.2b)

Theorem 4.1. Problem (4.2) has one and only one solution \( u^+_h; \) this solution depends continuously upon the datum \( f \in H^{s-m}(\Omega) \) (stability of the method):
\[
\|u^+_h\|_{m, \Omega} \leq c \|f\|_{s-m, \Omega}.
\]

Whether the subspace \( W^+_h(\Omega) \) of \( H^m_0(\Omega) \) is closed or not requires further investigations. Consequently, the hypothesis of the Lax–Milgram lemma may not be guaranteed for the proof of Theorem 4.1. Even if the lemma was applicable, the infinite dimension of \( W^+_h \) would still be an obstacle to a constructive solution of problem (4.2). Therefore, we use Propositions 4.3 and 4.8 below which, besides, provide a practical way of solving (4.2). To this end, we associate with any \( \mu \in \mathbb{C} \) and \( h > 0, \) the following finite-dimensional subspace of \( H^m_0(\Omega) \)
\[
V^+_h(\mu, \Omega) := V_h(\Omega) \oplus \left\{ \phi \sum_{(\lambda, \gamma) \in \mathcal{A}(s)} c^{\lambda, \gamma} Y^{\lambda, \gamma}(\mu); c^{\lambda, \gamma} \in \mathbb{C} \right\},
\]
(4.3)
where $Y_{\lambda \nu}(\mu)$ is defined by Eq. (2.30b) with $\mu$ in lieu of $\zeta$, the kernel $m$ being specified in Remark 2.10.

**Remark 4.2.** Contrary to the comments following Theorem 4.1, the subspace
\[ \tilde{W}_h(Q) := \{ w_h \in H^m_0(Q); \hat{w}_h(\zeta) \in V^+_h(\zeta, \Omega), \text{ for a.e. } \xi \in \mathbb{R} \} \]
which contains $W^+_h(Q)$, is closed in view of the proof of Proposition 3.1.

**Proposition 4.3.** For each $h > 0$ and $\xi \in \mathbb{R}$, there exists a function
\[ u_h^{\lambda \nu} = u_h^{\zeta} + \varphi \sum_{(\lambda, \nu) \in A(s)} b_{h}^{\lambda \nu \zeta} Y_{\lambda \nu}(\zeta) \in V^+_h(\zeta, \Omega), \tag{4.4a} \]
which is the unique solution of the discrete problem associated with (2.16):
\[ a(\hat{u}^{\lambda \nu \zeta}_h, \hat{v}^{\lambda \nu \zeta}_h) = \int_{\Omega} \hat{f}(\xi) \hat{v}^{\lambda \nu \zeta}_h \, dx, \quad \forall \hat{v}^{\lambda \nu \zeta}_h \in V^+_h(\zeta, \Omega). \tag{4.4b} \]
Furthermore, the solution obeys the inequality
\[ (1 + |\xi|)^2 \| u_h^{\lambda \nu} \|_{m, \Omega, 1 + |\xi|} \leq c \| \hat{f}(\xi) \|_{s - m, \Omega, 1 + |\xi|}. \tag{4.5} \]

**Proof.** Owing to the property (2.9), the proposition is essentially a straightforward consequence of the Lax–Milgram lemma. The estimate (4.5) is obtained as in the continuous case described in (2.17).

In order to get the solution of (4.2b) in the form (4.2a), the natural idea is to apply the inverse Fourier transform to (4.4a). This requires suitable estimates based on the next two lemmas inspired by Lemmas 7.1 and 8.1 in [10].

**Lemma 4.4.** For any $(\lambda, \nu) \in A(s)$ and all $w_h \in V_h(\Omega)$, we have
\[ \| \varphi Y_{\lambda \nu}(\xi) - w_h \|_{m, \Omega, 1 + |\xi|} \geq c h^{1 - m + \Re \lambda} (1 + |\xi|)^{m - 1 - \Re \lambda}. \tag{4.6} \]

**Proof.** We follow the proof of Lemma 7.1 of [10] with the necessary adaptation due to the dependence on $\xi$.

Let $\omega_0 > 0$ be the smallest angle of all triangles of $\bigcup_{h > 0} \mathcal{T}_h$. For each $h$, let us fix one $K \in \mathcal{T}_h$ containing 0. Denote by $\rho$ the interior diameter of $K$ and set
\[ C_\rho := \left\{ (r, \theta) : 0 < \theta < \omega_0, \ 0 < r < \frac{\rho}{2} \right\} \subset K. \]
Then, with $h$ small enough so that $\varphi = 1$ on $C_\rho$, we have
\[ \| \varphi Y_{\lambda \nu}(\xi) - w_h \|_{m, \Omega, 1 + |\xi|} \geq |Y_{\lambda \nu}(\xi) - w_h|_{m, C_\rho}. \tag{4.7} \]
If $|\xi| \leq \delta$, then the specificity of $Y_{\lambda \nu}(\xi)$ in (2.30b) and the argument of the proof of Lemma 7.1 in [10] yields, for small enough $\rho$,
\[ |Y_{\lambda \nu}(\xi) - w_h|_{m, C_\rho} \geq c \rho^{\Re \lambda - m + 1}. \tag{4.8} \]
In the case when $|\xi| > \delta$, the change of variable $F(s, \theta) = (|\xi|^{-1}s, \theta)$ which maps $C_{\rho|\xi|}$ into $C_{\rho}$ and the choice of $m$ in Remark 2.10 leads, as in the first case, to
\[
|Y^{\lambda, \rho}(\xi) - w_h|_{m,C_{\rho}} \geq c|\xi|^{m-1} - \text{Re}\, \lambda |^{1-m+\text{Re}\, \lambda}.
\] (4.9)
The inverse assumption insures that $\rho \geq c h$; therefore (4.8) and (4.9) prove (4.6). \qed

**Lemma 4.5.** Let us fix $(\lambda, \nu) \in \mathcal{A}(s)$ and define
\[ W^{\lambda, \nu}_h = V_h(\Omega) \oplus \text{Sp}\{\varphi Y^{\lambda, \nu}(\xi); (\lambda', \nu') \neq (\lambda, \nu) \text{ and } (\lambda', \nu') \in \mathcal{A}(s)\}.
\] Then there exists a constant $c(h) > 0$ independent of $\xi \in \mathbb{R}$ (but which depends upon $h$) such that for all $w^{\lambda, \nu}_h \in W^{\lambda, \nu}_h$, we have
\[
\|\varphi Y^{\lambda, \nu}(\xi) - w^{\lambda, \nu}_h\|_{m,\Omega,1+|\xi|} \geq c(h)(1 + |\xi|)^{m-1-\text{Re}\, \lambda}.
\] (4.10)

**Proof.** For small value of $\xi$, the estimate (4.10) follows from the continuous dependence with respect to $\xi$ and the fact that the singular functions $Y^{\lambda, \nu}(\xi)$ are linearly independent modulo $H^{s+m}$ [16, p. 85].

For large value of $\xi$, we proceed as in the previous Lemma: by the change of variable $F(s, \theta) = (|\xi|^{-1}s, \theta)$, one gets for small enough $\rho$ (see (2.22)):
\[
\|\varphi Y^{\lambda, \nu}(\xi) - w^{\lambda, \nu}_h\|_{m,\Omega,1+|\xi|} \geq c|\xi|^{m-1-\text{Re}\, \lambda} |\varphi^{\lambda, \nu}(1) - |\xi|^{\lambda}(w^{\lambda, \nu}_h \circ F)|_{m,C_{\rho}}.
\]
In view of the definition of the space $W^{\lambda, \nu}_h$, we have
\[
w^{\lambda, \nu}_h \circ F \in \tilde{W}^{\lambda, \nu}_h := P_{s+m-1}(C_{\rho}) \oplus \text{Sp}\{\tau^{\lambda, \nu}(1); (\lambda', \nu') \neq (\lambda, \nu) \text{ and } (\lambda', \nu') \in \mathcal{A}(s)\}.
\] Consequently, if $P^{\lambda, \nu}_h$ denotes the projection onto $\tilde{W}^{\lambda, \nu}_h$ for the inner product of $H^m(C_{\rho})/P_{m-1}(C_{\rho})$, we may write
\[
\|\varphi Y^{\lambda, \nu}(\xi) - w^{\lambda, \nu}_h\|_{m,\Omega,1+|\xi|} \geq c|\xi|^{m-1-\text{Re}\, \lambda} |(I - P^{\lambda, \nu}_h)\tau^{\lambda, \nu}(1)|_{m,C_{\rho}}.
\]
As above, since the $\tau^{\lambda, \nu}$'s are linearly independent, we deduce that
\[
|(I - P^{\lambda, \nu}_h)\tau^{\lambda, \nu}(1)|_{m,C_{\rho}} > 0,
\]
which leads to the estimate (4.10) for large value of $\xi$ (remark that the above constant depends upon $h$). \qed

**Remark 4.6.** We conjecture that the constant $c(h)$ in (4.10) is bounded from below by $c \cdot h^{1-m+\text{Re}\, \lambda}$, but the complex structure of $\tau^{\lambda, \nu}(1)$ forbids us to make the change of variable $s' = \rho s$ as in Lemma 8.1 of [10]. This holds, for instance, if $\tau^{\lambda, \nu}(1)$ coincides with the singular function $\sigma_0^{\lambda, \nu}$ of $L_0$, for all $(\lambda, \nu) \in \mathcal{A}(s)$ (note that this is always the case when $s = 1$).

Let us now come back to problem (4.4b):

**Lemma 4.7.** Let $u_h^{\lambda, \nu}$ of the form (4.4a) solve (4.4b). Then there exist two positive constants $c_1, c_2(h)$ independent of $\xi$ ($c_1$ is also independent of $h$, while $c_2$ depends on $h$) such that
\[
c_1\|u^{\lambda, \nu}_h\|_{m,\Omega,1+|\xi|} \leq \|u^{\lambda, \nu}_h\|_{m,\Omega,1+|\xi|} + \sum_{(\lambda', \nu') \in \mathcal{A}(s)} |b^{\lambda', \nu'}(\xi)| (1 + |\xi|)^{m-1-\text{Re}\, \lambda} \leq c_2(h)\|u^{\lambda, \nu}_h\|_{m,\Omega,1+|\xi|}.
\]
Proof. The left-hand side inequality is clear by the specificity of \( Y^{i,j}(\xi) \). As in the previous lemmas, we easily check that
\[
\| \phi Y^{i,j}(\xi) \|_{m,\Omega,1+|\xi|} \leq c (1 + |\xi|)^{m-1-\text{Re} \, i}.
\] (4.11)

For the second estimate, let us enumerate the (finite) set \( A(s) = \{(i_j, v_j)\}_{j=1,\ldots,I} \) (the order does not matter) and denote by \( P_i \) the orthogonal projection onto \( V_h(\Omega) \oplus Sp\{ \phi Y^{i_j,v_j}; j < i \} \) with respect to the norm \( \| \cdot \|_{m,\Omega,1+|\xi|} \). Then we show by induction that
\[
\| u_h^i \|_{m,\Omega,1+|\xi|} + \sum_{j<i} |b^j| (1 + |\xi|)^{1-m-\text{Re} \, i} \leq c_i(h) \left( u_h^i + \sum_{j<i} b^j(\phi Y^{i_j,v_j}) \right) \| \|_{m,\Omega,1+|\xi|}.
\] (4.12)
for all \( i = 0, 1, \ldots, I \), with the convention that the sum is zero if \( i = 0 \) and writing in short \( b^i \) for \( b_{h,j}^{i,v} \).

Clearly, the estimate (4.12) holds for \( i = 0 \). Let us now prove that if (4.12) holds for \( i-1 \), then it also holds for \( i \). Indeed by the induction hypothesis, the properties of the projection, the triangular inequality and (4.11), we have
\[
\| u_h^i \|_{m,\Omega,1+|\xi|} + \sum_{j<i} |b^j| (1 + |\xi|)^{1-m-\text{Re} \, i} \leq c_{i-1}(h) \left( u_h^i + \sum_{j<i} b^j(\phi Y^{i_j,v_j}) + b^i P_i(\phi Y^{i_j,v_j}) \right) \| \|_{m,\Omega,1+|\xi|}.
\] (4.13)
As \( P_i(\phi Y^{i_j,v_j}) \) belongs to the space \( W_{h,j}^{i_j,v_j} \) introduced in Lemma 4.5, the estimate (4.10) yields
\[
|b^j| (1 + |\xi|)^{1-m-\text{Re} \, i} \leq c_i(h)(I - P_i)(\phi Y^{i_j,v_j}) \| \|_{m,\Omega,1+|\xi|}
\]
This estimate in the second term of the right-hand side of (4.13) and the properties of the projection lead to (4.12) for \( i \).

The proof of the second estimate is complete since it corresponds to \( i = I \) in (4.12). ☐

Proposition 4.8. Problem (4.2) is equivalent to the family in \( \xi \in \mathbb{R} \) of problems (4.4), the solutions being, for almost every \( x, y \) and \( z \), related by the inverse Fourier transforms:
\[
u_h^+(x, y, z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} u_h^+(x, y) \, d\xi, \quad K_h^{i,j}(z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi z} b_h^{i,j}(z) \, d\xi.
\]

Proof. Observe first that, as in the proof of Proposition 4.3, any solution of (4.2) satisfies the inequality in Theorem 4.1.

Assume now that \( u_h^+ \) of the form (4.2a) is a solution of (4.2b). Then, clearly (cf. (2.30b) and (2.33)) the Fourier transform
\[
u_h^+(x, y, z) = \tilde{u}_h^+(\xi) = \hat{u}_h(\xi) + \phi(r) \sum_{(i,v) \in A(s)} \hat{K}_h^{i,j}(\xi) Y^{i,j}(\xi)
\]
Theorem 4.9. Under the conditions of Theorem 2.7, there holds the following error estimate:
\[ \| u - u_h^+ \|_{m,Q} \leq C h^r \| u_R \|_{s+m,Q}. \]

Proof. By Céa’s lemma, we have the quasi-optimal error estimate:
\[ \| u - u_h^+ \|_{m,Q} \leq \inf_{w_h^+ \in W_h^+(Q)} \| u - w_h^+ \|_{m,Q}. \]
Choose \( w_h^+ = R_{h,Q} u_R + q^+(\cdot) \) where \( u_R \) is the regular part of \( u \) and the sum is its singular part. Applying the approximation property of \( W_h^+(Q) \) (cf. Proposition 3.3) to \( u_R \), we obtain the theorem by decomposing \( u \) as in (2.32). 

Regarding the convergence in lower norms \( \| \cdot \|_{m-l} \), the optimal rates are restored under more restrictive conditions as we specify now.

Theorem 4.10. Fix a positive integer \( l \leq s \). Assume that, in addition to the conditions of Theorem 2.7, the hypothesis (2.23) holds also with \( l \) in lieu of \( s \). Then
\[ \| u - u_h^+ \|_{m-l,Q} \leq C h^{s+l} \| R_{h,Q} u_R \|_{s+m,Q}. \]

Proof. The result is based on Aubin-Nitsche lemma [14, Th. 3.2.4]. The proof works as that of Theorem 3.4 in [29], the conditions on \( l \) guaranteeing the \( O(h^l) \) convergence in \( H^m(Q) \) of the SFM (4.2) (cf. Theorem 4.9) for the adjoint problem to (2.15b). 

With an arbitrary \( \eta \equiv \eta(z) \) and the above \( v_{h}^{+\xi} \), we associate the function
\[ w_h^+ = v_h(x,y,z) = v_h(x,y) \eta(z) + q^+(\cdot) \]
and using this as a test function in (4.2b), it follows from Parseval’s identity that
\[ \| u_h^+ \|_{m,Q} \leq C h^r \| u_R \|_{s+m,Q}. \]

Proof. The result is based on Aubin-Nitsche lemma [14, Th. 3.2.4]. The proof works as that of Theorem 3.4 in [29], the conditions on \( l \) guaranteeing the \( O(h^l) \) convergence in \( H^m(Q) \) of the SFM (4.2) (cf. Theorem 4.9) for the adjoint problem to (2.15b). 

Integrating with respect to \( \xi \) both members of (4.4b), we obtain the claim by Parseval’s identity. 

\[ \Box \]
Concerning the error between the approximate intensity functions in (4.2a) and the exact ones in (2.32), we have the following partial result which extends [17] and [10].

**Theorem 4.11.** Assume that the poles \( \lambda \in \hat{A}(s) \) are simple (thus, the superscript \( v \) can be dropped) and that, with the notations from Lemma 4.5, the next estimate holds:

\[
\| \phi Y^j(\xi) - w_h^j \|_{m, \Omega, 1 + |\xi|} \geq c h^{1 + m + \Re \lambda} (1 + |\xi|)^{-m + 1 - \Re \lambda},
\]

for all \( w_h^j \in W_h^j \). Then

\[
\|K^\lambda - K_h^\lambda\|_{m, \Omega, 1 + |\xi|, \mathbb{R}} \leq c h^{m + 1 - \Re \lambda}.
\]

Proof. For the sake of simplicity, we prove (4.15) in the case when \( \hat{A}(s) \) is a singleton \( \{\lambda\} \). We refer to Theorem 9.1 of [10] for the general situation. Using the equivalence between problems (4.2) and (4.4) obtained in Proposition 4.8, we write

\[
\begin{align*}
\widehat{u_h^j}(\xi) &= \widehat{u_\lambda}(\xi) + b_h^{\lambda, j} \phi Y^j(\xi), \\
\widehat{u}(\xi) &= \widehat{u_\lambda}(\xi) + b^{\lambda, j} \phi Y^j(\xi),
\end{align*}
\]

with \( \widehat{u_\lambda}(\xi) \in V_h(\Omega) \), \( \widehat{u} \in H^{1+m}(\Omega, 1 + |\xi|) \), \( b_h^{\lambda, j} = K_h^\lambda(\xi) \) and \( b^{\lambda, j} = K^\lambda(\xi) \). As in Theorem 9.1 of [10], the above two identities yield the expression

\[
b_h^{\lambda, j} - b^{\lambda, j} = \frac{a_\lambda((I - G_h^\lambda)\widehat{u_R}, (I - G_h^\lambda)(\phi Y^j))}{a_\lambda((I - G_h^\lambda)(\phi Y^j), (I - G_h^\lambda)(\phi Y^j))},
\]

where \( G_h^\lambda u \) is the Galerkin approximation of \( u \) in \( V_h(\Omega) \), i.e.

\[
a_\lambda(u, v_h) = a_\lambda(G_h^\lambda u, v_h), \quad \forall v_h \in V_h(\Omega).
\]

Owing to the uniform coerciveness of \( a_\lambda \) (estimate (2.9)) and the uniform continuity, the above identity implies

\[
|b_h^{\lambda, j} - b^{\lambda, j}| \leq c \frac{\|((I - G_h^\lambda)\widehat{u_R})\|_{m, \Omega, 1 + |\xi|}}{\|((I - G_h^\lambda)(\phi Y^j))\|_{m, \Omega, 1 + |\xi|}}.
\]

(4.16)

By Proposition 3.3 (see (3.5)), we deduce that

\[
\begin{align*}
\frac{\|((I - G_h^\lambda)\widehat{u_R})\|_{m, \Omega, 1 + |\xi|}}{\|((I - G_h^\lambda)(\phi Y^j))\|_{m, \Omega, 1 + |\xi|}} &\leq c \frac{h^\lambda}{h^\lambda}, \\
&\leq c \frac{h^\lambda}{\|\widehat{f}\|_{m, \Omega, 1 + |\xi|}}.
\end{align*}
\]

With the help of (4.14), the inequality (4.16) then becomes

\[
|b_h^{\lambda, j} - b^{\lambda, j}| \leq c h^{1 + m - \Re \lambda - m - 1 + \Re \lambda} \|\widehat{f}\|_{m, \Omega, 1 + |\xi|}.
\]

By inverse Fourier transform, we get (4.15). \( \square \)

**Remark 4.12.** 1. The assumption (4.14) is realistic and actually holds if there is one simple eigenvalue in \( \Lambda(s) \) or if \( \chi^{\lambda, j}(1) = \sigma_0^{\lambda, v} \), for all \( \lambda \in \Lambda(s) \) (see Remark 4.6).

2. The error bound in Theorem 4.11 is obtained using the lower norm \( H^{m - 1 + \Re \lambda}(\mathbb{R}) \). The same estimate remains valid in the \( L_2(\mathbb{R}) \) norm (resp. \( H^{1 + m - 1 + \Re \lambda}(\mathbb{R}) \)) provided the datum \( f \) is more...
regular, namely $D_i^2f \in H^{2-m}(Q)$, for all $i = 0, \ldots, I$ with $I \geq \Re \lambda_{-m+1}$ (resp. $I = s$). It is in fact under this first assumption that the second estimate in Theorem 2.4 of [31] is valid. Anyway, Theorem 4.11 agrees with the classical poor convergence feature of the SFM. Therefore, it would be interesting to investigate whether better approximate stress intensity functions may be obtained by alternative techniques as, for instance, the dual singular function method used in [12] for two-dimensional problems.

4.2. Practical computations of discrete solutions

An alternative semi-discrete singular function method consists in considering the space $\tilde{W}_h(Q)$ (cf. Remark 4.2) as the space of test and trial functions in (4.2a) and (4.2b). The advantage of such an approach is that the well-posedness, i.e. the proof of Theorem 4.1, results from the Lax–Milgram lemma since $\tilde{W}_h(Q)$ is a complete space. In this section, we discuss in a more practical way a somewhat similar approach. To this end, let us assume that the cylinder $Q$ can be replaced with the prism $\Omega \times [0, 1]$ (or an axisymmetric domain).

In this case, it is more practical to replace the Fourier transform $\hat{w}(\xi)$, $\xi \in \mathbb{R}$, of a function $w(z) \equiv w(x, y, z)$ with the Fourier coefficients $w_k \equiv w_k(x, y)$, $k \in \mathbb{N}^*$, of its Fourier series

$$w(z) = \sum_{k=1}^{\infty} w_k \sin(k\pi z).$$

Performing the modifications whenever necessary in the previous section, Proposition 4.8 guarantees that the solution of the semi-discrete singular function method (4.2) is given by the Fourier series:

$$u^+_h(x, y, z) = \sum_{k=1}^{\infty} u^+_{h,k}(x, y) \sin(k\pi z); \quad K^{\lambda,\nu}_h(z) = \sum_{k=1}^{\infty} b^{\lambda,\nu,k}_h \sin(k\pi z). \quad (4.17)$$

The function $u^+_{h,k}(x, y)$ and the constants $b^{\lambda,\nu,k}_h$ are described in the analogue of Proposition 4.3 as solutions of problems (4.4) for $\xi = k$.

Our aim is to compute the functions in (4.17). Following the idea of [24,37], we fix an integer $N \in \mathbb{N}^*$ and solve the finite sequence of problems (4.4) corresponding to $\xi = k$, $k = 1, 2, \ldots, N$. This yields the following truncated Fourier series as approximations of $u^+_h$ and $K^{\lambda,\nu}_h$ in (4.17):

$$u^+_{N,h} = u_{N,h} + \varphi(r) \sum_{k=1}^{N} b^{\lambda,\nu,k}_h Y^{\lambda,\nu}(k) \sin(k\pi z), \quad (4.18a)$$

$$K^{\lambda,\nu}_{N,h} = \sum_{k=1}^{N} b^{\lambda,\nu,k}_h \sin(k\pi z); \quad u_{N,h} = \sum_{k=1}^{N} u^+_h \sin(k\pi z). \quad (4.18b)$$

**Theorem 4.13.** With the notation (4.17) and (4.18), we have the error estimate:

$$\|u - u^+_{N,h}\|_{\Omega \times [0, 1]} \leq c \left(h^r + N^{-s}\right),$$

whereas the following holds under the conditions of Theorem 4.11:

$$\|K^{\lambda,\nu} - K^{\lambda,\nu}_{N,h}\|_{m-1 - \Re \lambda, [0, 1]} \leq c \left(h^{r+m-1 - \Re \lambda} + N^{-s-m+1 - \Re \lambda}\right).$$
Proof. Denote by \(u_N\) and \(K^N\) the respective truncated Fourier series of the exact solutions \(u\) and \(K^x\).

On the one hand, Theorems 4.9 and 4.11 yield
\[
\|u_N - u^x_h\|_{m, \Omega \times [0, 1]} = O(h^s); \quad \|K^N - K^x_h\|_{m - 1 - \Re \lambda, [0, 1]} = O(h^{s + m - 1 - \Re \lambda}).
\]

On the other hand, the tangential regularity of \(u\) and \(K^x\) in Theorem 2.7 implies the classical error estimates below on truncated Fourier series (cf. [13, Section 9.1.2]):
\[
\|u - u_N\|_{m, \Omega \times [0, 1]} = O(N^{-s});
\]
\[
\|K^x - K^x_N\|_{m - 1 - \Re \lambda, [0, 1]} \leq c \|K^x - K^x_N\|_{0, [0, 1]} \leq c h^{1 + \Re \lambda - s - m}.
\]

Theorem 4.13 follows then by triangular inequality and the above asymptotic error estimates.

Remark 4.14. If the above conditions for the use of Fourier series are not met, the effective practical computation of \(u^x_h\) and \(K^x_h\) may be done by filtering and sampling the functions \(\xi \rightarrow u^x_h\) and \(\xi \rightarrow b^x_h\) (see [42]).

4.3. Other improved convergent schemes

As usual, with the classical finite element method where, in (4.2), \(W^+_h(Q)\) is replaced with \(W_h(Q)\) the convergence of the discrete solution \(u_h \in W_h(Q)\) to \(u\) is slow. More precisely, the non-integer version of Theorem 2.7 (see [16, Theorem 16.9]) yields the maximal regularity:
\[
u \in H^{m+\sigma - \varepsilon}(Q), \quad \forall \varepsilon > 0; \quad \sigma := \min \{\Re \lambda > m - 1; \lambda \in A\} - m + 1.
\]

Therefore, Céa’s lemma, Proposition 3.3 and interpolation theory in Sobolev spaces (cf. [7]) yield the estimate:
\[
\|u - u_h\|_{m, Q} \leq c \cdot h^{\sigma - \varepsilon} \|u\|_{m + \sigma - \varepsilon, Q}.
\]

We shall however be concerned with a method which is better than the classical finite element method. The algorithm is based on the tensor product decomposition in the first part of Corollary 2.8 and may be extended to polyhedrons thanks to the results in [21,40]. To this end, we introduce the functions \(\tilde{Y}^\varepsilon(\xi) \equiv \tilde{Y}^\varepsilon(x, y, \xi)\) defined as \(Y^\varepsilon(\xi)\) in (2.30) but without the kernel \(m(r, \xi)\). From the functions \(\tilde{Y}^\varepsilon(\xi)\), we define, according to formula (4.3), a finite-dimensional subspace \(\tilde{V}_h(\xi, \Omega)\) of \(H^m_\sigma(\Omega)\).

For \(\xi \in \mathbb{R}\), we denote by \(\tilde{u}_h^\varepsilon \in \tilde{V}_h(\xi, \Omega)\) the unique solution of problem (4.4b) where the space of trial and test functions is \(\tilde{V}_h(\xi, \Omega)\). Owing to the estimate (2.9), the solution satisfies \(\|\tilde{u}_h^\varepsilon\|_{m, \Omega} \leq c \|f(\xi)\|_{s-m, \Omega}\) and even (4.5). By Céa’s lemma, the approximation property (3.4) and the decomposition (2.34), the solution \(\tilde{u}_h^\varepsilon\) satisfies also the inequality:
\[
\|\tilde{u}(\xi) - \tilde{u}_h^\varepsilon\|_{m, \Omega} \leq c \cdot h^s \|\tilde{u}_h(\xi)\|_{s+m, \Omega}.
\]

By inverse Fourier transform, we have proven the following result:
Theorem 4.15. The function $\tilde{u}_h(x, y, z) := (1/\sqrt{2\pi}) \int_{\mathbb{R}} e^{i\xi \cdot \zeta} u_h(x, y) \, d\xi$ approximates the exact solution $u$ with the asymptotic error estimate

$$
\| u - \tilde{u}_h \|_{L_2(\mathbb{R}; H^{s+\varepsilon}(\Omega))} \leq c \cdot h^s \| \tilde{u}_R \|_{L_2(\mathbb{R}; H^{s+\varepsilon}(\Omega))}.
$$

4.4. FEM with finite-dimensional space of trial and test functions

The tensor product decomposition (2.34) in the second part of Corollary 2.8 may provide a singular function method which is similar to the classical situation in that the space of trial and test functions is finite-dimensional. To illustrate this, we assume, as in Section 4.2, that the cylinder $Q$ is replaced with the prism $\tilde{Q} = \Omega \times ]0, 1[$, using Fourier series instead of Fourier transform. The analysis below applies actually also to polyhedrons (see Remark 4.17).

Following line by line the construction of the approximate space $V_h(\Omega)$ in Section 3, we define here the finite element space $V_h(\Omega)$ by formula (3.2), where however, $\Omega$ is replaced by $\tilde{Q}$ and $(\mathcal{T}_h)$ by a suitable family of triangulations of $\Omega$ made up of tetrahedrons for example. It is also assumed that $V_h(\Omega)$ satisfies the analogue of the approximation property in Proposition 3.3.

On the other hand dividing $[0, 1]$ into subintervals of uniform length $h$, we may generate finite-element subspaces $I_h \subset H^{s+m-1}(0, 1)$ containing, in particular, piecewise polynomials of degree $\leq s + m - 1$ and satisfying approximation properties similar to that in Proposition 3.3.

The enriched approximation space to be considered now is, following the notation of the second part of Corollary 2.8, the finite-dimensional space:

$$
\tilde{V}_h(\tilde{Q}) := V_h(\tilde{Q}) \oplus \left\{ \varphi(r) \sum_{(\lambda, \gamma) \in A(s)} \tilde{Z}^{\lambda \gamma}(q_h^{\lambda \gamma}); q_h^{\lambda \gamma} \in I_h \right\}.
$$

Theorem 4.16. The singular function method (4.2b) with $\tilde{V}_h(\tilde{Q})$ in lieu of $W_h^+(Q)$ admits a unique solution $\tilde{u}_h \in \tilde{V}_h(\tilde{Q})$ which, under the conditions of the second part of Corollary 2.8, obeys the error estimate:

$$
\| u - \tilde{u}_h \|_{H^{s+\varepsilon} \tilde{Q}} \leq c \cdot h^s \| \tilde{u}_R \|_{H^{s+\varepsilon} \tilde{Q}}, \quad \forall \varepsilon > 0.
$$

Proof. The existence and uniqueness of $\tilde{u}_h$ is clear because the space $\tilde{V}_h(\tilde{Q})$ is finite-dimensional. Regarding the error estimate, by Céa’s lemma, and the decomposition (2.34) of $u$, we have

$$
\| u - \tilde{u}_h \|_{H^{s+\varepsilon} \tilde{Q}} \leq c \cdot \left\{ \| \tilde{u}_R - r_{h, \tilde{Q}} \|_{H^{s+\varepsilon} \tilde{Q}} + \sum_{(\lambda, \gamma) \in A(s)} \| \varphi(r) \tilde{Z}^{\lambda \gamma}(K^{\lambda \gamma}) - \tilde{Z}^{\lambda \gamma}(r_{h, 0, 1}[K^{\lambda \gamma}]) \|_{H^{s+\varepsilon} \tilde{Q}} \right\},
$$

where $r_{h, \tilde{Q}}$ and $r_{h, 0, 1}[K^{\lambda \gamma}]$ are appropriate interpolation operators as in Proposition 3.3. The regularity $\tilde{u}_R \in H^{s+m-\varepsilon}(\tilde{Q})$, an interpolation argument and Proposition 3.3 yield:

$$
\| \tilde{u}_R - r_{h, \tilde{Q}} \tilde{u}_R \|_{H^{s+\varepsilon} \tilde{Q}} \leq c \cdot h^s \| \tilde{u}_R \|_{H^{s+m-\varepsilon} \tilde{Q}}.
$$

Using Eq. (2.35) and the form of the pseudo-differential operators $\Psi^{\lambda \gamma, \eta}$, we deduce that

$$
\| \varphi(r) \tilde{Z}^{\lambda \gamma}(K^{\lambda \gamma}) - \tilde{Z}^{\lambda \gamma}(r_{h, 0, 1}[K^{\lambda \gamma}]) \|_{H^{s+\varepsilon} \tilde{Q}} \leq c \cdot \sum_{0 \leq p \leq s + m - 1 - \text{Re} \lambda} \| K^{\lambda \gamma} - r_{h, 0, 1}[K^{\lambda \gamma}] \|_{H^{s+p+\varepsilon} \tilde{Q}}.
$$
The regularity of $K^{\lambda,\nu}$ and Proposition 3.3 lead, as above, to
\[
\|K^{\lambda,\nu} - r_{h,h}[K^{\lambda,\nu}]\|_{m+p+\varepsilon,0,1} \leq c \cdot h^{s-1 - \Re\lambda - p - \varepsilon} \|K^{\lambda,\nu}\|_{s+m-1 - \Re\lambda,0,1}.
\]
The condition between $s$, $s_0$ and $s_1$ in Corollary 2.8 implies that
\[
s - 1 - \Re\lambda - p \geq s_0.
\]

The above five inequalities yield (4.19).  

**Remark 4.17.** The analysis of this subsection is inspired by [39]. The approach applies actually also to polyhedrons as described in this reference. Contrary to what is claimed by these authors, the convergence in Theorem 4.16 is not in $O(h^s)$. However, the current convergence in $O(h^{s-\varepsilon})$ is better than the slow convergent standard FEM. Furthermore, this result is not valid for small values of $s$ as for example $s = m$.

Anyway, it would be interesting to check whether the convergence is improved for a singular function method with a finite dimensional space of trial and test functions based on the decomposition (2.32), where $u_R$ has the optimal global regularity $H^{s+m}(Q)$. Notice however that the direct applicability of the optimally convergent SFM approach of Section 4.1 to domains with both vertex and edge singularities is not clear due to the interference of edge and vertex singular functions.

5. An illustrative example

To illustrate the results of the previous sections, we consider the solution $u \in H^1_0(Q)$ of the academic example

\[
Lu := -\Delta u = f \text{ in } Q, \quad u = 0 \text{ on } \partial Q,
\]
where $f \in L^2(Q)$ ($m = 1 = s$), the angle $\omega$ of the polygonal cross section $\Omega$ of the cylinder $Q$ in Section 2.1 being such that $\omega > \pi$. Here, $L_0 = -\Delta = r^{-2}L(rD_r)$ is the two-dimensional Laplace operator acting from $H^1_0(G)$ into $H^{-1}(G)$ where $L(\lambda) = D_0^2 - \lambda^2$. The singular functions of $L_0$ correspond to the poles
\[
\lambda = l\pi/\omega, \quad l \text{ natural integer},
\]
of the operator $L^{-1}(\lambda)$. Notice that all poles $\lambda$ are simple and $J(\lambda) = 1$ (cf. (2.14)). Because of the assumptions $s = 1 = m$ and $\omega > \pi$, there is only one pole (cf. (2.21)), namely $\lambda = \pi/\omega$, which generates the singularities listed in Table 1. (These assumptions exclude also pseudo-differential operators from the analogues of (2.30b) and (2.35) in the table as $p = 0$.)

Most of the numerical schemes presented in the previous sections reduce, for the example in question here, to the approximation of the family of Helmholtz problems

\[
-\Delta w + \xi^2 w = g \text{ in } \Omega, \quad w = 0 \text{ on } \Gamma
\]
by

\[
\text{find } w_h \in D^h_\Omega : \int_{\Omega} (\nabla w_h \nabla v_h + \xi^2 w_h v_h) \, dxdy = \int_{\Omega} g v_h \, dxdy, \quad \forall v_h \in D^h_\Omega,
\]
where $D^h_\Omega$ is a suitable finite-dimensional subspace of $H^1_0(\Omega)$ which contains the standard triangular finite element space $V_h(\Omega) \subset H^1_0(\Omega)$ of piecewise polynomials of degree at most 1.
Table 1
$H^2$ singularities and regularity

<table>
<thead>
<tr>
<th>Operators</th>
<th>Singular part</th>
<th>Regular part</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_0 = -\Delta$</td>
<td>$\sigma(x, y) = \varphi(r) r^{\frac{\alpha}{2} - 1} \sin(\pi/\omega) \theta$</td>
<td>$H^2(\Omega)$</td>
</tr>
<tr>
<td>$L(\xi) = -\Delta + \xi^2$</td>
<td>$b^2 e^{-r</td>
<td>\xi</td>
</tr>
<tr>
<td>$L = -\Delta$</td>
<td>$Z(K) = (\frac{1}{r^2 + z^2}) \ast K \sigma$</td>
<td>$H^2(Q)$</td>
</tr>
<tr>
<td>$L = -\Delta$</td>
<td>$\tilde{Z}(K) = K(z) \sigma$</td>
<td>$L_2(\Theta, H^2(\Omega))$</td>
</tr>
</tbody>
</table>

Table 2
Rates of convergence

<table>
<thead>
<tr>
<th>Numerical methods</th>
<th>Space $D_h^\xi$</th>
<th>Errors on solutions</th>
<th>Errors on coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>Classical FEM</td>
<td>$V_h(\Omega)$</td>
<td>$h^{\xi/\omega - 1}$</td>
<td>Not applicable</td>
</tr>
<tr>
<td>w = $\tilde{u}(\xi)$; g = $\tilde{f}(\xi)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Semi-discrete SFM</td>
<td>$V_h(\Omega) + \text{Sp}(e^{-r</td>
<td>\xi</td>
<td>} \sigma(x, y))$</td>
</tr>
<tr>
<td>w = $\tilde{u}(\xi)$; g = $\tilde{f}(\xi)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SFM</td>
<td>$V_h(\Omega) + \text{Sp}(e^{-r</td>
<td>\xi</td>
<td>} \sigma(x, y))$</td>
</tr>
<tr>
<td>w = $u_\xi$; g = $f_\xi$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fourier series-SFM</td>
<td>on $Q = \Omega \times (0, 1)$</td>
<td>$h + N^{-1}$</td>
<td>$h^{1-\xi/\omega} + N^{(\xi/\omega)-1}$</td>
</tr>
</tbody>
</table>

Table 2 specifies the expressions $D_h^\xi$, w, g in (5.3)–(5.4) as well as the obtained convergence results. It should be noted that (5.3)–(5.4) are popular examples in numerical analysis. Numerical tests confirming rates of convergence in Table 2 have indeed been provided by several authors among which [9,17,19,41]. (Other operators are considered in [18]). Likewise, numerical tests for the approximation of Fourier series by truncated Fourier series, as in the last row of Table 2, are available (see for instance [13,24]).

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